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PAPERS

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FOURIER CONSTANTS AND LEBESGUE CLASSES

By S. VERBLUNSKY.

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1. The object of this paper is to generalize the theorems of Hardy and Littlewood* concerning Fourier constants, by replacing the special sequence of orthogonal functions $e^{im\theta}$, m an integer, by any uniformly bounded sequence of orthogonal functions. The theorems of Hardy and Littlewood with which we are concerned are those numbered 1–11 in the paper cited. The relations between the theorems of the present paper and those of Hardy and Littlewood will be indicated at the appropriate points. In this introductory section we make some remarks about the method used.

The method is essentially an application of the arguments and the theorems of M. Riesz† on linear operations. We shall not require the theorems of M. Riesz in their full generality; we shall use only two special cases; but it seems necessary to give a short account of the ideas involved, and to state the results which we shall use.

Let d and δ denote two intervals, finite or infinite. The variable whose domain is d we denote by x; the variable whose domain is δ we denote by y. Let $a \geqslant 1$, and consider the class $L_d(a)$ of functions $f(x)$ such that $|f|^a$ is integrable in d.

* Hardy and Littlewood, *Math. Annalen*, 97 (1927), 159–209.

† M. Riesz, *Acta Math.*, 46 (1926), 465–497.

Suppose that to each function $f(x)$ of $L_d(a)$ there corresponds a function $T_f(y)$, defined in δ, such that

$$T_{f_1+f_2}(y) = T_{f_1}(y)+T_{f_2}(y);$$

$$\left[\int_\delta |T_f(y)|^c\right]^{1/c} \leqslant M\left[\int_d |f|^a\right]^{1/a}, \tag{1}$$

where $c \geqslant 1$ and M is a constant. We then have a linear functional transformation of the class $L_d(a)$ into a sub-class of $L_\delta(c)$. The least value of M such that (1) is true for all the functions f we denote by $M(a, c)$. It is called the bound of the transformation. We also admit the value $c = \infty$, and by the first member of (1) we then mean the effective maximum of $|T_f(y)|$ in δ, *i.e.* the maximum when sets of zero measure are neglected. By $L_\delta(\infty)$ we mean the class of functions which are effectively bounded in δ.

If we write $$\frac{1}{c}+\frac{1}{c'} = 1,$$

then the bilinear operation

$$A(f, g) = \int_\delta g(y)\, T_f(y)\, dy$$

is defined for all $g(y)$ of $L_\delta(c')$. To each such $g(y)$ there corresponds a function $\bar{T}_g(x)$ of $L_d(a')$, where

$$\frac{1}{a}+\frac{1}{a'} = 1.$$

The operation by which we obtain $\bar{T}_g(x)$ from $g(y)$ is the transposed operation to that by which we obtain $T_f(y)$ from $f(x)$. We have the relations

$$\int_d f(x)\, \bar{T}_g(x)\, dx = \int_\delta g(y)\, T_f(y)\, dy$$

and

$$\left[\int_d |\bar{T}_g(x)|^{a'}\right]^{1/a'} \leqslant M(a, c)\left[\int_\delta |g(y)|^{c'}\right]^{1/c'}. \tag{2}$$

Here $M(a, c)$ is the number defined above, and it is known to be the least number for which (2) is true for all $g(y)$ of $L_\delta(c')$.

Suppose now that $c \geqslant a$. In the class $L_d(a)$ we introduce the distance function

$$D(f_1, f_2) = \left[\int_d |f_1-f_2|^a\, dx\right]^{1/a}.$$

If we know the values of $T_f(y)$ for a class S of functions f which is everywhere dense in $L_d(a)$ for every $a \geqslant 1$, then we can decide whether or not, given $a_1 \geqslant 1$ and $c_1 \geqslant a_1$, there is a least number $M(a_1, c_1)$ such that

$$\left[\int_\delta |T_f(y)|^{c_1}\right]^{1/c_1} \leqslant M(a_1, c_1)\left[\int_\delta |f|^a dx\right]^{1/a}.$$

The first result of M. Riesz which we shall need states that, if for two pairs of numbers $c_1 \geqslant a_1 \geqslant 1$, $c_2 \geqslant a_2 \geqslant 1$ there exist the numbers $M(a_1, c_1)$, $M(a_2, c_2)$, then $M(a, c)$ exists for all pairs (a, c) such that $(1/a, 1/c)$ is a point on the segment joining $(1/a_1, 1/c_1)$ and $(1/a_2, 1/c_2)$, and that $\log M(a, c)$, considered as a function of a point on the segment, is convex.

The second result is obtained from the first by replacing the class $L_d(a)$ by a special sub-class. Let the interval d be expressed in some fixed way as the sum of a finite or enumerable set of non-overlapping intervals, and let $F_d(a)$ denote the class of functions which are constant in each sub-interval, and which belong to $L_d(a)$. Then all the above statements are true, if we replace the class $L_d(a)$ by the class $F_d(a)$.

In all that follows we suppose that we are concerned with a *complete* system of orthogonal functions. A little consideration of each theorem will show that this involves no loss of generality.

2. We consider Theorem 1 of Hardy and Littlewood (*loc. cit.*, p. 167), namely the case of equality in Hausdorff's inequality. We suppose in the first place that all the functions are real, and that the orthogonal functions are defined in a finite interval. It is desirable to separate the two halves of Hardy and Littlewood's theorem. We proceed to the generalization of the first half.

THEOREM 1. *Let* $\{\omega_k(x)\}$, $k = 1, 2, \ldots$, *be a system of bounded, normal orthogonal functions for the interval* $(0, 1)$. *Let* $1 < a < 2$. *Let* $f(x)$ *be a function of* $L(a)$ *in* $(0, 1)$, *and let*

$$c_k = \int_0^1 f(x)\,\omega_k(x)\,dx.$$

Let

$$\mu = \max_{(k,\ 0 \leqslant x \leqslant 1)} |\omega_k(x)|. \tag{3}$$

Then

$$\left[\sum_{k=1}^{\infty} |c_k|^{a'}\right]^{1/a'} \leqslant \mu^{(2-a)/a}\left[\int_0^1 |f(x)|^a\right]^{1/a}. \tag{4}$$

In this relation, equality can occur non-trivially only when $f(x)$ is of the form

$$C \sum_{r=1}^{\nu} \epsilon_r \omega_{n_r}(x). \tag{5}$$

where ϵ_r $(r = 1, \ldots, \nu)$ is a constant which satisfies $\epsilon_r^2 = 1$.

Equality does occur for an expression of the form (5) if and only if

$$\left| \sum_{r=1}^{\nu} \epsilon_r \omega_{n_r}(x) \right|$$

has a constant value in a set E of measure $1/(\nu\mu^2)$, and is zero in CE.

The enunciation requires some comment. The trivial case of equality is $f = 0$. The part of the theorem which asserts the inequality is due to F. Riesz*. This part of the theorem has been shown by M. Riesz to be an immediate deduction from his theorems on linear operations, referred to in § 1. The idea of the following proof is this: we apply the arguments by which M. Riesz proves his theorems to the particular case of Theorem 1, and we note in what circumstance any relevant inequalities can become equalities.

Consider the class of functions $f(x)$, defined in (0, 1), such that

$$\left[\int_0^1 |f|^a\right]^{1/a} < \infty.$$

To each function $\phi(x)$ which is integrable in (0, 1), we make a function $T_\phi(y)$ correspond in $0 \leqslant y < \infty$ by the definition

$$T_\phi(y) = \int_0^1 \phi(x)\,\omega_k(x)\,dx \quad (k-1 \leqslant y < k;\ k = 1, 2, \ldots).$$

Let b denote a variable in $1 \leqslant b \leqslant 2$. Then

$$\log \max \frac{\left[\int_0^\infty |T_\phi(y)|^{b'}\right]^{1/b'}}{\left[\int_0^1 |\phi(x)|^b\right]^{1/b}}, \tag{6}$$

where the maximum is taken for all ϕ of $L(b)$, is a convex function of $\beta = 1/b$. The maximum of the quotient in (6) has the value 1 when $b = 2$.

* F. Riesz, *Math. Zeitschrift*, 18 (1923), 117–124.

For $b = 1$ the quotient becomes

$$\max_{(k)} \left| \int_0^1 \phi(x)\, \omega_k(x)\, dx \right| \Big/ \int_0^1 |\phi(x)|\, dx,$$

and the maximum of this maximum is μ. Hence, if we denote the maximum in (6) by M_β, we have $M_\beta = \mu^{2\beta-1}$, and for $\beta = \alpha = 1/a$,

$$M_\alpha = \mu^{2\alpha-1}.$$

This is the inequality of Theorem I.

A function $y = f(x)$, which is convex in a closed interval, can take a value at an interior point which corresponds to a point on the chord joining the end points of the arc $y = f(x)$ only if $f(x)$ is linear. Thus we can have $M_\alpha = \mu^{2\alpha-1}$ only if

$$M_\beta = \mu^{2\beta-1} \tag{7}$$

for all b satisfying $1 \leqslant b \leqslant 2$.

We define the bilinear functional

$$A(\phi, \psi) = \int_0^\infty \psi(y)\, T_\phi(y)\, dy.$$

The functional transformation $\bar{T}_\psi(x)$ which is transposed to $T_\phi(y)$ satisfies the relation

$$\int_0^\infty \psi(y)\, T_\phi(y)\, dy = \int_0^1 \phi(x)\, \bar{T}_\psi(x)\, dx. \tag{8}$$

Take $\phi(x) = \omega_k(x)$. Then

$$T_\phi(y) = 1 \quad \text{in } (k-1, k),$$

$$= 0 \quad \text{elsewhere},$$

and (8) becomes

$$\int_{k-1}^k \psi(y)\, dy = \int_0^1 \bar{T}_\psi(x)\, \omega_k(x)\, dx.$$

Since the system of orthogonal functions is complete, we have

$$\bar{T}_\psi(x) \sim \sum_{k=1}^\infty \left(\int_{k-1}^k \psi(y)\, dy \right) \omega_k(x)\, dx. \tag{9}$$

We now suppose that there is a function $f_0(x)$ which gives equality in (4), so that (7) holds. On grounds of homogeneity we can suppose that

$$\int_0^1 |f_0|^a dx = 1. \tag{10}$$

We define a function $g_0(y)$ by the relations

$$|T_{f_0}(y)| = \mu^{2a-1} |g_0(y)|^{a-1}, \tag{11}$$

$$\operatorname{sgn} T_{f_0}(y)\, g_0(y) = 1. \tag{12}$$

Then

$$\int_0^\infty |g_0(y)|^a dy = 1. \tag{13}$$

We have

$$|A(f_0, g_0)| = \left| \int_0^\infty g_0(y)\, T_{f_0}(y)\, dy \right| = \mu^{2a-1}, \tag{14}$$

and

$$|A(f_0, g_0)| = \left| \int_0^1 f_0(x)\, \bar{T}_{g_0}(x)\, dx \right|$$

$$\leqslant \left[\int_0^1 |\bar{T}_{g_0}(x)|^{a'} \right]^{1/a'} \left[\int_0^1 |f_0|^a \right]^{1/a}. \tag{15}$$

We have further, for any $\phi \subset L(a)$ in $(0, 1)$, $\psi \subset L(a)$ in $(0, \infty)$,

$$\left| \int_0^1 \phi \bar{T}_\psi(x)\, dx \right| = |A(\phi, \psi)| = \left| \int_0^\infty \psi(y)\, T_\phi(y)\, dy \right|$$

$$\leqslant \left[\int_0^\infty |\psi(y)|^a \right]^{1/a} \left[\int_0^\infty |T_\phi(y)|^{a'} \right]^{1/a'}$$

$$\leqslant \mu^{2a-1} \left[\int_0^\infty |\psi(y)|^a \right]^{1/a} \left[\int_0^1 |\phi(x)|^a \right]^{1/a}.$$

If we keep ψ fixed and vary ϕ, we see that

$$\left[\int_0^1 |\bar{T}_\psi(x)|^{a'} \right]^{1/a'} \leqslant \mu^{2a-1} \left[\int_0^\infty |\psi(y)|^a \right]^{1/a}.$$

If for ψ we take g_0, we get

$$\left[\int_0^1 |\bar{T}_{g_0}(x)|^{a'} \right]^{1/a'} \leqslant \mu^{2a-1}. \tag{16}$$

If now we had inequality in (15), we should have, by (14), (15), (10), and (16),

$$\mu^{2\alpha-1} < \left[\int_0^1 |\bar{T}_{g_0}(x)|^{\alpha'}\right]^{1/\alpha'} \leqslant \mu^{2\alpha-1},$$

a contradiction. Thus there is equality in (15). The equation

$$\left|\int_0^1 f_0(x)\,\bar{T}_{g_0}(x)\,dx\right| = \left[\int_0^1 |\bar{T}_{g_0}(x)|^{\alpha'}\right]^{1/\alpha'}\left[\int_0^1 |f_0|^{\alpha}\right]^{1/\alpha}$$

implies that

$$|\bar{T}_{g_0}(x)| = \mu^{2\alpha-1}|f_0(x)|^{\alpha-1} \tag{17}$$

and

$$\operatorname{sgn}\bar{T}_{g_0}(x)\,f_0(x) = \epsilon \quad \text{for} \quad \bar{T}_{g_0}(x) \neq 0,$$

where ϵ is a constant such that $\epsilon^2 = 1$.

We choose α_1 and α_2 to satisfy $\alpha_1 < \alpha < \alpha_2$ and*

$$\frac{1-\alpha_1}{\alpha_2} \leqslant \frac{\alpha-\alpha_1}{\alpha_2-\alpha} \leqslant \frac{\alpha_1}{1-\alpha_2}.$$

We write

$$a_1 = \frac{1}{\alpha_1}, \quad a_1' = \frac{1}{1-\alpha_1},$$

and similarly for a_2, a_2'. Then

$$\mu^{2\alpha-1}\left[\int_0^1 |f_0|^{(\alpha-1)a_1'}\right]^{1/a_1'} = \left[\int_0^1 |\bar{T}_{g_0}(x)|^{\alpha_1'}\right]^{1/a_1'} \leqslant \mu^{2\alpha_1-1}\left[\int_0^\infty |g_0(y)|^{\alpha_1}\right]^{1/a_1}, \tag{18}$$

$$\mu^{2\alpha-1}\left[\int_0^\infty |g_0(y)|^{(\alpha-1)a_2'}\right]^{1/a_2'} = \left[\int_0^\infty |T_{f_0}(y)|^{\alpha_2'}\right]^{1/a_2'} \leqslant \mu^{2\alpha_2-1}\left[\int_0^1 |f_0(x)|^{\alpha_2}\right]^{1/a_2}. \tag{19}$$

By multiplying (18) and (19), raised respectively to the powers $\frac{\alpha_2-\alpha}{\alpha_2-\alpha_1}$ and $\frac{\alpha-\alpha_1}{\alpha_2-\alpha_1}$, and cancelling $\mu^{2\alpha-1}$ on both sides, we get

$$\left[\int_0^1 |f_0|^{(\alpha-1)a_1'}\right]^{\frac{1}{a_1'}\cdot\frac{\alpha_2-\alpha}{\alpha_2-\alpha_1}}\left[\int_0^\infty |g_0|^{(\alpha-1)a_2'}\right]^{\frac{1}{a_2'}\cdot\frac{\alpha-\alpha_1}{\alpha_2-\alpha_1}} \leqslant \left[\int_0^\infty |g_0(y)|^{\alpha_1}\right]^{\frac{1}{a_1}\cdot\frac{\alpha_2-\alpha}{\alpha_2-\alpha_1}}\left[\int_0^1 |f_0|^{\alpha_2}\right]^{\frac{1}{a_2}\cdot\frac{\alpha-\alpha_1}{\alpha_2-\alpha_1}}. \tag{20}$$

* If we write $\alpha_1 = \alpha-\delta$, $\alpha_2 = \alpha+\eta$, we see that the condition can be satisfied with small δ and η if $(1-\alpha)^2 < \alpha^2$, which is true since $\alpha > \frac{1}{2}$.

Since $a_2 < a$, (10) implies that

$$\left[\int_0^1 |f_0|^{a_2}\right]^{1/a_2} < \infty.$$

Hence, by (19),
$$\left[\int_0^\infty |g_0|^{(a-1)a_2'}\right]^{1/a_2'} < \infty. \tag{21}$$

Write
$$p = \frac{1-a_2}{a_1} \cdot \frac{a-a_1}{a_2-a},$$

so that $0 \leqslant p \leqslant 1$, and

$$a_1 = p(a-1)a_2' + (1-p)a.$$

Then

$$\left[\int_0^\infty |g_0|^{a_1}\right]^{\frac{1}{a_1}\cdot\frac{a_2-a}{a_2-a_1}} = \left[\int_0^\infty |g_0|^{p(a-1)a_2'+(1-p)a}\right]^{\frac{1}{a_1}\cdot\frac{a_2-a}{a_2-a_1}}$$

$$\leqslant \left[\int_0^\infty |g_0|^{(a-1)a_2'}\right]^{\frac{p}{a_1}\cdot\frac{a_2-a}{a_2-a_1}} \left[\int_0^\infty |g_0|^{a}\right]^{\frac{1-p}{a_1}\cdot\frac{a_2-a}{a_2-a_1}}$$

$$\leqslant \left[\int_0^\infty |g_0|^{(a-1)a_2'}\right]^{\frac{1}{a_1'}\cdot\frac{a-a_1}{a_2-a_1}}. \tag{22}$$

By (21) it follows that

$$\int_0^\infty |g_0|^{a_1} < \infty,$$

and hence, by (18), that

$$\int_0^1 |f_0|^{(a-1)a_1'} < \infty.$$

Thus each of the four factors in (20) is finite.

We write
$$q = \frac{1-a_1}{a_2} \cdot \frac{a_2-a}{a-a_1},$$

so that $0 \leqslant q \leqslant 1$, and

$$a_2 = q(a-1)a_1' + (1-q)a. \tag{23}$$

Then

$$\left[\int_0^1 |f_0|^{a_2}\right]^{\frac{1}{a_2}\cdot\frac{a-a_1}{a_2-a_1}} = \left[\int_0^1 |f_0|^{q(a-1)a_1'+(1-q)a}\right]^{\frac{1}{a_2}\cdot\frac{a-a_1}{a_2-a_1}}$$

$$\leqslant \left[\int_0^1 |f_0|^{(a-1)a_1'}\right]^{\frac{q}{a_2}\cdot\frac{a-a_1}{a_2-a_1}} \left[\int_0^1 |f_0|^a\right]^{\frac{1-q}{a_2}\cdot\frac{a-a_1}{a_2-a_1}} \tag{24}$$

$$\leqslant \left[\int_0^1 |f_0|^{(a-1)a_1'}\right]^{\frac{1}{a_1'}\cdot\frac{a_2-a}{a_2-a_1}}. \tag{25}$$

Multiplying (22) and (25), we obtain (20) with the sign of inequality reversed. Hence there is equality in (20), (22), and (25). We now observe that equality in (25) requires Hölder's inequality in (24) to be an equality, and this requires that

$$|f_0| = K \quad \text{in} \quad E \subset (0, 1), \tag{26}$$

$$= 0 \quad \text{in} \quad CE.$$

By (10), $$(mE)K^a = 1. \tag{27}$$

Similarly, equality in (22) requires that

$$|g_0| = L \quad \text{in} \quad F \subset (0, \infty),$$

$$= 0 \quad \text{in} \quad CF.$$

By (11), $g_0(y)$ differs from zero only where $T_{f_n}(y)$ does. By the definition of the operation $T_f(y)$, the set F must be the sum of a set of intervals whose end points are consecutive integers, and, by (13), there is only a finite number of these intervals. We denote them by

$$(k_{n_r}-1,\ k_{n_r}) \quad (r = 1, 2, \ldots \nu).$$

Then (13) gives $$\nu L^a = 1. \tag{28}$$

We have $$g_0(y) = \epsilon_r L \quad (k_{n_r}-1 \leqslant y < k_{n_r}),$$

where, by the definition of the operation $T_f(y)$ and by (12), ϵ_r is a constant such that $\epsilon_r^2 = 1$.

By (9), $$\bar{T}_{g_0}(x) = L \sum_{r=1}^{\nu} \epsilon_r \omega_{n_r}(x). \tag{29}$$

We have, by (11),

$$|T_{f_0}(y)| = \mu^{2\alpha-1} L^{\alpha-1} \quad \text{for} \quad (k_{n_r}-1 \leqslant y < k_{n_r}) \quad (r = 1, 2, \ldots \nu),$$

$$= 0 \qquad \text{elsewhere},$$

and, by (12),

$$\operatorname{sgn} T_{f_0}(y) = \epsilon_r \quad \text{for} \quad (k_{n_r}-1 \leqslant y < k_{n_r}).$$

From the definition of $T_{f_0}(y)$, it follows that

$$f_0(x) = \mu^{2\alpha-1} L^{\alpha-1} \sum_{r=1}^{\nu} \epsilon_r \omega_{n_r}(x). \tag{30}$$

By (17), (29), and (30), we have

$$L|\Sigma \epsilon_r \omega_{n_r}(x)| = \mu^{2\alpha-1} |\mu^{2\alpha-1} L^{\alpha-1} \Sigma \epsilon_r \omega_{n_r}(x)|^{\alpha-1}.$$

Hence
$$|\Sigma \epsilon_r \omega_{n_r}(x)|^{2-\alpha} = \mu^{2-\alpha} L^{\alpha(\alpha-2)} \quad \text{for} \quad x \subset E, \tag{31}$$

$$= 0 \qquad \text{for} \quad x \subset CE.$$

By (26) and (30),

$$|\Sigma \epsilon_r \omega_{n_r}(x)| = K\mu^{1-2\alpha} L^{1-\alpha} \quad \text{for} \quad x \subset E.$$

Hence, by (31), $(KL)^{\alpha} = \mu^2$, and so, by (27) and (28), $\nu . mE = 1/\mu^2$.

It is easily verified that a function of the form (30), where the sum possesses the properties stated in the enunciation of Theorem I, does give equality in (4). This completes the proof.

3. (*a*) Consider the case in which the functions $\{\omega_k(x)\}$ are defined in an infinite interval d. The enunciation of the theorem is unaltered, and the above proof remains the same up to the relation (20). We now require a modification of the argument, since

$$\int_d |f_0|^{\alpha} < \infty$$

does not imply that
$$\int_d |f_0|^{\alpha_2} < \infty. \tag{32}$$

If, however, we had

$$\int_d |f_0|^{\alpha_2} = \infty,$$

then, by the analogue of (25), we should have

$$\int_d |f_0|^{(a-1)a_1'} = \infty.$$

The analogue of (18) would then give

$$\int_0^\infty |g_0|^{a_1} = \infty. \tag{33}$$

Now $a_1 > a$, and

$$\int_0^\infty |g_0|^a = 1.$$

By (11), (12), and the definition of $T_{f_0}(y)$, g_0 is constant in each interval whose end points are consecutive integers. If the constant value in $(k-1, k)$ is u_k, then

$$\Sigma |u_k|^a = 1.$$

Hence

$$[\Sigma |u_k|^{a_1}]^{1/a_1} \leqslant 1,$$

which contradicts (33). Thus (32) is true, and the proof is then completed as before.

(*b*) For the system of functions

$$1, \quad \sqrt{2}\cos 2\pi x, \quad \sqrt{2}\sin 2\pi x, \quad \ldots$$

which are normal and orthogonal for the interval (0, 1), equality cannot occur non-trivially. We have $\mu = \sqrt{2}$. The only trigonometric polynomial which is constant in a set of positive measure is a constant. But when $f(x) = 1$, equality does not occur, since

$$1 < (\sqrt{2})^{(2-a)/a}$$

for $1 < a < 2$.

(*c*) The enunciation and proof of Theorem I need only small alterations when we consider complex functions. Let $\{\omega_k(x)\}$ be a set of complex functions which are normal and orthogonal for the interval (0, 1). We denote the function conjugate to $\omega_k(x)$ by $\bar{\omega}_k(x)$, so that

$$\int_0^1 \omega_k(x)\,\bar{\omega}_k(x)\,dx = 1, \quad \int_0^1 \omega_m(x)\,\bar{\omega}_n(x)\,dx = 0 \quad (m \neq n).$$

In the enunciation of Theorem I, we now understand ϵ_r to be a constant of the form $e^{i\theta_r}$. The only change in the proof is the definition of the linear

transformation $T_\phi(y)$. It is now defined by

$$T_\phi(y) = \int_0^1 \phi(x)\,\bar{\omega}_k(x)\,dx \quad (k-1 \leqslant y < k).$$

For the system

$$1,\ e^{2\pi ix},\ e^{-2\pi ix},\ e^{4\pi ix},\ e^{-4\pi ix},\ \ldots,$$

which is normal and orthogonal in (0, 1), we have $\mu = 1$. The only linear combination of these functions which has a constant modulus in a set of positive measure is a constant multiple of any one of them. There is then equality in (4). This is the result of Hardy and Littlewood.

(*d*) We now turn to the generalization of the second half of Theorem 1 of Hardy and Littlewood. The system $\{\omega_k(x)\}$ and the number μ are defined as in the enunciation of Theorem I.

THEOREM II. *Let* $1 < a < 2$. *If* (c_k), $k = 1, 2, \ldots$, *is a sequence of constants such that* $\Sigma |c_k|^a$ *is convergent, then there is a function* $f(x)$ *in* (0, 1) *such that*

$$c_k = \int_0^1 f(x)\,\omega_k(x)\,dx \quad (k = 1, 2, \ldots),$$

and such that

$$\left[\int_0^1 |f|^{a'}\right]^{1/a'} \leqslant \mu^{(2-a)/a}\left[\sum_1^\infty |c_k|^a\right]^{1/a}.$$

In this relation, equality can occur non-trivially only if a finite number of the c_k, *say* $c_{n_1}, c_{n_2}, \ldots, c_{n_\nu}$, *differ from zero, and are respectively of the form* $c\epsilon_1, c\epsilon_2, \ldots, c\epsilon_\nu$, *where* ϵ_r *is a constant which satisfies* $\epsilon_r^2 = 1$.

Equality occurs if and only if

$$\left|\sum_{r=1}^{\nu} \epsilon_r\,\omega_{n_r}(x)\right|$$

has a constant value in a set E *of measure* $1/(\nu\mu^2)$, *and is zero in* CE.

Consider the class of functions $g(y)$, defined in $(0, \infty)$, which have a constant value in each interval $(k-1 \leqslant y < k)$, and are such that

$$\int_0^\infty |g|^a < \infty.$$

To each function $\psi(y)$ which is defined in $(0, \infty)$, and which is constant in each interval $(k-1 \leqslant y < k)$, we make a function $T_\psi(x)$ correspond in $(0, 1)$, by

$$T_\psi(x) \sim \sum_{k=1}^{\infty} \left[\int_{k-1}^{k} \psi(y)\, dy \right] \omega_k(x).$$

The proof is a manifest transposition of the proof of Theorem I. We use the second result mentioned in § 1, instead of the first.

Concerning Theorem II, remarks similar to (a), (b), (c) may be made.

4. We are now concerned with the generalization of Theorems 5, 6, 7 of Hardy and Littlewood (*loc. cit.*, p. 189). A generalization of these theorems has been given by Paley* without an evaluation of the constant which enters into the inequalities. By a considerable elaboration of Paley's proof, we show that the constant is of the same order of magnitude as in the case considered by Hardy and Littlewood.

We first prove a preliminary theorem.

THEOREM A. *Let $\{\omega_k(x)\}$ be a sequence of normal orthogonal functions defined in the interval* $(0, 1)$. *Let*

$$\max_{(k,\ 0 \leqslant x \leqslant 1)} |\omega_k(x)| = \mu.$$

Let p denote a positive even integer. Given a sequence (a_n), $n = 1, 2, \ldots$, such that $\Sigma |a_n|^p n^{p-2}$ is convergent, there is a function $f(x)$ in $(0, 1)$ such that

$$a_n = \int_0^1 f(x)\, \omega_n(x)\, dx \quad (n = 1, 2, \ldots),$$

and such that

$$\int_0^1 |f|^p \leqslant \mu^{p-2} C^{p-2} (p-2)!\, \Sigma |a_n|^p n^{p-2},$$

where C is an absolute constant.

LEMMA 1. *If p is any integer greater than or equal to 2, and $r = 0, 1, \ldots$, then*

$$\int_0^1 \left| \sum_{2^r+1}^{2^{r+1}} a_\nu \omega_\nu(x) \right|^p dx \leqslant \mu^{p-2} \sum_{2^r+1}^{2^{r+1}} |a_\nu|^p \nu^{p-2}.$$

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* Paley, *Studia Math.*, 3 (1931), 226–238.

The proof is by induction. The lemma is true for $p = 2$. Assume it for all integers less than p. If p is even, $p = 2k$, we have

$$\int_0^1 \left| \sum_{2^r+1}^{2^{r+1}} a_\nu \omega_\nu(x) \right|^{2k} \leqslant \max_{0 \leqslant x \leqslant 1} |\Sigma a_\nu \omega_\nu(x)|^k \int_0^1 |\Sigma a_\nu \omega_\nu(x)|^k \, dx. \tag{34}$$

We can apply the lemma to the second factor in (34). The expression which we are considering does not exceed

$$\mu^k [\Sigma |a_\nu|]^k \mu^{k-2} \Sigma |a_\nu|^k \nu^{k-2}$$

$$= \mu^{2k-2} [\Sigma |a_\nu| \nu^{\frac{2k-2}{2k}} \nu^{-\frac{2k-2}{2k}}]^k [\Sigma |a_\nu|^k \nu^{k-1} \nu^{-1}]$$

$$\leqslant \mu^{2k-2} [\Sigma a_\nu^{2k} \nu^{2k-2}]^{\frac{1}{2}} [\Sigma \nu^{-\frac{2k-2}{2k-1}}]^{k-\frac{1}{2}} [\Sigma a_\nu^{2k} \nu^{2k-2}]^{\frac{1}{2}} [\Sigma \nu^{-2}]^{\frac{1}{2}}. \tag{35}$$

Now
$$\sum_{2^r+1}^{2^{r+1}} \nu^{-\frac{2k-2}{2k-1}} < \int_{2^r}^{2^{r+1}} x^{-\frac{2k-2}{2k-1}} \, dx$$

$$< (2k-1) 2^{\frac{r}{2k-1}} (2^{\frac{1}{2k-1}} - 1)$$

$$< 2^{\frac{r}{2k-1}};$$

and
$$\sum_{2^r+1}^{2^{r+1}} \nu^{-2} < \int_{2^r}^{2^{r+1}} x^{-2} \, dx$$

$$< \frac{1}{2^{r+1}},$$

so that the product of the third and fifth factors in (35) does not exceed 1.

If, however, p is odd, $p = 2k+1$, then

$$\int_0^1 |\Sigma|^{2k+1} \leqslant \max_{(0 \leqslant x \leqslant 1)} |\Sigma a_\nu \omega_\nu(x)|^k \int_0^1 |\Sigma|^{k+1}.$$

To the second factor we can apply the lemma, since $2 \leqslant k+1 < p$. Thus the expression which we are considering does not exceed

$$\mu^k [\Sigma |a_\nu|]^k \mu^{k-1} \Sigma |a_\nu|^{k+1} \nu^{k-1}$$

$$\leqslant \mu^{2k-1} [\Sigma |a_\nu| \nu^{\frac{2k-1}{2k+1}} \nu^{-\frac{2k-1}{2k+1}}]^k [\Sigma |a_\nu|^{k+1} \nu^{\frac{(2k-1)(k+1)}{2k+1}} \nu^{-\frac{2k}{2k+1}}]$$

$$\leqslant \mu^{2k-1} [\Sigma |a_\nu|^{2k+1} \nu^{2k-1}]^{-\frac{k}{2k+1}} [\Sigma \nu^{-\frac{2k-1}{2k}}]^{\frac{2k \cdot k}{2k+1}} [\Sigma |a_\nu|^{2k+1} \nu^{2k-1}]^{\frac{k+1}{2k+1}} [\Sigma \nu^{-2}]^{\frac{k}{2k+1}},$$

and the product of the third and fifth factors is again less than 1. This proves the lemma.

We write

$$f_1(x) = a_1 \omega_1(x),$$

$$f_2(x) = a_2 \omega_2(x) = \sum_{2^0+1}^{2^1} a_\nu \omega_\nu(x),$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$f_k(x) = \sum_{2^{k-2}+1}^{2^{k-1}} a_\nu \omega_\nu(x).$$

We write further

$$\epsilon_1 = a_1{}^p,$$

$$\epsilon_2 = a_2{}^p\, 2^{p-2},$$

$$\cdots \quad \cdots \quad \cdots$$

$$\epsilon_k = \sum_{2^{k-2}+1}^{2^{k-1}} a_\nu{}^p\, \nu^{p-2},$$

where p is an even integer.

LEMMA 2. *For* $p = 2k$,

$$\int_0^1 |f_{m_1} f_{m_2}|^k\, dx \leqslant \mu^{p-2}\, \epsilon_{m_1}^{\frac{1}{2}}\, \epsilon_{m_2}^{\frac{1}{2}}\, 2^{-\frac{1}{2}|m_1 - m_2|}.$$

We may suppose that $m_1 \leqslant m_2$. Suppose first that $m_1 > 1$. Then

$$\int |f_{m_1} f_{m_2}|^k \leqslant \max_{0 \leqslant x \leqslant 1} |f_{m_1}|^k \int |f_{m_2}|^k$$

$$\leqslant \mu^k \left[\sum_{2^{m_1-2}+1}^{2^{m_1-1}} |a_\nu| \right]^k \mu^{k-2} \sum_{2^{m_2-2}+1}^{2^{m_2-1}} |a_\nu|^k\, \nu^{k-2} \quad \text{(by Lemma 1)}$$

$$\leqslant \mu^{p-2} \left[\sum_{2^{m_1-2}+1}^{2^{m_1-1}} |a_\nu|\, \nu^{\frac{2k-2}{2k}}\, \nu^{-\frac{2k-2}{2k}} \right]^k \left[\sum_{2^{m_2-2}+1}^{2^{m_2-1}} |a_\nu|^k\, \nu^{k-1}\, \nu^{-1} \right]$$

$$\leqslant \mu^{p-2}\, \epsilon_{m_1}^{\frac{1}{2}}\, 2^{\frac{1}{2}(m_1-2)}\, \epsilon_{m_2}^{\frac{1}{2}}\, 2^{-\frac{1}{2}(m_2-1)} \qquad (36)$$

$$\leqslant \mu^{p-2}\, \epsilon_{m_1}^{\frac{1}{2}}\, \epsilon_{m_2}^{\frac{1}{2}}\, 2^{-\frac{1}{2}|m_1 - m_2|}.$$

If $m_1 = 1$, $m_2 > 1$, then $\max |f_{m_1}|^k \leqslant \mu^k |a_1|^k$, and the factor $2^{\frac{1}{2}(m_1-2)}$ does not occur in (36), so that the lemma still holds. Finally, the lemma is true if $m_1 = m_2 = 1$.

LEMMA 3. *For* $p = 2k$,

$$\int_0^1 \Big(\sum_{m=1}^{\infty} |f_m| \Big)^{2k} dx < \mu^{p-2} C^p \, p\,! \, \Sigma |a_n|^p \, n^{p-2},$$

where C *is an absolute constant.*

We have

$$\int_0^1 \Big(\sum_1^{\infty} |f_m| \Big)^{2k} dx = \Sigma \int_0^1 |f_{m_1} f_{m_2} \dots f_{m_{2k}}| \, dx,$$

where the sum is taken over all permutations of positive integers $m_1, \dots, m_{2k}$. For a given set of such integers, there are $k(2k-1)$ combinations two at a time. Hence

$$\int_0^1 |f_{m_1} f_{m_2} \dots f_{m_{2k}}| \, dx = \int_0^1 \Pi |f_{m_r} f_{m_s}|^{\frac{1}{2k-1}} dx,$$

where m_r, m_s is a combination, and the product is taken over the $k(2k-1)$ such combinations. Then

$$\int_0^1 \Pi |f_{m_r} f_{m_s}|^{\frac{1}{2k-1}} \leqslant \Pi \Big[\int_0^1 |f_{m_r} f_{m_s}|^k dx \Big]^{\frac{1}{k(2k-1)}}$$

$$\leqslant \mu^{2k-2} \, [\Pi (\epsilon_{m_r} \epsilon_{m_s})^{\frac{1}{2k(2k-1)}}] \, 2^{-\frac{1}{2k(2k-1)} \Sigma |m_r - m_s|}$$

by Lemma 2. In the last expression

$$\Sigma |m_r - m_s| \tag{37}$$

is the sum of $k(2k-1)$ terms, each of which arises from the corresponding combination m_r, m_s. We write (37) as

$$\sum_{(c)} (m_1, \dots, m_{2k}).$$

We have

$$\Pi (\epsilon_{m_r} \epsilon_{m_s})^{\frac{1}{2k(2k-1)}} < \frac{\Sigma (\epsilon_{m_r} \epsilon_{m_s})^{\frac{1}{2}}}{k(2k-1)} < \frac{\Sigma (\epsilon_{m_r} + \epsilon_{m_s})}{2k(2k-1)},$$

the sum being taken over the $k(2k-1)$ combinations. Each ϵ_{m_r} occurs equally often, and since there are $2 \times k(2k-1)$ terms in the last sum, each ϵ_{m_r} occurs $2k-1$ times. Thus

$$\Pi (\epsilon_{m_r} \epsilon_{m_s})^{\frac{1}{2k(2k-1)}} < \frac{\epsilon_{m_s} + \dots + \epsilon_{m_{2k}}}{2k},$$

and

$$\int_0^1 \Big(\sum_1^{\infty} |f_m| \Big)^{2k} < \frac{1}{2k} \, \mu^{2k-2} \, \Sigma (\epsilon_{m_1} + \dots + \epsilon_{m_{2k}}) \, 2^{-\frac{1}{2k(2k-1)} \sum_{(c)} (m_1, \dots, m_{2k})}, \tag{38}$$

the first summation being over all the permutations of positive integers $m_1, \ldots, m_{2k}$.

We write $$\rho = 2^{-\frac{1}{2k(2k-1)}},$$

and $$\Sigma(\epsilon_{m_1}+\ldots+\epsilon_{m_{2k}})\rho^{\sum\limits_{(c)}(m_1, \ldots, m_{2k})} = \sum_1^\infty A_m \epsilon_m. \tag{39}$$

To find A_m, we observe that, when $m_1 = m$, there arises a contribution to A_m which is equal to

$$B_m = \Sigma \rho^{\sum\limits_{(c)}(m, m_2, \ldots, m_{2k})},$$

where the first summation is taken over all permutations of the $2k-1$ positive integers $m_2, \ldots, m_{2k}$. If $m_2 = m$, we have, on grounds of symmetry, an equal contribution; and so on. Thus

$$A_m = 2k\, B_m. \tag{40}$$

We alter the notation for B_m, and write

$$B_m = \Sigma \rho^{\sum\limits_{(c)}(m, m_1, \ldots, m_{2k-1})}, \tag{41}$$

the first summation being over all permutations of positive integers $m_1, \ldots, m_{2k-1}$. We write

$$B_m = \sum_{n=1}^\infty C_n^{(m)} \rho^n. \tag{42}$$

Then $$C_n^{(m)} = \Sigma\, 1,$$

where the sum is taken over all permutations subject to

$$\sum_{(c)}(m, m_1, \ldots, m_{2k-1}) = n. \tag{43}$$

If we take $2k-1$ numbers which satisfy (43), then (43) remains satisfied when the numbers are permuted. Hence

$$C_n^{(m)} = (2k-1)!\,\Sigma\, 1, \tag{44}$$

where the sum is taken over all combinations $m_1, \ldots, m_{2k-1}$ which satisfy

$$\sum_{(c)}(m, m_1, \ldots, m_{2k-1}) = n; \quad m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{2k-1}. \tag{45}$$

The condition (45) can be written

$$|m_1-m|+|m_2-m|+\ldots+|m_{2k-1}-m|+\sum_{(c)}(m_1, \ldots, m_{2k-1}) = n;$$

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{2k-1}. \tag{46}$$

If $\lambda(n, m)$ is the number of solutions of (46), we have

$$B_m = (2k-1)!\,\Sigma\lambda(n, m)\rho^n.$$

The number of solutions of (46) is less than the number of solutions of

$$\sum_{(c)}(m_1, \ldots, m_{2k-1}) \leqslant n; \quad m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{2k-1}. \tag{47}$$

This last number is independent of m. We denote it by Λ_n. Then $\lambda(n, m) \leqslant \Lambda_n$, and

$$B_m \leqslant (2k-1)!\,\Sigma\Lambda_n\rho^n. \tag{48}$$

Write

$$\begin{aligned} m_1 - m_2 &= \mu_1, \\ m_2 - m_3 &= \mu_2, \\ \ldots \quad &\ldots \quad \ldots \\ m_{2k-2} - m_{2k-1} &= \mu_{2k-2}, \\ m_{2k-1} &= \mu_{2k-1}. \end{aligned}$$

Then

$$\sum_{(c)}(m_1, \ldots, m_{2k-1}) = (2k-2)\mu_1 + 2(2k-3)\mu_2 + 3(2k-4)\mu_3 + \ldots + (2k-2)\mu_{2k-1}.$$

If r is a non-negative integer, the number of solutions of

$$(2k-2)\mu_1 + 2(2k-3)\mu_2 + \ldots + (2k-2)\mu_{2k-1} = r \tag{49}$$

is the coefficient of x^r in

$$\frac{1}{(1-x^{2k-2})}\,\frac{1}{(1-x^{2(2k-3)})}\cdots\frac{1}{(1-x^{2k-2})}.$$

Hence Λ_n is the coefficient of x^n in

$$\frac{1}{(1-x)}\,\frac{1}{(1-x^{2k-2})}\,\frac{1}{(1-x^{2(2k-3)})}\cdots\frac{1}{(1-x^{2k-2})}.$$

By (48),

$$B_m \leqslant (2k-1)!\,\frac{1}{(1-\rho)}\,\frac{1}{(1-\rho^{2k-2})}\,\frac{1}{(1-\rho^{2(2k-3)})}\cdots\frac{1}{(1-\rho^{2k-2})}.$$

Now

$$\begin{aligned} 1-\rho &= 1-e^{-\frac{\log 2}{2k(2k-1)}} \\ &> \frac{\log 2}{2k(2k-1)}, \end{aligned}$$

and

$$1-\rho^{r(2k-1-r)} > \frac{r(2k-1-r)}{2k(2k-1)}\log 2.$$

Hence
$$B_m \leqslant 2k!\,\frac{2k-1}{\log 2}\prod_{r=1}^{2k-2}\frac{2k(2k-1)}{r(2k-1-r)}\,\frac{1}{\log 2}$$
$$\leqslant \frac{2k!(2k-1)\,[2k(2k-1)]^{2k-2}}{[(2k-2)!]^2\,(\log 2)^{2k-1}}.$$

By (40), we infer the existence of an absolute constant C such that
$$A_m \leqslant C^{2k}(2k)!,$$
which proves the lemma.

Theorem A follows from Lemma 3 in a familiar way, by the theory of strong convergence. We can now prove the theorems referred to at the beginning of § 4. The meaning of $\{\omega_k(z)\}$ and of μ is the same as in Theorem A.

THEOREM III. (i) *Let $2 \leqslant c < \infty$. Let (d_k) be a sequence of numbers such that $\Sigma |d_k|^c k^{c-2}$ is convergent. Then there is a function $h(x)$, defined in $(0, 1)$, such that*
$$d_k = \int_0^1 h(x)\,\omega_k(x)\,dx,$$
and such that
$$\int_0^1 |h(x)|^c\,dx \leqslant \mu^{c-2}(c-1)^c\,K^{c-2}\,\Sigma |d_k|^c\,k^{c-2}.$$

(ii) *Let $1 < a \leqslant 2$. Let $g(y)$ be of class $L(a)$ in $(0, 1)$, and let*
$$c_k = \int_0^1 g(y)\,\omega_k(y)\,dy.$$

Then
$$\sum_1^\infty |c_n|^a\,n^{a-2} \leqslant \mu^{2-a}\left(\frac{1}{a-1}\right)^a K^{2-a}\int_0^1 |g|^a\,dy.$$

In both parts of the theorem, K denotes the same absolute constant. We may suppose a and c to be connected by
$$\frac{1}{a}+\frac{1}{c}=1.$$

Consider the class $F(c)$ of functions $f(x)$ with the properties

(α) $f(x)$ is defined in $(0, \pi^2/6)$;

(β) $f(x)$ is constant in each sub-interval $\delta_k \equiv \left(\sum_{n=1}^{k-1}\frac{1}{n^2} \leqslant x < \sum_{n=1}^{k}\frac{1}{n^2}\right)$;

(γ) $f(x)$ is of class $L(c)$.

We denote the value of $f(x)$ in δ_k by $f(\delta_k)$. Let $T_f(y)$ be defined in $(0, 1)$ by

$$T_f(y) \sim \Sigma\, \omega_k(y) f(\delta_k)/k.$$

By Theorem A,

$$\frac{\left[\int_0^1 |T_f(y)|^c\right]^{1/c}}{\left[\int_0^{\frac{1}{6}\pi^2} |f|^c\right]^{1/c}} \leqslant \mu^{1-2/c}\,(c-1)\,C^{1-2/c}$$

if c is an even integer. By the second result mentioned in § 1,

$$\log M(c) = \log \max \frac{\left[\int_0^1 |T_f(y)|^c\right]^{1/c}}{\left[\int_0^{\frac{1}{6}\pi^2} |f|^c\right]^{1/c}}$$

is a convex function of $1/c$. Hence, if $c \geqslant 2$, and k is a positive integer such that $2k \leqslant c < 2k+2$, then

$$\begin{aligned} M(c) &\leqslant M(2k)^{k(2k+2-c)/c}\, M(2k+2)^{(c-2k)(k+1)/c} \\ &\leqslant [(\mu C)^{1-1/k}(2k-1)]^{k(2k+2-c)/c}\, [(\mu C)^{1-1/(k+1)}(2k+1)]^{(c-2k)(k+1)/c} \\ &\leqslant (\mu C)^{1-2/c}\, [(2k-1)^{k(2k+2-c)/c}\,(2k+1)^{(c-2k)(k+1)/c}]. \end{aligned}$$

Since $2k-1 < 2c$, $2k+1 < 2c$, the expression in square brackets does not exceed $2c$. On the other hand, the expression in square brackets has the value 1 for $c = 2$. Hence it does not exceed $A^{\frac{1}{2}}(c-1)$, where $A > 1$ is an absolute constant, and, *a fortiori*, does not exceed $A^{1-2/c}(c-1)$. This proves (i) with $K = AC$.

We now observe that if $g(y)$ denotes any function of $L(a)$ in $(0, 1)$, then there is a function $\bar{T}_g(x)$ which is subject to

$$\int_0^{\frac{1}{6}\pi^2} \bar{T}_g(x) f(x)\, dx = \int_0^1 g(y)\, T_f(y)\, dy \tag{50}$$

and

$$\max \frac{\left[\int_0^{\frac{1}{6}\pi^2} |\bar{T}_g(x)|^a\right]^{1/a}}{\left[\int_0^1 |g(y)|^a\right]^{1/a}} = M(c). \tag{51}$$

If, in (50), we take

$$\begin{aligned} f(x) &= 1 \quad \text{in} \quad \delta_k, \\ &= 0 \quad \text{elsewhere}, \end{aligned}$$

then $$T_f(y) = \omega_k(y)/k,$$

and (50) gives

$$\int_{\delta_k} \bar{T}_g(x)\,dx = \frac{1}{k}\int_0^1 g(y)\,\omega_k(y)\,dy$$

$$= c_k/k.$$

Then $$\Sigma |c_n|^a n^{a-2} = \Sigma \frac{1}{n^2}\left| n^2 \int_{\delta_n} \bar{T}_g(x)\,dx \right|^a$$

$$\leqslant \int_0^{\frac{1}{4}\pi^2} |\bar{T}_g(x)|^a\,dx,$$

so that, by (51),

$$\Sigma |c_n|^a n^{a-2} \leqslant [M(c)]^a \int_0^1 |g(y)|^a\,dy.$$

If we use the bound previously obtained for $M(c)$, and substitute for c from $1/a+1/c=1$, we get (ii).

The advantage of dealing with a general system of normal orthogonal functions becomes apparent when we observe that, in Theorem III, the functions $\{\omega_k(x)\}$ can be taken in any order. If, for example, we arrange them in (i) of Theorem III so that the corresponding sequence $|d_k|$ is non-increasing, we obtain a more precise inequality. It should, however, be noticed that it is not permissible to suppose that the sequence $|d_k|$ is non-increasing before proving Theorem III. For then we should not be led to consider the class $F(c)$, but a particular sub-class; namely the sub-class which satisfies the additional condition

$$|f(\delta_{k+1})| \leqslant |f(\delta_k)|,$$

and the results of M.Riesz have not been shown to apply to such a class of functions. This has been overlooked by Paley in his proof (*loc. cit.*, p. 227).

Theorem III remains true for an infinite interval, and for complex functions.

5. A further advantage of considering any bounded system of normal orthogonal functions will become apparent in the investigation of the generalization of Theorems 2, 3, 4 of Hardy and Littlewood (*loc.cit.*, p. 175), to which we now proceed. The system $\{\omega_k(x)\}$ and μ have the same meaning as before.

THEOREM IV. (i) *If $q \geqslant 2$ and $|f(\theta)|^2 \theta^{q-2}$ is integrable in $(0, 1)$, and*

$$d_n = \int_0^1 f(\theta)\, \omega_n(\theta)\, d\theta,$$

then

$$\sum_1^\infty |d_n|^q \leqslant \mu^{q-2}\, (q-2)!\; C^{q-2} \int_0^1 |f|^q\, \theta^{q-2}\, d\theta.$$

(ii) *Let $1 < p \leqslant 2$. If (c_n) is a sequence such that $\Sigma |c_n|^p$ is convergent, then there is a function $h(\theta)$ in $(0, 1)$ such that*

$$c_m = \int_0^1 h(\theta)\, \omega_m(\theta)\, d\theta,$$

and such that

$$\int_0^1 |h(\theta)|^\nu\, \theta^{\nu-2}\, d\theta \leqslant \mu^{2-\nu} \left(\frac{1}{p-1}\right)^p C^{2-\nu} \Sigma |c_n|^\nu.$$

Let $\phi = 1/\theta$. Then

$$\int_0^1 |f(\theta)|^q\, \theta^{q-2}\, d\theta = \int_1^\infty f\left(\frac{1}{\phi}\right) \left(\frac{1}{\phi}\right)^{q-2} \frac{d\phi}{\phi^2}$$

$$= \int_1^\infty |g(\phi)|^q\, d\phi,$$

where

$$g(\phi) = \frac{1}{\phi} f\left(\frac{1}{\phi}\right) = \theta f(\theta).$$

The equation

$$d_m = \int_0^1 f(\theta)\, \omega_m(\theta)\, d\theta$$

becomes

$$d_m = \int_1^\infty f\left(\frac{1}{\phi}\right) \omega_m\left(\frac{1}{\phi}\right) \frac{d\phi}{\phi^2}$$

$$= \int_1^\infty g(\phi)\, \omega_m\left(\frac{1}{\phi}\right) \frac{d\phi}{\phi}.$$

Write

$$\gamma_m(\phi) = \omega_m\left(\frac{1}{\phi}\right) \frac{1}{\phi}.$$

Then

$$\int_1^\infty \gamma_m(\phi)\, \gamma_n(\phi)\, d\phi = \int_0^1 \omega_m(\theta)\, \omega_n(\theta)\, d\theta.$$

The functions $\gamma_m(\phi)$ thus form a normal set of orthogonal functions for the

interval $(1, \infty)$. And $|\omega_m(\theta)| \leqslant \mu$ becomes

$$|\gamma_m(\phi)| \leqslant \frac{\mu}{\phi}.$$

Thus the above theorem is the same as the following.

THEOREM V. *Let $\{\gamma_m(\theta)\}$ be a set of normal orthogonal functions for the interval $(1, \infty)$ such that*

$$|\gamma_m(\theta)| \leqslant \frac{\mu}{\theta} \quad (1 \leqslant \theta < \infty). \tag{52}$$

(i) *If $q \geqslant 2$ and $|f(\theta)|^q$ is integrable in $(1, \infty)$, and*

$$d_m = \int_1^\infty f(\theta)\gamma_m(\theta)\,d\theta,$$

then

$$\sum_1^\infty |d_m|^q \leqslant (\mu C)^{q-2}(q-2)! \int_1^\infty |f|^q.$$

(ii) *Let $1 < p \leqslant 2$. If (c_m) is a sequence such that $\Sigma|c_m|^p$ is convergent, then there is a function $h(\theta)$ such that*

$$c_m = \int_1^\infty h(\theta)\gamma_m(\theta)\,d\theta,$$

and

$$\int_1^\infty |h(\theta)|^p\,d\theta \leqslant (\mu C)^{2-p}\left(\frac{1}{p-1}\right)^p \Sigma|c_m|^p.$$

The proof of this theorem is somewhat analogous to the proof of Theorem III. We begin by proving two lemmas which lead up to the proof of (i) for an even integer.

LEMMA 4. *Let δ be a sub-interval of $(1, \infty)$, and let $\mu(\delta)$ denote a common bound for $|\gamma_m(\theta)|$, $m = 1, 2, \ldots$, in δ. Then, given a function $F(\theta)$ which is of class L^p in δ, where $1 < p \leqslant 2$, and writing*

$$c_{m,\delta} = \int_\delta F(\theta)\gamma_m(\theta)\,d\theta,$$

we have the inequality

$$[\Sigma|c_{m,\delta}|^q]^{1/q} \leqslant \{\mu(\delta)\}^{(2-p)/p}\left[\int_\delta |F(\theta)|^p\right]^{1/p} \quad \left(\frac{1}{p}+\frac{1}{q}=1\right).$$

The proof is the same as for the inequality of F. Riesz, which was proved in § 2.

LEMMA 5. *If k is an integer greater than or equal to 2, and*

$$c_{m,n} = \int_{2^n}^{2^{n+1}} f(\theta)\,\gamma_m(\theta)\,d\theta \quad (n = 1, 2, \ldots),$$

then

$$\sum_m \left| \sum_n c_{m,n} \right|^{2k} \leqslant \mu^{2k-2}(Ck)^{2k} \int_2^\infty |f|^{2k}\,d\theta.$$

We have

$$\sum_m \left| \sum_n c_{m,n} \right|^{2k} = \sum_m \sum c_{m,n_1} c_{m,n_2} \ldots c_{m,n_{2k}},$$

where the inner sum is taken over all permutations of $2k$ positive integers $n_1, \ldots, n_{2k}$. The preceding expression is

$$\sum_m \sum c_{m,n_1} c_{m,n_2} \ldots c_{m,n_{2k}}.$$

We have

$$|c_{m,n_1} \ldots c_{m,n_{2k}}| = \Pi\, |c_{m,n_r} c_{m,n_s}|^{1/(2k-1)},$$

the product being taken over the $k(2k-1)$ combinations two at a time of $n_1, \ldots, n_{2k}$. Hence

$$\sum_m |c_{m,n_1} \ldots c_{m,n_{2k}}| = \sum_m \Pi\, |c_{m,n_r} c_{m,n_s}|^{1/(2k-1)}$$

$$\leqslant \Pi \left(\sum_m |c_{m,n_r} c_{m,n_s}|^k \right)^{\frac{1}{k(2k-1)}}.$$

For $n' \geqslant n''$ we have

$$\sum_m |c_{m,n'} c_{m,n''}|^k \leqslant \max_{(m)} |c_{m,n'}|^k \sum_{(m)} |c_{m,n''}|^k.$$

Now

$$|c_{m,n'}| = \left| \int_{2^{n'}}^{2^{n'+}} f(\theta)\,\gamma_m(\theta)\,d\theta \right|$$

$$\leqslant \frac{\mu}{2^{n'}} \int |f|\,d\theta$$

by (52). Thus

$$\max_{(m)} |c_{m,n'}|^k \leqslant \frac{\mu^k}{2^{n'k}} \left[\int |f| \right]^k$$

$$\leqslant \frac{\mu^k}{2^{n'k}} \left[\int |f|^{2k} \right]^{\frac{1}{2}} [2^{n'}]^{k-\frac{1}{2}}$$

$$\leqslant \frac{\mu^k}{2^{\frac{1}{2}n'}} \left[\int |f|^{2k} \right]^{\frac{1}{2}}.$$

Further, by Lemma 4,

$$\left(\sum_m |c_{m,n''}|^k\right)^{1/k} \leqslant \left[\frac{\mu}{2^{n''}}\right]^{(2/k')-1} \left[\int_{2^{n''}}^{2^{n''+1}} |f|^{k'}\right]^{1/k'}$$

$$\leqslant \mu^{(2/k')-1}(2^{n''})^{1/k} \left[\frac{1}{2^{n''}} \int |f|^{k'}\right]^{1/k'}$$

$$\leqslant \mu^{(2/k')-1}(2^{n''})^{1/k} \left[\frac{1}{2^{n''}} \int |f|^{k}\right]^{1/k}$$

$$\leqslant \mu^{(2/k')-1} \left[\int |f|^k\right]^{1/k},$$

so that

$$\sum_m |c_{m,n''}|^k \leqslant \mu^{(2k/k')-k} \int |f|^k$$

$$\leqslant \mu^{k-2}\, 2^{\frac{1}{2}n''} \left[\int |f|^{2k}\right]^{\frac{1}{2}}.$$

With the notation

$$\epsilon_n = \int_{2^n}^{2^{n+1}} |f|^{2-k},$$

we have

$$\sum_m |c_{m,n'}\, c_{m,n''}|^k \leqslant \mu^{2k-2}(\epsilon_{n'}\, \epsilon_{n''})^{\frac{1}{2}}\, 2^{-\frac{1}{2}|n'-n''|}.$$

Thus, by the argument of Lemma 3,

$$\sum_m \left|\sum_n c_{m,n}\right|^{2k} \leqslant \frac{1}{2k}\, \mu^{2k-2} \sum (\epsilon_{n_1} + \ldots + \epsilon_{n_{2k}})\, 2^{-\frac{1}{2k(2k-1)} \sum\limits_{(\epsilon)} (n_1, \ldots, n_{2k})}$$

$$\leqslant \mu^{2k-2}(Ck)^{2k} \int_2^\infty |f|^{2k}.$$

We can now prove (i) for $q = 2k$. We may suppose that $k \geqslant 2$. We write

$$d_m = d_m{}^0 + \sum_{n=1}^{\infty} c_{m,n},$$

where

$$d_m{}^0 = \int_1^2 f(\theta)\, \gamma_m(\theta)\, d\theta.$$

Then

$$\sum |d_m|^{2k} \leqslant 2^{2k} \left[\sum_m |d_m{}^0|^{2k} + \sum_m \left|\sum_n c_{m,n}\right|^{2k}\right]. \qquad (53)$$

Now $$[\Sigma|d_m^{\,0}|^{2k}]^{1/(2k)} \leqslant \mu^{(2/k')-1}\left[\int_1^2 |f|^{k'}\right]^{1/k'} \quad \text{(by Lemma 4)}$$

$$\leqslant \mu^{(2/k')-1}\left[\int_1^2 |f|^{2k}\right]^{1/(2k)},$$

so that $$\Sigma|d_m^{\,0}|^{2k} \leqslant \mu^{2k-2}\int_1^2 |f|^{2k}.$$

By (53) and Lemma 5, the result follows.

We can now complete the proof of Theorem V. We consider the class of functions $f(x)$ such that $|f|^q$ is integrable in $(1, \infty)$. We define $T_f(y)$ by

$$T_f(y) = \int_1^\infty f(x)\,\gamma_m(x)\,dx$$

$$(m \leqslant y < m+1; \quad m = 1, 2, \ldots).$$

Then

$$\log \max \frac{\left[\int_1^x |T_f(y)|^q\right]^{1/q}}{\left[\int_1^x |f(x)|^q\right]^{1/q}} \tag{54}$$

is a convex function of $1/q$, and by the same argument as in Theorem III, (i) follows for all $q \geqslant 2$.

We denote the maximum in (54) by $M(q)$. We know that there is a transposed transformation which satisfies

$$\int_1^\infty h(y)\,T_f(y)\,dy = \int_1^\infty f(x)\,\bar{T}_h(x)\,dx.$$

Take $f(x) = \gamma_m(x)$. Then

$$\int_m^{m+1} h(y)\,dy = \int_1^\infty \bar{T}_h(x)\,\gamma_m(x)\,dx.$$

Since we are supposing that the system of orthogonal functions is complete, we have

$$\bar{T}_h(x) \sim \Sigma\,\gamma_m(x)\int_m^{m+1} h(y)\,dy.$$

If $h(y) = c_m$ in $(m, m+1)$ for $m = 1, 2, \ldots$, then

$$\bar{T}_h(x) \sim \Sigma\,c_m\,\gamma_m(x),$$

and, writing $$\frac{1}{p}+\frac{1}{q} = 1,$$

we have

$$\max \frac{\left[\int_1^\infty |\bar{T}_h(x)|^p\right]^{1/p}}{\left[\int_1^\infty |h(y)|^p\right]^{1/\mu}} = M(q).$$

Using the bound for $M(q)$ given by (i), and substituting for q in terms of p, we get the desired result.

6. We finally consider the generalization of Theorems 8-11 of Hardy and Littlewood (*loc. cit.*, p. 198). The generalization of Theorems 10, 11 is as follows.

THEOREM VI. *Let a and c be connected by*

$$\frac{1}{a}+\frac{1}{c}=1.$$

Let $a \leqslant r \leqslant c$, and

$$\mu=\frac{1}{r}-\frac{1}{c}, \quad \nu=\frac{1}{\rho}-\frac{1}{a}, \quad \frac{1}{r}+\frac{1}{\rho}=1.$$

(i) *If $|f(x)|^a$ is integrable in $(0, 1)$, and*

$$c_k=\int_0^1 f(x)\,\omega_k(x)\,dx, \tag{55}$$

then

$$\left[\sum_1^\infty |c_m|^r m^{-\mu r}\right]^{1/r} \leqslant M(a, r)\left[\int_0^1 |f(x)|^a\right]^{1/a}.$$

(ii) *If (d_k) is a sequence of numbers such that $\Sigma |d_k|^\rho k^{-\nu\rho}$ is convergent, then there is a function $h(x)$ such that*

$$d_k=\int_0^1 h(x)\,\omega_k(x)\,dx,$$

and

$$\left[\int_0^1 |h(x)|^c\right]^{1/c} \leqslant M(a, r)\,[\Sigma |d_k|^\rho k^{\nu\rho}]^{1/\rho}.$$

The first part is a consequence of Theorem III (ii) and of Theorem I. We have

$$\begin{aligned}\sum_1^\infty |c_m|^r m^{-\mu r} &= \Sigma |c_m|^r m^{\frac{r-c}{c}} \\ &= \Sigma |c_m|^{a\cdot\frac{c-r}{c-a}} m^{(a-2)\frac{c-r}{c-a}} c_m^{c\cdot\frac{r-a}{c-a}} \\ &\leqslant [\Sigma |c_m|^a m^{a-2}]^{\frac{c-r}{c-a}} [\Sigma |c_m|^c]^{\frac{r-a}{c-a}}. \end{aligned} \tag{56}$$

To the first factor we apply Theorem III (ii); to the second factor we apply Theorem I. The expression in (56) does not exceed

$$\left[\mu^{2-a}\left(\frac{1}{a-1}\right)^a K^{2-a}\int|f|^a\right]^{\frac{c-r}{c-a}}\left[\mu^{\frac{2-a}{a}}\int|f|^a\right]^{\frac{r-a}{c-a}\cdot\frac{c}{a}},$$

the desired result.

Write $$r(1+\mu)=\delta.$$

Let I denote the interval $\left(0\leqslant y<\sum_1^x n^{-\delta}\right)$. To each $f(x)$ of $L(a)$ we make a function $T_f(y)$ correspond in I by

$$T_f(y)=kc_k \quad \left(\sum^{k-1} n^{-\delta}\leqslant y<\sum^{k} n^{-\delta}\right) \quad (k=1,\ 2,\ \ldots),$$

where c_k is defined by (55). Then (i) gives

$$\max\frac{\left[\int_I|T_f(y)|^r\right]^{1/r}}{\left[\int_0^1|f(x)|^a\right]^{1/a}}=M(a,\ r),$$

where $M(a,\ r)$ denotes the bound of the linear transformation. We know that there is a transposed transformation $\bar{T}_g(x)$, where $g(x)$ is of class $L(\rho)$ in I. We have

$$\int_I g(y)\,T_f(y)\,dy=\int_0^1 f(x)\,\bar{T}_g(x)\,dx. \tag{57}$$

Let $f(x)=\omega_k(x)$. Then

$$T_f(y)=k \quad \text{for} \quad \sum^{k-1} n^{-\delta}\leqslant y<\sum^{k} n^{-\delta} \quad (\equiv I_k)$$
$$=0 \quad \text{elsewhere},$$

and (57) gives

$$k\int_{I_k} g(y)\,dy=\int_0^1 \bar{T}_g(x)\,\omega_k(x)\,dx.$$

If then $g(y)$ has the constant value $k^{\delta-1}d_k$ in I_k $(k=1,\ 2,\ \ldots)$, the preceding relation gives

$$d_k=\int_0^1 \bar{T}_g(x)\,\omega_k(x)\,dx.$$

We know that

$$\frac{\left[\int_0^1 |\bar{T}_g(x)|^c\right]^{1/c}}{\left[\int_I |g(y)|^\rho\right]^{1/\rho}} \leqslant M(a, r).$$

It remains to prove that

$$\int_I |g(y)|^\rho = \Sigma |d_k|^\rho k^{-\nu\rho}.$$

We have

$$\int_I |g|^\rho = \Sigma k^{(\delta-1)\rho} |d_k|^\rho k^{-\delta},$$

and

$$\begin{aligned}\delta\rho - \delta - \rho &= \frac{\delta(\rho-1)}{\rho}\rho - \rho \\ &= \frac{\delta}{r}\rho - \rho \\ &= (1+\mu)\rho - \rho \\ &= -\nu\rho.\end{aligned}$$

The generalization of Theorems 8 and 9 of Hardy and Littlewood is as follows.

THEOREM VII. *Let p and q be connected by*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Let $p \leqslant r \leqslant q$, and

$$\mu = \frac{1}{r} - \frac{1}{q}, \quad \nu = \frac{1}{\rho} - \frac{1}{p}, \quad \frac{1}{r} + \frac{1}{\rho} = 1.$$

(i) *If (c_m) is a sequence such that $\Sigma |c_m|^p$ converges, then there is a function $f(\theta)$ such that*

$$c_m = \int_0^1 f(\theta)\, \omega_m(\theta)\, d\theta,$$

and

$$\left[\int_0^1 |f|^r \theta^{-\mu r}\right]^{1/r} \leqslant M(p, r)[\Sigma |c_m|^p]^{1/p}.$$

(ii) *If $|f(\theta)|\theta^{-\nu}$ is of class $L(\rho)$ in $(0, 1)$, and*

$$d_m = \int_0^1 f(\theta)\, \omega_m(\theta)\, d\theta,$$

then

$$[\Sigma |d_m|^q]^{1/q} \leqslant M(p, r)\left[\int_0^1 |f|^\rho \theta^{-\nu\rho}\right]^{1/\rho}.$$

The first part is a consequence of Theorem IV (ii) and of Theorem II. We have

$$\int_0^1 |f|^r \theta^{-\mu r} d\theta = \int |f|^r \theta^{\frac{r-q}{q}}$$

$$= \int |f|^{p \cdot \frac{q-r}{q-p}} \theta^{(p-2)\frac{q-r}{q-p}} |f|^{q \cdot \frac{r-p}{q-p}} d\theta$$

$$\leqslant \left[\int |f|^p \theta^{p-2}\right]^{\frac{q-r}{q-p}} \left[\int |f|^q\right]^{\frac{r-p}{q-p}}. \qquad (58)$$

To the first factor we apply Theorem IV (ii); to the second factor we apply Theorem II. The expression (58) does not exceed

$$\left[\mu^{2-p}\left(\frac{1}{p-1}\right)^p c^{2-p} \Sigma |c_m|^p\right]^{\frac{q-r}{q-p}} [\mu^{\frac{2-p}{p}} \Sigma |c_m|^p]^{\frac{q}{p} \cdot \frac{r-p}{q-p}},$$

which is the desired result.

Consider the class of functions $\phi(x)$, defined in $(1, \infty)$, which satisfy

(*a*) $\phi(x)$ is constant in each interval $(m, m+1)$,

(*b*) $\phi(x)$ is of class $L(p)$ in $(1, \infty)$.

Denote by ϕ_m the constant value of $\phi(x)$ in $(m, m+1)$. To each $\phi(x)$ there corresponds a function $\psi(\theta)$ defined in $(0, 1)$ by

$$\psi(\theta) \sim \Sigma \phi_m \omega_m(\theta).$$

We define the transformation $T_\phi(\theta)$ by

$$T_\phi(\theta) = \psi(\theta)\theta^{-\mu} \quad (0 < \theta \leqslant 1).$$

Then

$$\max \frac{\left[\int_0^1 |T_\phi(\theta)|^r\right]^{1/r}}{\left[\int_1^\infty |\phi(x)|^p\right]^{1/p}} = M(p, r),$$

where $M(p, r)$ denotes the bound of the transformation. We know that there is a transposed transformation $\bar{T}_g(x)$, where $g(\theta)$ is any function of class $L(\rho)$ in $(0, 1)$. We have

$$\int_0^1 g(\theta) T_\phi(\theta) d\theta = \int_1^\infty \phi(x) \bar{T}_g(x) dx.$$

If we take $\phi(x) = 1$ in $(m, m+1)$, and $\phi(x) = 0$ elsewhere, we get

$$\int_0^1 g(\theta)\,\theta^{-\mu}\,\omega_m(\theta)\,d\theta = \int_m^{m+1} \bar{T}_g(x)\,dx;$$

so that, if $$g(\theta) = f(\theta)\,\theta^{-\nu},$$

then $$\int_m^{m+1} \bar{T}_g(x)\,dx = \int_0^1 f(\theta)\,\omega_m(\theta)\,d\theta$$

$$= d_m.$$

We know that

$$\max \frac{\left[\int_1^\infty |\bar{T}_g(x)|^q\right]^{1/q}}{\left[\int_0^1 |g(\theta)|^p\right]^{1/p}} = M(p, r). \tag{59}$$

But $$\Sigma |d_m|^q = \Sigma \left| \int_m^{m+1} \bar{T}_g(x) \right|^q$$

$$\leqslant \int_1^\infty |\bar{T}_g(x)|^q\,dx.$$

By (59), (ii) follows.

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THE BENDING OF MODERATELY THICK CIRCULAR PLATES

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The problem which we discuss in this paper is that of the bending of a moderately thick uniform circular plate for the case of complete axial symmetry, when the opposite faces of the plate are subject to arbitrary normal tractions.

In § 1 we consider the equations governing the problem, and in § 2 we obtain the usual plane stress solution when the opposite faces of the plate are entirely free from traction. In the next section we determine a solution which satisfies a certain assumed symmetrical law of distribution of normal stress over the two faces of the plate, and we call this the normal solution. These two particular solutions in §§ 2, 3 are then adequate for the construction of the solution when the normal surface tractions are arbitrary, and this is carried out in the two remaining sections for the cases of the complete circular plate and the ring-shaped circular plate.

1. *The equations.*

When there is complete symmetry about the z axis, the displacements u, v, w in the directions of the cylindrical coordinates r, θ, z respectively are independent of θ, and then the strain relations* break up into two sets:

$$\text{(i)} \quad e_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r}, \quad e_{\theta z} = \frac{\partial v}{\partial z}, \quad 2\varpi_r = -\frac{\partial v}{\partial z}, \quad 2\varpi_z = \frac{1}{r}\frac{\partial}{\partial r}(rv),$$

which depends on v only, and

$$\text{(ii)} \quad \left.\begin{aligned} e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{u}{r}, \quad e_{zz} = \frac{\partial w}{\partial z}, \\ e_{zr} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \quad 2\varpi_\theta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}, \end{aligned}\right\}$$

which depends on u and w alone.

* Love, *Elasticity*, 4th ed. (Cambridge, 1927), § 22.

The same division arises in the equations of equilibrium in terms of the stresses*, one of which yields the equation

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}-\frac{v}{r^2}=0$$

for v.

In this investigation we are confining our attention to types of problems for which v vanishes†, and then

$$\left.\begin{gathered} e_{rr}=\frac{\partial u}{\partial r},\quad e_{\theta\theta}=\frac{u}{r},\quad e_{zz}=\frac{\partial w}{\partial z},\\ e_{r\theta}=0,\quad e_{\theta z}=0,\quad e_{zr}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r},\\ \Delta=\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z},\\ 2\varpi_r=0,\quad 2\varpi_\theta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r},\quad 2\varpi_z=0. \end{gathered}\right\} \qquad (1)$$

The stress-strain relations are

$$\left.\begin{gathered} \widehat{rr}=\lambda\Delta+2\mu e_{rr},\quad \widehat{\theta\theta}=\lambda\Delta+2\mu e_{\theta\theta},\quad \widehat{zz}=\lambda\Delta+2\mu e_{zz},\\ \widehat{r\theta}=\mu e_{r\theta},\quad \widehat{\theta z}=\mu e_{\theta z},\quad \widehat{zr}=\mu e_{zr}. \end{gathered}\right\} \qquad (2)$$

where $$\lambda=\frac{E\sigma}{(1+\sigma)(1-2\sigma)},\quad \mu=\frac{E}{2(1+\sigma)},$$

and E and σ are Young's modulus and Poisson's ratio respectively.

The equations of equilibrium‡, when expressed in terms of the stresses, are

$$\left.\begin{gathered} \frac{\partial}{\partial r}\widehat{rr}+\frac{\partial}{\partial z}\widehat{rz}+\frac{1}{r}(\widehat{rr}-\widehat{\theta\theta})=0,\\ \frac{\partial}{\partial r}\widehat{rz}+\frac{\partial}{\partial z}\widehat{zz}+\frac{1}{r}\widehat{rz}=0, \end{gathered}\right\} \qquad (3)$$

and, when expressed in terms of the strains,

$$\left.\begin{gathered} (\lambda+2\mu)\frac{\partial\Delta}{\partial r}+\mu\frac{\partial}{\partial z}(2\varpi_\theta)=0,\\ (\lambda+2\mu)\frac{\partial\Delta}{\partial z}-\frac{\mu}{r}\frac{\partial}{\partial r}(2r\varpi_\theta)=0. \end{gathered}\right\} \qquad (4)$$

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* Love, *loc. cit.*, § 59, (i).

† If desired, a solution for v may be superposed on those obtained in this paper.

‡ Love, *loc. cit.*, § 99.

In rearranging these equations (1), (2), (3), (4) in forms suitable for finding the stresses, we find it convenient to introduce the function

$$\Theta = \widehat{rr}+\widehat{\theta\theta}+\widehat{zz} = (3\lambda+2\mu)\Delta = E\Delta/(1-2\sigma),$$

and it readily follows from (4) that

$$\nabla^2\Delta = \nabla^2\Theta = 0,$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.$$

The second equation of (4) may be written in the form

$$(\lambda+\mu)\frac{\partial\Delta}{\partial z}+\mu\left[\frac{\partial\Delta}{\partial z}-\frac{1}{r}\frac{\partial}{\partial r}(2r\varpi_\theta)\right] = 0,$$

and, when we substitute for Δ and ϖ_θ from (1) in the second term, we obtain

$$2\frac{\lambda+\mu}{3\lambda+2\mu}\frac{\partial\Theta}{\partial z}+\nabla^2(2\mu w) = 0. \tag{5}$$

On differentiating this equation with respect to z, we have

$$2\frac{\lambda+\mu}{3\lambda+2\mu}\frac{\partial^2\Theta}{\partial z^2}+\nabla^2\left(2\mu\frac{\partial w}{\partial z}\right) = 0,$$

$$\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z^2}+\nabla^2(\widehat{zz}-\lambda\Delta) = 0,$$

and, since Δ is a harmonic function, we have finally

$$\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z^2}+\nabla^2\widehat{zz} = 0. \tag{6}$$

From (5) and (6) we see that

$$\nabla^4 w = 0,$$

and

$$\nabla^4\widehat{zz} = 0.$$

When we write the first equation of (4) in the form

$$(\lambda+\mu)\frac{\partial\Delta}{\partial r}+\mu\left[\frac{\partial\Delta}{\partial r}+\frac{\partial}{\partial z}(2\varpi_\theta)\right] = 0,$$

we find in similar fashion that

$$\frac{\partial\Theta}{\partial r}+E\left(\nabla^2-\frac{1}{r^2}\right)(u) = 0. \tag{7}$$

Again, when we differentiate (5) with respect to r and (7) with respect to z and add, we have

$$\frac{2}{1+\sigma}\frac{\partial^2\Theta}{\partial r\,\partial z}+\frac{E}{1+\sigma}\left[\frac{\partial}{\partial r}(\nabla^2 w)+\frac{\partial}{\partial z}\left(\nabla^2-\frac{1}{r^2}\right)(u)\right]=0,$$

$$\frac{2}{1+\sigma}\frac{\partial^2\Theta}{\partial r\,\partial z}+\frac{E}{1+\sigma}\left[\left(\nabla^2-\frac{1}{r^2}\right)\left(\frac{\partial w}{\partial r}\right)+\left(\nabla^2-\frac{1}{r^2}\right)\left(\frac{\partial u}{\partial z}\right)\right]=0,$$

$$\frac{2}{1+\sigma}\frac{\partial^2\Theta}{\partial r\,\partial z}+\left(\nabla^2-\frac{1}{r^2}\right)(2\mu e_{zr})=0,$$

$$\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial r\,\partial z}+\left(\nabla^2-\frac{1}{r^2}\right)(\widehat{zr})=0. \tag{8}$$

2. The plane stress solution.

The plane stress solution* is that for which $\widehat{zz}$ is zero everywhere and $\widehat{rz}$ vanishes at $z=\pm h$, the opposite faces of the plate. In this case, equation (6) becomes

$$\frac{\partial^2\Theta}{\partial z^2}=0,$$

and so

$$\Theta=zR_1+R_0,$$

where R_1 and R_0 are functions of r alone. Further, Θ must be harmonic, and it readily follows that

$$\Theta=(C_1z+C_0)\log r+D_1z+D_0,$$

where C_1, C_0, D_1, D_0 are constants.

From the equation

$$\frac{\partial}{\partial r}\widehat{rz}+\frac{\partial}{\partial z}\widehat{zz}+\frac{1}{r}\widehat{rz}=0,$$

we find that

$$\widehat{rz}=Z_1/r,$$

where Z_1 is a function of z only.

* Love, *loc. cit.*, §§ 299–303.

When we substitute these expressions for Θ and $\widehat{rz}$ in equation (8), we see that Z_1 must satisfy the equation

$$\frac{d^2 Z_1}{dz^2} = -\frac{C_1}{1+\sigma}.$$

Also Z_1 must vanish for $z = \pm h$, and so it follows that

$$\widehat{rz} = -\frac{C_1}{2(1+\sigma)} \frac{z^2-h^2}{r}.$$

The equation

$$\frac{\partial}{\partial r} \widehat{rr} + \frac{\partial}{\partial z} \widehat{rz} + \frac{1}{r} (\widehat{rr} - \widehat{\theta\theta}) = 0$$

may be put in the form

$$r^2 \frac{\partial}{\partial r} \widehat{rr} + 2r \widehat{rr} = -r^2 \frac{\partial}{\partial z} \widehat{rz} + r(\widehat{rr} + \widehat{\theta\theta}),$$

or

$$\frac{\partial}{\partial r} (r^2 \widehat{rr}) = -r^2 \frac{\partial}{\partial z} \widehat{rz} + r\Theta.$$

When we substitute the above expressions for Θ and $\widehat{rz}$, we obtain

$$\frac{\partial}{\partial r} (r^2 \widehat{rr}) = \frac{C_1}{1+\sigma} zr + (C_1 z + C_0) r \log r + (D_1 z + D_0) r,$$

which leads to

$$\widehat{rr} = \tfrac{1}{2} \frac{C_1}{1+\sigma} z + \tfrac{1}{2}(C_1 z + C_0) \log r - \tfrac{1}{4}(C_1 z + C_0) + \tfrac{1}{2}(D_1 z + D_0) + \frac{Z_2}{r^2},$$

where Z_2 is a function of z only.

Next, we have

$$\widehat{\theta\theta} = \Theta - \widehat{rr}$$

$$= -\tfrac{1}{2} \frac{C_1}{1+\sigma} z + \tfrac{1}{2}(C_1 z + C_0) \log r + \tfrac{1}{4}(C_1 z + C_0) + \tfrac{1}{2}(D_1 z + D_0) - \frac{Z_2}{r^2}.$$

The stress-strain relation for $\widehat{\theta\theta}$ may be written in the form

$$E \frac{u}{r} = (1+\sigma) \widehat{\theta\theta} - \sigma\Theta,$$

which yields

$$Eu = \tfrac{1}{2}(1-\sigma)(C_1 z + C_0) r \log r + \tfrac{1}{2}(1-\sigma)(D_1 z + D_0) r$$

$$-\tfrac{1}{2} C_1 zr + \tfrac{1}{4}(1+\sigma)(C_1 z + C_0) r - (1+\sigma) \frac{Z_2}{r}.$$

Moreover, the stress-strain equation for $\widehat{rr}$ may be written in the form

$$E\frac{\partial u}{\partial r} = (1+\sigma)\widehat{rr} - \sigma\Theta,$$

and from this equation we find that

$$E\frac{\partial u}{\partial r} = \tfrac{1}{2}(1-\sigma)(C_1 z + C_0)\log r + \tfrac{1}{2}(1-\sigma)(D_1 z + D_0) + \tfrac{1}{2}C_1 z - \tfrac{1}{4}(1+\sigma)(C_1 z + C_0) + (1+\sigma)\frac{Z_2}{r^2}.$$

In order that these expressions for Eu and $E\partial u/\partial r$ may be consistent, C_0 must vanish, and so we omit C_0 from the remainder of our work. The stress-strain equation for $\widehat{zz}$ in this case may be written in the form

$$E\frac{\partial w}{\partial z} = -\sigma\Theta = -\sigma[C_1 z\log r + D_1 z + D_0],$$

and so we have

$$Ew = -\tfrac{1}{2}\sigma C_1 z^2 \log r - \tfrac{1}{2}\sigma D_1 z^2 - \sigma D_0 z + R_2,$$

where R_2 is a function of r alone.

The remaining stress-strain equation is

$$2(1+\sigma)\widehat{rz} = E\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right),$$

and it is satisfied by the expressions for $\widehat{rz}$, u, and w if

$$\frac{1}{r}\left[(1-\tfrac{1}{2}\sigma)C_1 z^2 - (1+\sigma)\frac{dZ_2}{dz}\right] + \left[-\frac{C_1 h^2}{r} + \tfrac{1}{2}(1-\sigma)C_1 r\log r - \tfrac{1}{4}(1-\sigma)C_1 r + \tfrac{1}{2}(1-\sigma)D_1 r + \frac{dR_2}{dr}\right] = 0.$$

Since R_2 is a function of r only, and Z_2 a function of z only, it follows that we must have

$$(1-\tfrac{1}{2}\sigma)C_1 z^2 - (1+\sigma)\frac{dZ_2}{dz} = K_1,$$

and

$$(K_1 - C_1 h^2)\frac{1}{r} + \tfrac{1}{2}(1-\sigma)C_1 r\log r - \tfrac{1}{4}(1-\sigma)C_1 r + \tfrac{1}{2}(1-\sigma)D_1 r + \frac{dR_2}{dr} = 0,$$

where K_1 is a constant.

On integration, these two equations give

$$(1+\sigma)\,Z_2 = \tfrac{1}{6}(2-\sigma)\,C_1 z^3 - K_1 z + K_2,$$

$$R_2 = \tfrac{1}{4}(1-\sigma)\,C_1 r^2 - \tfrac{1}{4}(1-\sigma)\,C_1 r^2 \log r - (K_1 - C_1 h^2)\log r - \tfrac{1}{4}(1-\sigma)\,D_1 r^2 + K_3,$$

where K_2 and K_3 are constants.

Finally, the displacements and stresses of this solution are

$$Eu = \tfrac{1}{2}(1-\sigma)\,C_1 zr \log r - \tfrac{1}{4}(1-\sigma)\,C_1 zr + \tfrac{1}{2}(1-\sigma)(D_1 z + D_0) - \frac{1}{r}\left[\tfrac{1}{6}(2-\sigma)\,C_1 z^3 - K_1 z + K_2\right],$$

$$Ew = -\tfrac{1}{2}\sigma\,C_1 z^2 \log r - \tfrac{1}{2}\sigma\,D_1 z^2 - \sigma\,D_0 z + \tfrac{1}{4}(1-\sigma)\,C_1 r^2 - \tfrac{1}{4}(1-\sigma)\,C_1 r^2 \log r - (K_1 - C_1 h^2)\log r - \tfrac{1}{4}(1-\sigma)\,D_1 r^2 + K_3,$$

$$\widehat{zz} = 0, \quad \widehat{rz} = -\tfrac{1}{2}\frac{C_1}{1+\sigma}\,\frac{z^2-h^2}{r},$$

$$\widehat{rr} = \tfrac{1}{2}\frac{C_1}{1+\sigma}\,z - \tfrac{1}{4}C_1 z + \tfrac{1}{2}C_1 z \log r + \tfrac{1}{2}(D_1 z + D_0) + \frac{1}{1+\sigma}\left[\tfrac{1}{6}(2-\sigma)\,C_1 z^3 - K_1 z + K_2\right]\frac{1}{r^2},$$

$$\widehat{\theta\theta} = -\tfrac{1}{2}\frac{C_1}{1+\sigma}\,z + \tfrac{1}{4}C_1 z + \tfrac{1}{2}C_1 z \log r + \tfrac{1}{2}(D_1 z + D_0) - \frac{1}{1+\sigma}\left[\tfrac{1}{6}(2-\sigma)\,C_1 z^3 - K_1 z + K_2\right]\frac{1}{r^2},$$

and it may be noted that the solution involves six arbitrary constants.

3. *The normal solution.*

In this section we determine a solution of equations (1), (2), and (3) when $\widehat{zz}$ takes certain prescribed values and $\widehat{rz}$ vanishes at $z = \pm h$, the faces of the plate. On this assumption, the second equation of (3) shows that

$$\frac{\partial}{\partial z}\,\widehat{zz} = 0 \quad \text{at} \quad z = \pm h.$$

Since $\widehat{zz}$ must satisfy the equation

$$\nabla^4 \widehat{zz} = 0,$$

let us assume a solution of the form

$$\widehat{zz} = \psi J_0(kr),$$

where ψ is a function of z only, $J_0(kr)$ is a Bessel function of order zero, and k is a constant. Further, let us assume that

$$\begin{aligned} \widehat{zz} &= b_0 J_0(kr) \quad \text{at} \quad z = -h \\ &= b_1 J_0(kr) \quad \text{at} \quad z = +h, \end{aligned}$$

$$\frac{\partial}{\partial z}\widehat{zz} = 0 \qquad \text{at} \quad z = \pm h.$$

Since $J_0(kr)$ satisfies the equation

$$\frac{d^2 J}{dr^2} + \frac{1}{r}\frac{dJ}{dr} + k^2 J = 0,$$

we see that ψ must satisfy the differential equation

$$\psi^{\text{iv}} - 2k^2\psi'' + k^4\psi = 0,$$

and the conditions

$$\begin{aligned} \psi = b_0, \quad \psi' = 0 \quad \text{at} \quad z = -h, \\ \psi = b_1, \quad \psi' = 0 \quad \text{at} \quad z = +h. \end{aligned}$$

The expression for ψ is then

$$\psi = (A_1 + B_1 z)\cosh kz + (A_2 + B_2 z)\sinh kz,$$

where

$$\left.\begin{aligned} A_1 &= \tfrac{1}{2}(b_0 + b_1)\frac{kh\cosh kh + \sinh kh}{kh + \sinh kh\cosh kh}, \\ B_1 &= -\tfrac{1}{2}(b_0 - b_1)\frac{k\cosh kh}{kh - \sinh kh\cosh kh}, \\ A_2 &= \tfrac{1}{2}(b_0 - b_1)\frac{kh\sinh kh + \cosh kh}{kh - \sinh kh\cosh kh}, \\ B_2 &= -\tfrac{1}{2}(b_0 + b_1)\frac{k\sinh kh}{kh + \sinh kh\cosh kh}. \end{aligned}\right\} \tag{9}$$

On substituting the form $\psi J_0(kr)$ for $\widehat{zz}$ in (6), we have

$$\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z^2} + (\psi'' - k^2\psi)J_0(kr) = 0,$$

and so the form of Θ for this solution is

$$\Theta = -(1+\sigma)\, J_0(kr)\iint(\psi''-k^2\psi)\,dz\,dz.$$

In this expression for Θ we have omitted the terms

$$zR_1+R_0,$$

where R_1 and R_0 are functions of r only, since these terms contribute to the plane stress which is dealt with in the previous section.

When we perform the integration and make use of (9), we find that

$$\Theta = -\frac{2(1+\sigma)}{k}\,(B_1 \sinh kz + B_2 \cosh kz)\, J_0(kr),$$

which is harmonic, as required.

$\widehat{rz}$ must satisfy the equation

$$\frac{\partial}{\partial r}\widehat{rz}+\frac{1}{r}\widehat{rz}+\frac{\partial}{\partial z}\widehat{zz}=0,$$

which we can write in the form

$$\frac{\partial}{\partial r}(r\,\widehat{rz}) = -r\psi'\, J_0(kr).$$

Then, omitting a function* of z as above, we have

$$\begin{aligned} r\,\widehat{rz} &= -\psi'\int r\, J_0(kr)\,dr \\ &= -\frac{1}{k}\,\psi' r\, J_1(kr), \\ \widehat{rz} &= -\frac{1}{k}\,\psi'\, J_1(kr), \end{aligned}$$

where $J_1(kr)$ is the Bessel function of order one. Also, this expression for $\widehat{rz}$ vanishes at $z = \pm h$, since ψ' vanishes there†.

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* *Cf.* Z_1 in § 2.

† In similar fashion, we can determine a solution for $\widehat{zz}$ and $\widehat{rz}$ which satisfies the conditions

$$\widehat{zz} = b_0 J_0(kr), \quad \widehat{rz} = b_2 J_1(kr) \quad \text{at} \quad z=-h,$$
$$\widehat{zz} = b_1 J_0(kr), \quad \widehat{rz} = b_3 J_1(kr) \quad \text{at} \quad z=+h,$$

by choosing ψ to satisfy the conditions

$$\psi = b_0, \quad \psi' = -kb_2 \quad \text{at} \quad z=-h,$$
$$\psi = b_1, \quad \psi' = -kb_3 \quad \text{at} \quad z=+h.$$

To determine $\widehat{rr}$ we have

$$\frac{\partial}{\partial r}\widehat{rr}+\frac{\partial}{\partial z}\widehat{rz}+\frac{1}{r}(\widehat{rr}-\widehat{\theta\theta})=0,$$

$$r^2\frac{\partial}{\partial r}\widehat{rr}+2r\,\widehat{rr}=-r^2\frac{\partial}{\partial z}\widehat{rz}+r(\widehat{rr}+\widehat{\theta\theta}),$$

$$\frac{\partial}{\partial r}(r^2\widehat{rr})=-r^2\frac{\partial}{\partial z}\widehat{rz}+r(\Theta-\widehat{zz}).$$

When we substitute the above expressions for $\widehat{zz}$, Θ, and $\widehat{rz}$, this last equation becomes

$$\frac{\partial}{\partial r}(r^2\widehat{rr})=\frac{1}{k}\psi''r^2J_1(kr)-\left[\psi+\frac{2(1+\sigma)}{k}(B_1\sinh kz+B_2\cosh kz)\right]rJ_0(kr),$$

which gives, on integration*,

$$r^2\widehat{rr}=\frac{1}{k^2}\psi''r^2J_2(kr)-\frac{1}{k}\left[\psi+\frac{2(1+\sigma)}{k}(B_1\sinh kz+B_2\cosh kz)\right]rJ_1(kr).$$

From the formula

$$J_2(kr)+J_0(kr)=\frac{2}{kr}J_1(kr),$$

the expression for $\widehat{rr}$ reduces to

$$\widehat{rr}=\left[\psi+\frac{2(1-\sigma)}{k}(B_1\sinh kz+B_2\cosh kz)\right]\frac{1}{kr}J_1(kr)-\frac{1}{k^2}\psi''J_0(kr).$$

From the equation

$$\Theta=\widehat{rr}+\widehat{\theta\theta}+\widehat{zz},$$

we find that

$$\widehat{\theta\theta}=\left[\frac{1}{k^2}\psi''-\psi-\frac{2(1+\sigma)}{k}(B_1\sinh kz+B_2\cosh kz)\right]J_0(kr)$$

$$-\left[\psi+\frac{2(1-\sigma)}{k}(B_1\sinh kz+B_2\cosh kz)\right]\frac{1}{kr}J_1(kr)$$

$$=-\frac{2\sigma}{k}(B_1\sinh kz+B_2\cosh kz)J_0(kr)$$

$$-\left[\psi+\frac{2(1-\sigma)}{k}(B_1\sinh kz+B_2\cosh kz)\right]\frac{1}{kr}J_1(kr).$$

* Omitting a function of z. *Cf.* Z_2 in §2.

When we write the equation

$$\widehat{\theta\theta} = \lambda\Delta + 2\mu e_{\theta\theta}$$

in the form

$$Eu = (1+\sigma)\, r\,\widehat{\theta\theta} - \sigma r\Theta,$$

it leads to

$$Eu = -\frac{1+\sigma}{k}\left[\psi + \frac{2(1-\sigma)}{k}\,(B_1 \sinh kz + B_2 \cosh kz)\right] J_1(kr).$$

Also, when the equation

$$\widehat{rr} = \lambda\Delta + 2\mu e_{rr}$$

is written in the form

$$E\,\frac{\partial u}{\partial r} = (1+\sigma)\,\widehat{rr} - \sigma\Theta,$$

we obtain an expression for $E\,\partial u/\partial r$ which is consistent with the expression just obtained for $E\,u$. Writing the stress-strain equation for $\widehat{zz}$ in the form

$$E\,\frac{\partial w}{\partial z} = (1+\sigma)\,\widehat{zz} - \sigma\Theta,$$

we find that

$$E\,\frac{\partial w}{\partial z} = (1+\sigma)\left[\psi + \frac{2\sigma}{k}\,(B_1 \sinh kz + B_2 \cosh kz)\right] J_0(kr).$$

On integrating* this equation, we have

$$Ew = (1+\sigma)\left[\int \psi\, dz + \frac{2\sigma}{k^2}\,(B_1 \cosh kz + B_2 \sinh kz)\right] J_0(kr)$$

$$= \frac{1+\sigma}{k^2}\,\{[k(A_1 + B_1 z) - (1-2\sigma)\,B_2] \sinh kz$$

$$+ [k(A_2 + B_2 z) - (1-2\sigma)\,B_1] \cosh kz\}\, J_0(kr).$$

It is now easily verified that the above expressions for $\widehat{rz}$, Eu, and Ew satisfy the remaining stress-strain equation

$$2(1+\sigma)\,\widehat{rz} = E\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right).$$

So we see that the solution of this section is self-consistent.

* Omitting a function of z. *Cf.* R_2 in § 2.

For this normal solution, it is quite a straightforward matter to show that the formulae

$$G_r = \int_{-h}^{h} z\widehat{rr}\,dz, \quad G_\theta = \int_{-h}^{h} z\widehat{\theta\theta}\,dz,$$

for the resultant couples over the cylindrical and radial sections of the plate, agree approximately with the corresponding formulae expressed in terms of the curvatures* of the plane $z = 0$ of the plate, provided that h is small†.

4. *The complete circular plate.*

Let us now consider a complete circular plate of radius a, whose faces, $z = \pm h$, are subject to arbitrary normal stresses which are functions of r, say:

$$\widehat{zz} = f_0(r) \quad \text{at} \quad z = -h.$$

$$= f_1(r) \quad \text{at} \quad z = +h.$$

If $J_0(kr)$ is the Bessel function of the first kind, and $k_1, k_2, k_3, \ldots$ denote the roots of the equation

$$J_0(ka) = 0,$$

we can write down the Fourier-Bessel expansions

$$f_0(r) = \sum_{n=1}^{\infty} b_{0n} J_0(k_n r),$$

$$f_1(r) = \sum_{n=1}^{\infty} b_{1n} J_0(k_n r).$$

Let us now take the normal solution of the previous section, and replace k by k_n, b_0 by b_{0n}, and b_1 by b_{1n}. When this has been done, let the expressions for A_1, A_2, B_1, B_2, and ψ be denoted by A_{1n}, A_{2n}, B_{1n}, B_{2n}, and ψ_n respectively, and then the normal solution takes the form given by

$$\widehat{zz} = \psi_n J_0(k_n r), \quad \widehat{rz} = -\frac{1}{k_n}\psi_n' J_1(k_n r), \quad \ldots,$$

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* Love, *loc. cit.*, 465.

† Love, *loc. cit.*, 479.

for which $\widehat{zz}$ assumes the values

$$b_{0n} J_0(k_n r) \quad \text{at} \quad z = -h,$$

$$b_{1n} J_0(k_n r) \quad \text{at} \quad z = +h,$$

while $\widehat{rz}$ vanishes at $z = \pm h$.

Next, let us take the plane stress solution and make C_1, K_1, K_2 all zero in order to keep the solution finite at $r = 0$.

Then we can build up the general solution.

$$\widehat{zz} = \sum_{n=1}^{\infty} \psi_n J_0(k_n r),$$

$$\widehat{rz} = -\sum_{n=1}^{\infty} \frac{1}{k_n} \psi_n' J_1(k_n r),$$

$$\widehat{rr} = \tfrac{1}{2}(D_1 z + D_0) + \sum_{n=1}^{\infty} \left\{ \left[\psi_n + \frac{2(1-\sigma)}{k_n} (B_{1n} \sinh k_n z + B_{2n} \cosh k_n z) \right] \frac{1}{k_n r} J_1(k_n r) - \frac{1}{k_n^2} \psi_n'' J_0(k_n r) \right\},$$

$$\widehat{\theta\theta} = \tfrac{1}{2}(D_1 z + D_0) - \sum_{n=1}^{\infty} \left\{ \frac{2\sigma}{k_n} (B_{1n} \sinh k_n z + B_{2n} \cosh k_n z) J_0(k_n r) + \left[\psi_n + \frac{2(1-\sigma)}{k_n} (B_{1n} \sinh k_n z + B_{2n} \cosh k_n z) \right] \frac{1}{k_n r} J_1(k_n r) \right\},$$

$$Eu = \tfrac{1}{2}(1-\sigma)(D_1 z + D_0) r - (1+\sigma) \sum_{n=1}^{\infty} \left\{ \frac{1}{k_n} \left[\psi_n + \frac{2(1-\sigma)}{k_n} (B_{1n} \sinh k_n z + B_{2n} \cosh k_n z) \right] J_1(k_n r) \right\},$$

$$Ew = K_3 - \sigma(\tfrac{1}{2} D_1 z^2 + D_0 z) - \tfrac{1}{4}(1-\sigma) D_1 r^2 + (1+\sigma) \sum_{n=1}^{\infty} \frac{1}{k_n^2} \{ [k_n (A_{1n} + B_{1n} z) - (1-2\sigma) B_{2n}] \sinh k_n z + [k_n (A_{2n} + B_{2n} z) - (1-2\sigma) B_{1n}] \cosh k_n z \} J_0(k_n r),$$

where D_0, D_1, and K_3 are arbitrary constants. This is a solution for which

$$\widehat{zz} = \sum_{n=1}^{\infty} b_{0n} J_0(k_n r) = f_0(r) \quad \text{at} \quad z = -h$$

$$= \sum_{n=1}^{\infty} b_{1n} J_0(k_n r) = f_1(r) \quad \text{at} \quad z = +h,$$

while* $$\widehat{rz} = 0 \quad \text{at} \quad z = \pm h$$

We now consider the case where a uniform pressure p is applied over the circle $r = c$ on the face $z = -h$, while the face $z = +h$ is free from traction and the rim of the plate is supported. Then

$$f_0(r) = \begin{cases} p & (0 < r < c) \\ 0 & (c < r < a) \end{cases}$$

$$= \frac{2pc}{a^2} \sum_{n=1}^{\infty} \frac{J_1(k_n c)}{k_n \{[J_0(k_n a)]^2 + [J_1(k_n a)]^2\}} J_0(k_n r),$$

$$f_1(r) = 0,$$

and so

$$b_{0n} = \frac{2pc\, J_1(k_n c)}{k_n a^2 \{[J_0(k_n a)]^2 + [J_1(k_n a)]^2\}}, \quad b_{1n} = 0.$$

The conditions to be satisfied at the rim are

$$\left.\begin{aligned} \int_{-h}^{h} \widehat{rr}\, dz = 0 \\ \int_{-h}^{h} z\, \widehat{rr}\, dz = 0 \end{aligned}\right\} \quad \text{when} \quad r = a,$$

$$w = 0 \quad \text{when} \quad z = 0 \text{ and } r = a,$$

which evidently determine the values of the constants D_0, D_1, and K_3.

An interesting special case of this example arises when the uniform pressure p extends over the whole face $z = -h$ of the plate.

Then

$$b_{0n} = \frac{2p}{k_n a J_1(k_n a)}, \quad b_{1n} = 0,$$

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* A similar solution may be built up for which

$$\widehat{zz} = f_0(r), \quad \widehat{rz} = f_2(r) \quad \text{at} \quad z = -h,$$

$$\widehat{zz} = f_1(r), \quad \widehat{rz} = f_3(r) \quad \text{at} \quad z = +h,$$

by using a normal solution in which

$$\psi_n = b_{0n}, \quad \psi_n' = -k_n b_{2n} \quad \text{at} \quad z = -h,$$

$$\psi_n = b_{1n}, \quad \psi_n' = -k_n b_{3n} \quad \text{at} \quad z = +h,$$

where b_{2n} and b_{3n} are the coefficients of the Dini series

$$f_2(r) = \sum_{n=1}^{\infty} b_{2n} J_1(k_n r)$$

and $$f_3(r) = \sum_{n=1}^{\infty} b_{3n} J_1(k_n r),$$

respectively.

and the resulting solution is a more general one than that* in which the stress $\widehat{zz}$ is a cubic expression in z alone.

Next, we consider the case where there is an arbitrary normal stress $f_0(r)$ over the face $z = -h$, while the face $z = +h$ bears on an elastic bed which exerts a normal reaction proportional to the displacement w of that face.

In the normal solution it is evident that the solution still holds when A_1, A_2, B_1, B_2 have other suitable values. So we can still build up the solution of this section when all the A's and B's are properly chosen constants.

If the Fourier-Bessel series for

$$[K_3-\sigma(\tfrac{1}{2}D_1h^2+D_0h)-\tfrac{1}{4}(1-\sigma)D_1r^2]$$

is $$\sum_{n=1}^{\infty}\beta_n J_0(k_n r)$$

in the interval $(0, a)$, it is easy to see that the coefficients β_n are linear expressions in D_0, D_1, and K_3. The condition to be satisfied at $z = +h$ is

$$\widehat{zz}+\alpha Ew = 0,$$

where α is a constant.

When we write out this condition and introduce the series

$$\sum_{n=1}^{\infty}\beta_n J_0(k_n r),$$

it readily follows that

$$(A_{1n}+B_{1n}h)\sinh k_n h+(A_{2n}+B_{2n}h)\cosh k_n h+\alpha\beta_n$$
$$+\frac{\alpha(1+\sigma)}{k_n^2}\{k_n[(A_{1n}+B_{1n}h)-(1-2\sigma)B_{2n}]\sinh k_n h$$
$$+k_n[(A_{2n}+B_{2n}h)-(1-2\sigma)B_{1n}]\cosh k_n h\} = 0$$

for all values of n.

This equation together with the three conditions

$$\psi_n = b_{0n} \quad \text{at} \quad z = -h,$$
$$\psi_n' = 0 \quad \text{at} \quad z = \pm h,$$

are then sufficient to determine A_{1n}, A_{2n}, B_{1n}, and B_{2n} as linear expressions in D_0, D_1, and K_3, and so ensure that $\widehat{zz}$ and $\widehat{rz}$ satisfy the conditions at the faces $z = \pm h$.

* Love, *loc. cit.*, § 309.

Finally, the conditions which must be satisfied at the rim of the plate are

$$\left.\begin{aligned} \int_{-h}^{h} \widehat{rr}\, dz &= 0 \\ \int_{-h}^{h} \widehat{rz}\, dz &= 0 \\ \int_{-h}^{h} z\,\widehat{rr}\, dz &= 0 \end{aligned}\right\} \quad \text{when} \quad r = a,$$

and it is easy to see from the expressions for $\widehat{rr}$ and $\widehat{rz}$ that these give expressions linear in the A's, B's, D_0, D_1, and K_3, and so lead to three linear equations for the determination of D_0, D_1, and K_3 of the form

$$\gamma_{1s} D_0 + \gamma_{2s} D_1 + \gamma_{3s} K_3 + \gamma_{4s} = 0 \quad (s = 1, 2, 3),$$

where the γ's are determinate series expressions.

5. *The ring-shaped circular plate.*

Let us consider the ring-shaped circular plate bounded by $r = c$ and $r = a\,(c < a)$, whose faces are subject to arbitrary normal stresses. In this case we must introduce the Bessel function of order zero of the second kind. Instead of using the Bessel function $Y_0(kr)$ we employ the Bessel function* of the third kind, $H_0^{(1)}(kr)$, in the form†

$$U_0(kr) = J_0(kr)\, H_0^{(1)}(ka) - J_0(ka)\, H_0^{(1)}(kr).$$

Then the arbitrary functions $f_0(r)$ and $f_1(r)$ may be expanded in the region $r = c$ to $r = a$ in the series

$$f_0(r) = \sum_{n=1}^{\infty} b_{0n}\, U_0(\kappa_n r),$$

$$f_1(r) = \sum_{n=1}^{\infty} b_{1n}\, U_0(\kappa_n r),$$

where

$$b_{0n} = \frac{\pi^2 \kappa_n^2}{2} \frac{[J_0(\kappa_n c)]^2}{[J_0(\kappa_n a)]^2 - [J_0(\kappa_n c)]^2} \int_c^a r f_0(r)\, U_0(\kappa_n r)\, dr, \quad \text{etc.},$$

and $\pm\kappa_n$ are the real roots of

$$J_0(kc)\, H_0^{(1)}(ka) - J_0(ka)\, H_0^{(1)}(kc) = 0.$$

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* Nielsen, *Cylinderfunktionen* (1904), Ch. I, §§ 1–4. Nielsen's notation for the functions of the three kinds is

$$J^0(kr), \quad Y^0(kr), \quad H_1{}^0(kr)$$

respectively, where

$$H_1{}^0(kr) = J^0(kr) + iY^0(kr).$$

† Carslaw, *Conduction of heat*, 2nd ed.(1921), § 62.

Let us now take the solution

$$\widehat{zz} = \Psi_n \{H_0^{(1)}(\kappa_n a) J_0(\kappa_n r) - J_0(\kappa_n a) H_0^{(1)}(\kappa_n r)\}, \text{ etc.},$$

formed by combining the two normal solutions

$$\widehat{zz} = \Psi_n J_0(\kappa_n r), \quad \text{etc.},$$

$$\widehat{zz} = \Psi_n H_0^{(1)}(\kappa_n r), \quad \text{etc.},$$

where Ψ_n is the form assumed by ψ when κ_n replaces k. From this solution and the complete plane stress solution we can build up the solution

$$\widehat{zz} = \sum_{n=1}^{\infty} \Psi_n \{H_0^{(1)}(\kappa_n a) J_0(\kappa_n r) - J_0(\kappa_n a) H_0^{(1)}(\kappa_n r)\}, \quad \text{etc.}$$

for which

$$\widehat{zz} = \sum_{n=1}^{\infty} b_{0n} U_0(\kappa_n r) = f_0(r) \quad \text{at} \quad z = -h,$$

$$= \sum_{n=1}^{\infty} b_{1n} U_0(\kappa_n r) = f_1(r) \quad \text{at} \quad z = +h.$$

and* $$\widehat{rz} = 0 \quad \text{at} \quad z = \pm h.$$

In the complete plane stress solution there are six arbitrary constants, and in this solution the six constants arise linearly in the resultant stresses and stress couples over a cylindrical section of the plate of radius r, and may be determined by the boundary conditions.

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* As noted for the previous section, a similar solution may be built up for the arbitrary conditions

$$\widehat{zz} = f_0(r), \quad \widehat{rz} = f_2(r) \quad \text{at} \quad z = -h,$$

$$\widehat{zz} = f_1(r), \quad \widehat{rz} = f_3(r) \quad \text{at} \quad z = +h,$$

by using a normal solution in which

$$\Psi_n = b_{0n}, \quad \Psi_n' = -\kappa_n b_{2n} \quad \text{at} \quad z = -h,$$

$$\Psi_n = b_{1n}, \quad \Psi_n' = -\kappa_n b_{3n} \quad \text{at} \quad z = +h,$$

where b_{2n} and b_{3n} are the coefficients in the expansions

$$f_2(r) = \sum_{n=1}^{\infty} b_{2n} W_1(\kappa_n r),$$

$$f_3(r) = \sum_{n=1}^{\infty} b_{3n} W_1(\kappa_n r),$$

respectively and

$$W_1(kr) = J_1(kr) H_0^{(1)}(ka) - J_0(ka) H_1^{(1)}(kr).$$

These are series for the arbitrary functions $f_2(r)$ and $f_3(r)$ in the region $r = c$ to $r = a$ which are analogous to Dini series.

ON HANKEL TRANSFORMS

By A. C. Offord.

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Let us write

$$S_\nu(x) = x^{\frac{1}{2}} J_\nu(x),$$

where $J_\nu(x)$ is Bessel's function and $\mathfrak{R}(\nu) \geqslant -\frac{1}{2}$. Then two functions $f(x)$ and $F(x)$ are said to be Hankel transforms† of one another when they are connected by the formulae

$$F(x) = \int_0^\infty S_\nu(xu)\, f(u)\, du,$$

$$f(x) = \int_0^\infty S_\nu(xu)\, F(u)\, du,$$

where the integrals are usually interpreted in some generalized sense. When $\nu = -\frac{1}{2}$, $S_\nu(xu)$ becomes $\sqrt{(2/\pi)} \cos xu$, and these formulae reduce to the formulae for Fourier cosine transforms. In a recent paper‡ on Fourier transforms the following classes of functions were considered.

Definition A. *A function $f(x)$ belongs to the class H if it belongs to $L(0, X)$§ for all finite X and is such that*

$$\left| \int_0^\omega \left(1 - \frac{u}{\omega}\right) \cos xu\, f(u)\, du \right| \leqslant M,$$

for all $\omega \geqslant 0$ and $x \geqslant 0$, M being a number‖.

† *Cf.* Bochner, **1**, 180; Plancherel, **7**; Titchmarsh, **8** and **9**; Watson, **10**.

‡ Offord, **6**. A knowledge of this paper is not assumed.

§ By $L^p(a, b)$ we mean the class of functions whose p-th powers are integrable L in (a, b).

‖ We use M to denote a number independent of the variables under consideration but not necessarily the same at each occurrence.

DEFINITION B. *A function $F(x)$ belongs to the class B^* if it is bounded in $(0, \infty)$ and is such that the integral*

$$\int_0^\infty \frac{\sin xu}{u} F(u)\,du,$$

known to be summable $(C, 1)$ almost everywhere†, is uniformly summable $(C, 1)$ to an indefinite integral.

It was then shown that

(i) *a function of the class H has a cosine transform in B^*,*

(ii) *a function of the class B^* has a cosine transform in H.*

The methods employed to establish these results belong to the classical theory of Fourier transforms rather than to the theory which is based on Plancherel's theorem and employs the notion of limit in mean. One of the especially interesting properties of Fourier transforms, brought to light by the latter theory, is the reciprocity in the class L^2, *i.e.* a function of L^2 has a cosine transform in L^2. It is, therefore, interesting to find a similar reciprocity in the classical theory. The precise result is as follows :

If HB is the class of all the bounded functions which belong to H, then a function of HB has a cosine transform in HB.

Further we found a class of self-reciprocal functions‡ in the class HB. This result is similar to that discovered by Hardy and Titchmarsh§ for the class L^2.

In the present paper we extend these results to Hankel transforms. However, unlike the theory for the class L^2, this extension is by no means obvious. Indeed, at one or two stages we have found it convenient to employ a different argument. The following definitions replace definitions A and B.

DEFINITION 1. *A function $f(x)$ belongs to the class H_ν, $\mathfrak{R}(\nu) \geqslant -\frac{1}{2}$, if it belongs to $L(0, X)$ for all finite X and is such that*

$$\left| \int_0^\omega \left(1 - \frac{u}{\omega}\right) S_\nu(xu) f(u)\,du \right| \leqslant M,$$

for all $\omega \geqslant 0$, $x \geqslant 0$, M being independent of x and ω.

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† Since $F(u)/u$ belongs to $L^2(1, \infty)$, it follows that this integral will be summable $(C, 1)$ almost everywhere and it will be uniformly summable in any interval in which the sum function is continuous. *Cf.* Hobson, **4**, 748.

‡ Offord, **6**, Theorem 9.

§ Hardy and Titchmarsh, **3**, 201.

DEFINITION 2. *A function $F(x)$ belongs to the class B_ν^*, $\Re(\nu) \geqslant -\frac{1}{2}$, if it is bounded in $(0, \infty)$ and such that the integral*

$$\int_1^\infty \frac{\sin(xu - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}{u} F(u)\,du$$

is uniformly summable $(C, 1)$, in every finite range $0 \leqslant x \leqslant X$, to an indefinite integral.

The main results are as follows:—

(i) *A function of the class H_ν has a Hankel transform in B_ν^*.*

(ii) *A function of the class B_ν^* has a Hankel transform in H_ν.*

The first result is contained in Theorems 1 and 5 of this paper while the second follows from Theorem 4. The proof given here of Theorems 1 and 3 is essentially different from the proofs of the corresponding theorems for cosine transforms given in the previous paper. There we based the argument on harmonic functions and Fatou's theorem. Here the proof depends upon Arzelà's theorem. It is possible to prove these theorems in a manner strictly analogous to that used for the corresponding theorems for cosine transforms. The present argument, however, seems more suited for dealing with the more complicated Hankel transforms.

There is a reciprocal class $H_\nu B$ as in the case of the cosine transforms, and a corresponding class of self-reciprocal Hankel transforms.

The class H_ν.

2. In this section we prove our fundamental results.

THEOREM 1. *If $f(x)$ belongs to H_ν, then the integral*

$$\int_0^\infty S_\nu(xu)\, f(u)\,du$$

is summable $(C, 1)$ almost everywhere to a bounded function which we denote by $F(x)$. Further,

$$f(x) = \int_0^\infty S_\nu(xu)\, F(u)\,du \quad (C, 2) \tag{2.1}$$

almost everywhere.

THEOREM 2. *If $f(x)$ belongs to H_ν and if $\int_1^\infty \frac{|f(u)|}{u^2}\,du$ is convergent, then the integral (2.1) is summable $(C, 1)$ almost everywhere to $f(x)$.*

The next theorem is Parseval's theorem for Hankel transforms.

THEOREM 3. *If $f(x)$ belongs to H_ν and $G(x)$ belongs to $L(0, \infty)$, and if $g(x)$ is the Hankel transform of $G(x)$, then*

$$\int_0^\infty F(x)\, G(x)\, dx = \int_0^\infty f(x)\, g(x)\, dx,$$

the first integral being convergent and the second being summable $(C, 1)$.

In order to prove these theorems we have recourse to a series of lemmas. The proof of Theorem 3 follows that of Lemma 3. Theorem 1 is a consequence of Lemmas 3 and 6, and Theorem 2 is a consequence of Lemmas 3 and 4.

LEMMA 1. *Let $f(x)$ belong to H_ν and write*

$$F(x, \eta) = \int_0^\eta \left(1 - \frac{u}{\eta}\right) S_\nu(xu) f(u)\, du,$$

$$\Phi(x, \eta) = \int_0^x F(\xi, \eta)\, d\xi;$$

then there exists a sequence $\{\eta_i\}$ such that $\Phi(x, \eta_i)$ tends to a limit $\Phi(x)$ as η_i tends to infinity for all positive x. The function $\Phi(x)$ has almost everywhere a bounded differential coefficient.

Proof. Let $\{X_m\}$ be an increasing sequence of positive numbers such that X_m tends to infinity. Since $f(x)$ belongs to H_ν, we have

$$|\Phi(x_2, \eta) - \Phi(x_1, \eta)| = \left| \int_{x_1}^{x_2} F(\xi, \eta)\, d\xi \right| \leqslant M |x_2 - x_1|. \tag{2.2}$$

Hence the functions $\Phi(x, \eta)$, considered as functions of x in the interval $0 \leqslant x \leqslant X_1$, form a bounded equi-continuous family. Therefore, by Arzelà's theorem†, we can choose an infinite sequence $\{\eta_{1,n}\}$ such that $\Phi(x, \eta_{1,n})$ tends uniformly to a limit $\Phi(x)$ for $0 \leqslant x \leqslant X_1$. But the set $\{\Phi(x, \eta_{1,n})\}$ is bounded and equi-continuous in $(0, X_2)$, $X_2 > X_1$, and therefore from the infinite sequence $\{\eta_{1,n}\}$ we can pick out a subsequence $\{\eta_{2,n}\}$ such that $\Phi(x, \eta_{2,n})$ tends to a limit in $(0, X_2)$. Since this limit must coincide with $\Phi(x)$ in $(0, X_1)$, we can denote it by $\Phi(x)$. Repeating this process, we can find a subsequence $\{\eta_{m,n}\}$ and a function $\Phi(x)$ such that

† Hobson, **4**, 168. There are wide generalizations of Arzelà's theorem, *cf.* Fréchet, **2**, 361.

$\Phi(x,\ \eta_{m,n})$ tends to $\Phi(x)$ as n tends to infinity uniformly for $0 \leqslant x \leqslant X_m$. Put $\eta_i = \eta_{n,n}$ and we see at once that $\Phi(x,\ \eta_i)$ tends to $\Phi(x)$ as i tends to infinity for all positive values of x, and moreover the convergence will be uniform in every finite range.

From (2.2) it follows at once that $\Phi(x)$ satisfies the inequality

$$|\Phi(x_2)-\Phi(x_1)| \leqslant M|x_2-x_1|.$$

Hence $\Phi(x)$ has a bounded differential coefficient almost everywhere.

LEMMA 2. *Let a sequence η_i and a function $\Phi(x)$ be defined as in Lemma 1. Let $G(x)$ be integrable L in $(0,\ \infty)$, and let $g(x)$ be its Hankel transform. Then*

$$\int_0^\infty \Phi'(u)\,G(u)\,du = \lim_{i\to\infty} \int_0^{\eta_i} \left(1-\frac{u}{\eta_i}\right) f(u)\,g(u)\,du.$$

Proof. We have, since $G(x)$ is absolutely integrable in $(0,\ \infty)$,

$$\begin{aligned}\int_0^{\eta_i}\left(1-\frac{u}{\eta_i}\right) f(u)\,g(u)\,du &= \int_0^{\eta_i}\left(1-\frac{u}{\eta_i}\right) f(u)\,du \int_0^\infty S_\nu(ux)\,G(x)\,dx \\ &= \int_0^\infty G(x)\,dx \int_0^{\eta_i}\left(1-\frac{u}{\eta_i}\right) S_\nu(xu)\,f(u)\,du \\ &= \int_0^\infty F(x,\ \eta_i)\,G(x)\,dx.\end{aligned}$$

Now $|F(x,\ \eta_i)| \leqslant M$, for all x and η_i. Let $X = X(\epsilon)$ be chosen so that

$$M\int_X^\infty |G(x)|\,dx \leqslant \epsilon.$$

Now

$$\lim_{i\to\infty} \int_0^x F(\xi,\ \eta_i)\,d\xi = \int_0^x \Phi'(\xi)\,d\xi,$$

uniformly for $0 \leqslant x \leqslant X$, and $F(x,\ \eta_i)$ is bounded. Hence, by a theorem of Hobson and Lebesgue†,

$$\lim_{i\to\infty} \int_0^X F(x,\ \eta_i)\,G(x)\,dx = \int_0^X \Phi'(x)\,G(x)\,dx.$$

Therefore, by choice of $i > i_0(\epsilon)$, we have

$$\left|\int_0^\eta \left(1-\frac{u}{\eta_i}\right) f(u)\,g(u)\,du - \int_0^\infty \Phi'(x)\,G(x)\,dx\right| < 3\epsilon.$$

This completes the proof of the lemma.

† Hobson, **4**, 422; Lebesgue, **5**, 52.

3. LEMMA 3. *If $f(x)$ belongs to H_ν, then the integral*

$$\int_0^\infty S_\nu(xu)\, f(u)\, du \tag{3.1}$$

is summable $(C, 1)$ almost everywhere to a bounded function, which we denote by $F(x)$.

Proof. We suppose throughout that $x > 0$. Let the sequence $\{\eta_i\}$ and the function $\Phi(x)$ be defined as in Lemma 1. Since $\Phi(x)$ is absolutely continuous, we can write

$$F(x) = \Phi'(x).$$

We now show that the integral (3.1) is summable $(C, 1)$ to $F(x)$ almost everywhere. This will, incidentally, prove that $\Phi(x)$ is independent of the particular sequence $\{\eta_i\}$. Write

$$g_1(u) = \begin{cases} \left(1 - \frac{u}{\omega}\right) S_\nu(xu) & (0 \leqslant u \leqslant \omega), \\ 0 & (\omega \leqslant u). \end{cases} \tag{3.31}$$

Then the Hankel transform of $g_1(u)$ is

$$G_1(y) = \int_0^\omega \left(1 - \frac{u}{\omega}\right) S_\nu(xu)\, S_\nu(yu)\, du. \tag{3.32}$$

We have first to show that $G_1(y)$ satisfies the hypotheses of Lemma 1, *i.e.* that it is absolutely integrable in $(0, \infty)$. Let us write

$$\alpha = \tfrac{1}{2}\nu\pi + \tfrac{1}{4}\pi, \quad A = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{4\nu^2 - 1}{8}. \tag{3.41}$$

The asymptotic expansion† for $J_\nu(z)$ gives

$$S_\nu(z) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos(z - \alpha) - \frac{A}{z} \sin(z - \alpha) + R_1(z), \tag{3.42}$$

$$S_\nu'(z) = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(z - \alpha) - \frac{A}{z} \cos(z - \alpha) + R_2(z), \tag{3.43}$$

where $|z| \geqslant \delta > 0$, and

$$R_1(z) = O\left(\frac{1}{z^2}\right), \quad R_2(z) = O\left(\frac{1}{z^2}\right)$$

† Whittaker and Watson, **11**, 368.

as $z \to \infty$. Again, from the expansion† of $J_\nu(z)$ as a power series, *i.e.*

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-)^n z^{\nu+2n}}{2^{\nu+2n} n! \, \Gamma(\nu+n+1)},$$

it is easily verified that, for $\Re(\nu) \geqslant -\frac{1}{2}$ and $x \neq y$,

$$(3.5) \quad Q(x, y; \omega) = \int_0^\omega (\omega - u) S_\nu(xu) S_\nu(yu) \, du$$

$$= \frac{1}{x^2 - y^2} \int_0^\omega \{y S_\nu(xu) S_\nu'(yu) - x S_\nu'(xu) S_\nu(yu)\} \, du.$$

We now prove three inequalities for $Q(x, y; \omega)$. In each case x is fixed and positive.

$$(3.61) \qquad |Q(x, y; \omega)| \leqslant \frac{M}{|x^2 - y^2|} \quad (|x-y| \geqslant \delta > 0, \quad y \geqslant \delta > 0).$$

$$(3.62) \qquad |Q(x, y; \omega)| \leqslant \frac{M\omega}{|x^2 - y^2|} \quad (|x-y| \geqslant \delta > 0, \quad 0 \leqslant y \leqslant K).$$

$$(3.63) \qquad \left| Q(x, y; \omega) - \frac{1 - \cos \omega(x-y)}{\pi (x-y)^2} \right| \leqslant M\omega \quad (y \geqslant \delta > 0).$$

We have

$$(3.7) \qquad S_\nu'(z) = \frac{\frac{1}{2} + \nu}{z} S_\nu(z) - S_{\nu+1}(z).$$

Hence, if $y \geqslant \delta > 0$,

$$Q(x, y; \omega) = \frac{1}{x^2 - y^2} \int_0^{1/y} \{x S_{\nu+1}(xu) S_\nu(yu) - y S_\nu(xu) S_{\nu+1}(yu)\} \, du$$

$$+ \frac{1}{x^2 - y^2} \left[S_\nu(x\omega) S_\nu(y\omega) - S_\nu\left(\frac{x}{y}\right) S_\nu(1) - 2x \int_{1/y}^{\omega} S_\nu'(xu) S_\nu(yu) \, du \right].$$

Therefore

$$|Q| \leqslant \frac{2x}{|x^2 - y^2|} \left| \int_{1/y}^{K} + \int_K^\omega S_\nu'(xu) S_\nu(yu) \, du \right| + \frac{M}{|x^2 - y^2|}$$

$$= \frac{2x}{|x^2 - y^2|} \{I_1 + I_2\} + \frac{M}{|x^2 - y^2|}.$$

Hence, when $\Re(\nu) > -\frac{1}{2}$, it follows at once from (3.7) that I_1 is bounded. When $\nu = -\frac{1}{2}$, $S_\nu(z)$ reduces to $\cos z$ and I_1 is again bounded. If

† Whittaker and Watson, **11**, 358.

$\nu = -\frac{1}{2}+ik$, $k \neq 0$ we can write

$$S_\nu'(xu)\,S_\nu(yu) = (\tfrac{1}{2}+\nu)(xu)^{-1}\,S_\nu(xu)\,S_\nu(yu)+O(1)$$

$$= ik\,x^{-1}\,(xy)^{ik}\,u^{-1+2ik}+O(u^{-\frac{1}{2}})+O(1).$$

Therefore

$$|I_1| \leqslant k\left|\int_{1/y}^{K}\frac{du}{u^{1-2ik}}\right|+M \leqslant M.$$

Now, obviously, $|I_2| \leqslant M\omega$, and so (3.62) follows at once from (3.5).

To prove (3.61), choose K so that $K\delta > 1$, where $y \geqslant \delta$, and substitute the asymptotic expansions (3.42) and (3.43) in I_2. This gives

$$I_2(x, y;\, \omega) = -\frac{1}{\pi}\int_K^\omega [\sin\{(x+y)u-2\alpha\}+\sin(x-y)u]\,du$$

$$-\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\frac{A}{x}\int_K^\omega [\cos\{(x+y)u-2\alpha\}+\cos(x-y)u]\,\frac{1}{u}\,du$$

$$+\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\frac{A}{y}\int_K^\omega [\cos(x-y)u-\cos\{(x+y)u-2\alpha\}]\,\frac{1}{u}\,du$$

$$+\int_K^\omega\left\{O\left(\frac{1}{x^2u^2}\right)+O\left(\frac{1}{y^2u^2}\right)\right\}du.$$

Hence I_2 is bounded, and, since we have just shown that I_1 is bounded, we get

$$|I_1+I_2| = \left|\int_{1/y}^{\omega} S_\nu'(xu)\,S_\nu(yu)\,du\right| \leqslant M, \tag{3.8}$$

provided that $y \geqslant \delta > 0$ and $|x-y| \geqslant \delta > 0$. The inequality (3.61) now follows at once from (3.5). Finally (3.63) is easily verified by substituting (3.42) in the first member of (3.5).

Therefore $G_1(y)$ satisfies the hypotheses of Lemma 2, and so we have

$$\lim_{i\to\infty}\int_0^{\eta_i}\left(1-\frac{u}{\eta_i}\right)f(u)\,g_1(u)\,du = \int_0^\omega\left(1-\frac{u}{\omega}\right)S_\nu(xu)\,f(u)\,du$$

$$= \int_0^\infty F(y)\,G_1(y)\,dy$$

$$= \int_0^\delta+\int_\delta^{x-\delta}+\int_{x-\delta}^{x+\delta}+\int_{x+\delta}^\infty$$

$$= I_3+I_4+I_5+I_6.$$

By (3.62), since x is fixed and positive,

$$|G_1(y)| \leqslant M \quad (0 \leqslant y \leqslant \delta < x).$$

Therefore

$$|I_3| \leqslant M \int_0^\delta dy < \epsilon,$$

by choice of δ. Again, by (3.61),

$$|I_4| + |I_6| \leqslant \frac{M}{\omega} + \frac{M}{\omega} \int_{x+\delta}^\infty \frac{dy}{y^2} \leqslant \frac{M}{\omega}.$$

Also by (3.63), if δ is sufficiently small,

$$I_5 = \frac{2}{\pi} \int_{x-\delta}^{x+\delta} F(y) \frac{\sin^2 \frac{1}{2}\omega(x-y)}{\omega(x-y)^2} dy + o(1).$$

Therefore

$$\int_0^\omega \left(1 - \frac{u}{\omega}\right) S_\nu(xu) f(u)\, du = \frac{2}{\pi\omega} \int_0^\delta \{F(x+\xi) + F(x-\xi)\} \frac{\sin^2 \frac{1}{2}\omega\xi}{\xi^2} d\xi + o(1),$$

and it is well known that this tends to $F(x)$ as ω tends to infinity for almost all values of x. This completes the proof of the lemma.

Proof of Theorem 3. Since $G(x)$ belongs to $L(0, \infty)$ by Lemma 3, we have

$$\begin{aligned}
\int_0^\infty F(x)\, G(x)\, dx &= \lim_{\omega \to \infty} \int_0^\infty G(x)\, dx \int_0^\omega \left(1 - \frac{u}{\omega}\right) S_\nu(xu) f(u)\, du \\
&= \lim_{\omega \to \infty} \int_0^\omega \left(1 - \frac{u}{\omega}\right) f(u)\, du \int_0^\infty S_\nu(ux)\, G(x)\, dx \\
&= \lim_{\omega \to \infty} \int_0^\omega \left(1 - \frac{u}{\omega}\right) f(u) g(u)\, du,
\end{aligned}$$

the required result.

4. The next lemma, together with Lemma 3, completes the proof of Theorem 2.

LEMMA 4. *If $f(x)$ belongs to H_ν and is such that*

$$\int_1^\infty \frac{|f(x)|}{x^2} dx \tag{4.1}$$

converges, and if $F(x)$ is defined as in Lemma 3, then

$$f(x) = \lim_{\omega\to\infty} \int_0^\omega \left(1-\frac{u}{\omega}\right) S_\nu(xu)\, F(u)\, du,$$

almost everywhere.

Proof. In Theorem 3 take

$$G(u) = \begin{cases} \left(1-\dfrac{u}{\omega}\right) S_\nu(xu) & (0 \leqslant u \leqslant \omega), \\ 0 & (u \leqslant \omega). \end{cases}$$

Then, with the notation of (3.5),

$$g(y) = \omega^{-1} Q(x, y;\ \omega)$$

and

$$\int_0^\omega \left(1-\frac{u}{\omega}\right) S_\nu(xu)\, F(u)\, du = \frac{1}{\omega}\int_0^\infty Q(x, y;\ \omega) f(y)\, dy \quad (C, 1).$$

But, in virtue of (3.61) and (4.1), the integral in the second member converges. We can now show, exactly as in the case of Lemma 2, that the second member tends almost everywhere to $f(x)$ as ω tends to infinity.

5. We require two further lemmas to establish the reciprocal formula of Theorem 1.

LEMMA 5. *If $f(u)$ belongs to H_ν, then*

$$\left|\int_0^\omega (\omega-u)\frac{u}{1+u} f(u)\, du\right| \leqslant M\omega \log^2 \omega,$$

for all $\omega \geqslant 2$.

Proof. Write

$$G_2(y) = \int_0^\omega (\omega-u)\frac{u}{1+u} S_\nu(yu)\, du.$$

Then we shall prove that, if $\omega \geqslant 2$,

$$|G_2(y)| \leqslant M\omega^2 \quad \text{for all } y. \tag{5.1}$$

$$|G_2(y)| \leqslant My^{-1}\omega \log \omega \qquad (0 < y \leqslant 1). \tag{5.2}$$

$$|G_2(y)| \leqslant My^{-2}\omega\{|\log y|+1\} \quad (y > 0). \tag{5.3}$$

The inequality (5.1) follows at once from the fact that $S_\nu(yu)$ is bounded. To prove (5.2) we use

$$S_\nu(z) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos(yu-\alpha) + R(yu),$$

where

$$|R(yu)| \leqslant M \qquad (0 \leqslant yu \leqslant 1),$$

$$|R(yu)| \leqslant M(yu)^{-1} \quad (yu \geqslant 1).$$

and α is given by (3.41). Then it follows that

$$G_2(y) = \int_0^\omega (\omega-u)\cos(yu-\alpha)\,du - \int_0^\omega \frac{\omega-u}{1+u}\cos(yu-\alpha)\,du$$

$$+\left\{\int_0^{1/y} + \int_{1/y}^\omega\right\}(\omega-u)\frac{u}{1+u}R(yu)\,du$$

$$= I_1+I_2+I_3+I_4.$$

But

$$|I_1| = \left|\frac{\omega}{y}\sin\alpha + \frac{1}{y}\int_0^\omega \sin(yu-\alpha)\,du\right| \leqslant M\frac{\omega}{y},$$

$$|I_2| \leqslant M\int_0^\omega \frac{\omega-u}{1+u}\,du \leqslant M\omega\log\omega,$$

$$|I_3| \leqslant M\int_0^{1/y}(\omega-u)\,du \leqslant M\frac{\omega}{y},$$

$$|I_4| \leqslant \frac{M}{y}\int_{1/y}^\omega \frac{\omega-u}{1+u}\,du \leqslant \frac{M\omega\log\omega}{y}.$$

Hence

$$|G_2(y)| \leqslant \frac{M\omega\log\omega}{y}$$

for $0 < y \leqslant 1$, as required.

By (3.42),

$$G_2(y) = \int_0^{1/y}(\omega-u)\frac{u}{1+u}\{S_\nu(yu)-\cos(yu-\alpha)\}\,du$$

$$+\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\omega(\omega-u)\frac{u}{1+u}\cos(yu-\alpha)\,du$$

$$-A\int_{1/y}^\omega(\omega-u)\frac{u}{1+u}\frac{\sin(yu-\alpha)}{yu}\,du + \int_{1/y}^\omega(\omega-u)\frac{u}{1+u}R_1(yu)\,du$$

$$= I_5+I_6+I_7+I_8.$$

Now $$|I_5| \leqslant M\omega \int_0^{1/y} u\,du \leqslant M\frac{\omega}{y^2},$$

$$|I_6| = \left|\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\omega} \frac{d}{du}\left\{(\omega-u)\frac{u}{1+u}\right\}\frac{\sin(yu-a)}{y}\,du\right| \leqslant M\frac{\omega}{y^2},$$

on integrating by parts again. Further,

$$|I_7| \leqslant M\frac{\omega}{y^2} + \left|\int_{1/y}^{\omega} \frac{d}{du}\left(\frac{\omega-u}{1+u}\right)\frac{\cos(yu-a)}{y^2}\,du\right| \leqslant M\frac{\omega}{y^2},$$

$$|I_8| \leqslant \frac{M}{y^2}\int_{1/y}^{\omega} \frac{\omega-u}{u(1+u)}\,du \leqslant M\frac{\omega\{|\log y|+1\}}{y^2}.$$

Hence $$|G_2(y)| \leqslant M\omega y^{-2}\{|\log y|+1\},$$

which is (5.3).

It follows at once from (5.1) and (5.3) that $G_2(y)$ belongs to $L(0, \infty)$. Hence, by Theorem 3,

$$\int_0^{\omega}(\omega-u)\frac{u}{1+u}f(u)\,du = \int_0^{\infty} F(y)\,G_2(y)\,dy.$$

Therefore, applying the above inequalities,

$$\left|\int_0^{\omega}(\omega-u)\frac{u}{1+u}f(u)\,du\right|$$

$$\leqslant M\omega^2\int_0^{1/\omega} dy + M\omega\log\omega\int_{1/\omega}^{1}\frac{dy}{y} + M\omega\int_1^{\omega}\frac{\log y+1}{y^2}\,dy$$

$$\leqslant M\omega\log^2\omega,$$

the required result.

6. LEMMA 6. *If $f(x)$ belongs to H_ν and $F(x)$ is defined as in Lemma 3, then*

$$f(x) = \lim_{\omega\to\infty}\int_0^{\omega}\left(1-\frac{u}{\omega}\right)^2 S_\nu(xu)\,F(u)\,du,$$

almost everywhere.

Proof. As in the case of Lemma 3 we shall suppose throughout that $x > 0$. In Theorem 3 take

$$G(u) = G_3(u) = \begin{cases} \left(1-\frac{u}{\omega}\right)^2 S_\nu(xu) & (0 \leqslant u \leqslant \omega), \\ 0 & (\omega \leqslant u). \end{cases}$$

Then $G_3(u)$ obviously satisfies the hypotheses of Theorem 3 and

$$(6.1) \qquad g_3(y) = \frac{1}{\omega^2}\int_0^\omega (\omega-u)^2 S_\nu(xu)\, S_\nu(yu)\, du = \frac{2}{\omega^2}\int_0^\omega Q(x, y;\ u)\, du,$$

by (3.5). We now prove the following inequalities for $g_3(y)$ and its derivatives. In each case x is fixed and positive.

$$(6.21) \qquad |g_3(y)| \leqslant M \quad (|x-y| \geqslant \delta > 0,\ \ 0 \leqslant y \leqslant K),$$

$$(6.22) \qquad |g_3(y)| \leqslant \frac{M}{\omega|x^2-y^2|} \quad (|x-y| \geqslant \delta > 0,\ \ y \geqslant \delta > 0),$$

$$(6.23) \qquad \left| g_3(y) - \frac{2}{\pi\omega^2}\left\{\frac{\omega}{(x-y)^2} - \frac{\sin\omega(x-y)}{(x-y)^3}\right\}\right| \leqslant M \quad (y \geqslant \delta > 0),$$

$$(6.24) \qquad |g_3'(y)| \leqslant \frac{M}{\omega y^2} \quad (y \geqslant x+\delta,\ \ \delta > 0),$$

$$(6.25) \qquad |g_3''(y) - y^{-3}\{U \sin\omega y + V\cos\omega y\}| \leqslant \frac{M}{\omega y^3} \quad (y \geqslant x+\delta,\ \ \delta > 0),$$

where $U(x, y;\ \omega)$ and $V(x, y;\ \omega)$ are bounded functions of x, y, and ω.

The inequalities (6.21), (6.22), and (6.23) follow at once from (3.62), (3.61), and (3.63) respectively. To prove (6.24) observe that, from (3.5),

$$\frac{\partial Q}{\partial y} = \frac{2y}{x^2-y^2}\, Q + \frac{1}{x^2-y^2}\int_0^\omega \frac{\partial}{\partial y}\{y\, S_\nu(xu)\, S_\nu'(yu) - xS_\nu'(xu)\, S_\nu(yu)\}\, du$$

$$= \frac{2y}{x^2-y^2}\, Q + \frac{1}{x^2-y^2}\int_0^{1/y} \frac{\partial}{\partial y}\{x\, S_{\nu+1}(xu)\, S_\nu(yu) - y\, S_\nu(xu)\, S_{\nu+1}(yu)\}\, du$$

$$+ \frac{1}{x^2-y^2}\left[\int_{1/y}^\omega S_\nu(xu)\, S_\nu'(yu)\, du + \omega\, S_\nu(x\omega)\, S_\nu'(y\omega) - \frac{1}{y}\, S_\nu\left(\frac{x}{y}\right) S_\nu'(1)\right]$$

$$- \frac{1}{x^2-y^2}\left[2x\int_{1/y}^\omega u\, S_\nu'(xu)\, S_\nu'(yu)\, du + \int_{1/y}^\omega S_\nu(xu)\, S_\nu'(yu)\, du\right].$$

Hence, on using (3.61) and (3.7),

$$\frac{\partial Q}{\partial y} = -\frac{2\omega}{\pi(x^2-y^2)}\cos(x\omega-\alpha)\sin(y\omega-\alpha)$$

$$- \frac{2x}{x^2-y^2}\int_{1/y}^\omega u\, S_\nu'(xu)\, S_\nu'(yu)\, du + O\left(\frac{1}{x^2-y^2}\right).$$

Now, by a very similar argument to that employed to prove (3.8), we can dispose of the integral in the second member and so obtain

$$\left|\int_0^\omega \frac{\partial Q(x,y;u)}{\partial y}\,du\right| \leqslant \frac{M\omega}{|x^2-y^2|}. \tag{6.3}$$

The inequality (6.24) now follows from (6.1) and (6.3).

Again, it is easy to see that

$$\frac{\partial^2 Q}{\partial y^2} = \frac{2}{x^2-y^2}\,Q + \frac{4y}{x^2-y^2}\,\frac{\partial Q}{\partial y} + \frac{1}{x^2-y^2}\int_0^\omega \frac{\partial^2}{\partial y^2}\{y\,S_\nu(xu)\,S_\nu'(yu) - x\,S_\nu'(xu)\,S_\nu(yu)\}\,du.$$

Hence, by (3.61) and (6.3),

$$\begin{aligned}\frac{\partial^2 Q}{\partial y^2} = \frac{4y}{x^2-y^2}\,\frac{\partial Q}{\partial y} &+ \frac{1}{x^2-y^2}\int_0^{1/y}\{xu^2\,S_{\nu+1}(xu)\,S_\nu''(yu) \\ &- 2u\,S_\nu(xu)\,S_{\nu+1}'(yu) - yu^2\,S_\nu(xu)\,S_{\nu+1}''(yu)\}\,du \\ &+ \frac{1}{x^2-y^2}\Big[\omega^2\,S_\nu(x\omega)\,S_\nu''(y\omega) - \frac{1}{y}\,S_\nu\Big(\frac{x}{y}\Big)\,S_\nu''(1) \\ &- 2x\int_{1/y}^\omega u^2\,S_\nu'(xu)\,S_\nu''(yu)\,du\Big] + O\Big(\frac{1}{y^3}\Big).\end{aligned}$$

Now
$$S_\nu''(z) = \Big\{\frac{(\frac{1}{2}+\nu)^2}{z^2} - \frac{\frac{1}{2}+\nu}{z} - 1\Big\}\,S_\nu(z).$$

Therefore

$$\begin{aligned}\frac{\partial^2 Q}{\partial y^2} = \frac{4y}{x^2-y^2}\,\frac{\partial Q}{\partial y} &- \frac{\omega^2}{x^2-y^2}\,S_\nu(x\omega)\,S_\nu(y\omega) \\ &- \frac{(\frac{1}{2}+\nu)\omega}{(x^2-y^2)\,y}\,S_\nu(x\omega)\,S_\nu(y\omega) + \frac{x(1+2\nu)}{(x^2-y^2)y}\int_{1/y}^\omega u\,S_\nu'(xu)\,S_\nu(yu)\,du \\ &+ \frac{2x}{x^2-y^2}\int_{1/y}^\omega u^2\,S_\nu'(xu)\,S_\nu(yu)\,du + O\Big(\frac{1}{y^3}\Big).\end{aligned}$$

Hence, employing the usual argument with the asymptotic expansion, we see that

$$\begin{aligned}\int_0^\omega \frac{\partial^2 Q}{\partial y^2}\,du = \frac{4y}{x^2-y^2}\int_0^\omega \frac{\partial Q}{\partial y}\,du &- \frac{2}{\pi(x^2-y^2)}\int_0^\omega u^2\cos(xu-\alpha)\cos(yu-\alpha)\,du \\ &- \frac{4x}{\pi(x^2-y^2)}\int_0^\omega dv\int_0^v u^2\sin(xu-\alpha)\sin(yu-\alpha)\,du + O\Big(\frac{\omega}{y^3}\Big),\end{aligned}$$

and this together with (6.3) yields the inequality (6.25).

We can now complete the proof of the lemma. We have, by Theorem 2,

$$\int_0^\omega \left(1-\frac{u}{\omega}\right) S_\nu(xu)\, F(u)\, du$$

$$= \lim_{Y\to\infty} \int_0^Y \left(1-\frac{y}{Y}\right) f(y)\, g_3(y)\, dy$$

$$= \left\{\int_0^\delta + \int_\delta^{x-\delta} + \int_{x-\delta}^{x+\delta}\right\} f(y)\, g_3(y)\, dy + \lim_{Y\to\infty} \int_{x+\delta}^Y \left(1-\frac{y}{Y}\right) f(y)\, g_3(y)\, dy$$

$$= I_1+I_2+I_3+I_4.$$

By (6.21),

$$|I_1| \leqslant M\int_0^\delta |f(y)|\, dy < \epsilon,$$

by choice of δ. Again, by (6.22),

$$|I_2| \leqslant \frac{M}{\omega}\int_\delta^{x-\delta} |f(y)|\, dy < \epsilon,$$

if $\omega \geqslant \omega_0(\epsilon)$. Further, by (6.23), if δ is sufficiently small,

$$I_3 = \frac{2}{\pi\omega}\int_{x-\delta}^{x+\delta} f(y)\left\{\frac{1}{(x-y)^2} - \frac{\sin\omega(x-y)}{\omega(x-y)^3}\right\} dy + o(1)$$

$$= \frac{2}{\pi\omega}\int_0^\delta \{f(x+\xi)+f(x-\xi)-2f(x)\}\left\{\frac{1}{\xi^2} - \frac{\sin\omega\xi}{\omega\xi^3}\right\} d\xi$$

$$+\frac{4}{\pi}\, f(x)\int_0^{\omega\delta}\left\{\frac{1}{\xi^2}-\frac{\sin\xi}{\xi^3}\right\} d\xi + o(1).$$

Of the two terms in the second member, the second tends to $f(x)$ as ω tends to infinity, while the first tends to zero at every point at which

$$\frac{1}{\eta}\int_0^\eta \{f(x+\xi)+f(x-\xi)-2f(x)\}\, d\xi \to 0,$$

and so almost everywhere†. Hence

$$\lim_{\omega\to\infty} I_3 = f(x)$$

almost everywhere.

Write

$$\phi(y) = \frac{y}{1+y}\, f(y)$$

† Hobson, 4, 563.

and integrate I_4 by parts twice, using (6.22), (6.24), and Lemma 5. We get

$$I_4 = \lim_{Y\to\infty} \frac{2}{Y} \int_{x+\delta}^{Y} \left\{ \frac{1}{y^2} g_3(y) - \frac{1+y}{y} g_3'(y) \right\} \phi_2(y)\, dy$$

$$+2 \lim_{Y\to\infty} \int_{x+\delta}^{Y} \left(1-\frac{y}{Y}\right) \left\{ \frac{2}{y^3} g_3(y) - \frac{2}{y^2} g_3'(y) + \frac{1+y}{y} g_3''(y) \right\} \phi_2(y)\, dy + O\left(\frac{1}{\omega}\right).$$

Hence, in virtue of Lemma 5, and (6.22), (6.24), (6.25),

$$I_4 = 2 \int_{x+\delta}^{\infty} \frac{1+y}{y^4} \{U \sin \omega y + V \cos \omega y\} \phi_2(y)\, dy + O\left(\frac{1}{\omega}\right),$$

and this term tends to zero as ω tends to infinity by the Riemann-Lebesgue† theorem. This completes the proof of the lemma.

The class B_ν^*.

7. We have already defined the class B^* as the class of all the bounded functions $F(x)$ for which

$$\int_1^\infty \frac{\sin(xu-\alpha)}{u} F(u)\, du$$

is uniformly summable $(C, 1)$ to an integral.

Write

$$(7.1) \qquad T_\nu(z) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(z-\alpha) - \int_z^\infty \left\{ S_\nu(t) - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos(t-\alpha) \right\} dt,$$

so that

$$(7.2) \qquad T_\nu'(z) = S_\nu(z).$$

Substituting (3.42) into (7.1), we easily see that the integral converges, and, in fact,

$$(7.3) \qquad T_\nu(z) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(z-\alpha) + O\left(\frac{1}{z}\right).$$

Hence, if $F(u)$ belongs to B_ν^*, it is easily inferred that the integral

$$\int_1^\infty \frac{T_\nu(xu)}{u} F(u)\, du$$

is uniformly summable $(C, 1)$ to an indefinite integral.

† Hobson, **4**, 515.

THEOREM 4. *If $F(x)$ belongs to $B_\nu{}^*$, and if*

$$\mathfrak{F}(x) = \frac{d}{dx}\int_1^\infty \frac{T_\nu(xu)}{u} F(u)\,du,$$

then $\mathfrak{F}(x)$ belongs to H_ν and is the Hankel transform of the function which is equal to $F(x)$ for $x>1$ and zero for $0<x<1$.

Proof. Let

$$\mathfrak{F}_1(x) = \int_0^x \mathfrak{F}(t)\,dt = \int_1^\infty \frac{T_\nu(xu)}{u} F(u)\,du + C.$$

Then

$$\begin{aligned}
&\int_0^\omega (\omega-u)\, S_\nu(xu)\, \mathfrak{F}(u)\,du \\
&\quad = -\int_0^\omega \frac{d}{du}\{(\omega-u)\, S_\nu(xu)\}\, \mathfrak{F}_1(u)\,du \\
&\quad = \lim_{\delta\to 0}\left[(\omega-\delta)\, S_\nu(x\delta)\, C - \int_\delta^\omega \frac{d}{du}\{(\omega-u)\, S_\nu(xu)\}\,du \int_1^\infty \frac{T_\nu(uv)}{v} F(v)\,dv\right] \\
&\quad = \int_1^\infty F(v)\,dv \int_0^\omega (\omega-u)\, S_\nu(xu)\, S_\nu(vu)\,du + \lim_{\delta\to 0}(\omega-\delta)\, S_\nu(x\delta)\, \mathfrak{F}_1(\delta) \\
&\quad = \int_1^\infty F(v)\, Q(x, v;\ \omega)\,dv,
\end{aligned}$$

with the notation of (3.5). It follows from (3.61) and (3.63) that the second member is less than a constant multiple of ω. Hence $\mathfrak{F}(x)$ belongs to H_ν. Also, from Lemma 3 it follows that the Hankel transform of $\mathfrak{F}(x)$ is equal to $F(x)$ if $x>1$ and zero if $0<x<1$. This completes the proof of the theorem.

THEOREM 5. *If $f(x)$ belongs to H_ν, then its Hankel transform $F(x)$ belongs to $B_\nu{}^*$.*

Proof. We have

$$f(x) = \lim_{\omega\to\infty}\int_0^\omega \left(1-\frac{u}{\omega}\right)^2 S_\nu(xu)\, F(u)\,du,$$

almost everywhere. Further, it is easy to see from the proof of Lemma 6 that, if x lies in a finite range $0\leqslant x\leqslant X$, then the above integral behaves as regards summability $(C, 2)$ exactly like the Fourier series of a function which is equal to $f(x)$ in $(0, X)$. Also, from (7.2),

$$\int_0^z S_\nu(t)\,dt = T_\nu(z) - T_\nu(0).$$

Therefore

$$\int_0^x \left\{ f(t) - \int_0^1 S_\nu(tu)\, F(u)\, du \right\} dt = \lim_{\omega\to\infty} \int_1^\omega \left(1-\frac{u}{\omega}\right)^2 \frac{T_\nu(xu)-T_\nu(0)}{u}\, F(u)\, du.$$

But, since $F(x)$ is bounded, it follows from (7.3) that

$$\int_1^\infty \frac{T_\nu(xu)}{u}\, F(u)\, du$$

is summable $(C, 1)$ almost everywhere, and this requires

$$\int_1^\infty \frac{F(u)}{u}\, du$$

to be summable $(C, 2)$. Hence

$$\lim_{\omega\to\infty} \int_1^\omega \left(1-\frac{u}{\omega}\right)^2 \frac{T_\nu(xu)}{u}\, F(u)\, du = \int_0^x \left\{ f(t) - \int_0^1 S_\nu(tu)\, F(u)\, du \right\} dt + \text{const.},$$

and it follows that $F(x)$ belongs to $B_\nu{}^*$.

The class $H_\nu B$.

THEOREM 6. *If $f(x)$ belongs to H_ν and is bounded, then its Hankel transform $F(x)$ also belongs to H_ν and is bounded.*

Proof. By Theorem 3,

$$\int_0^\omega \left(1-\frac{u}{\omega}\right) S_\nu(xu)\, F(u)\, du = \frac{1}{\omega} \int_0^\infty f(y)\, Q(x, y;\ \omega)\, dy,$$

and, as has been remarked in Theorem 4, the second member is bounded. Hence $F(x)$ belongs to H_ν, and the theorem is proved.

We conclude by stating a theorem on self-reciprocal functions of the class $H_\nu B$ analogous to that given by Hardy and Titchmarsh† for functions of the class $L^2(0, \infty)$. It is easy to construct a proof on the lines of the proof for the special case of cosine transforms which has already been published‡.

Suppose that $\chi(t)$ is a function, real or complex, of the real variable t which is integrable L in every finite range and such that

$$\left| \int_{-T}^{T} \left(1-\frac{|t|}{T}\right) \chi(t)\, x^{-\frac{1}{2}-it}\, dt \right| \leqslant M, \tag{8.1}$$

† Hardy and Titchmarsh, 3, 208.

‡ Offord, 6.

for all $T \geqslant 0$ and $x \geqslant 0$. Further, let

$$\frac{\chi(t)}{2^{\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}it}\,\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}it)} = \text{an even function of } t. \tag{8.2}$$

THEOREM 7. *A necessary and sufficient condition that a function $f(x)$ of $H_\nu B$ should be its own Hankel transform is that it should be of the form*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi(t)\, x^{-\frac{1}{2}-it}\, dt, \tag{8.3}$$

where the integral is summable $(C, 1)$ almost everywhere and $\chi(t)$ satisfies (8.1) *and* (8.2).

In virtue of Theorem 5, we may also express the above result in the following form.

THEOREM 8. *A necessary and sufficient condition that a bounded function $f(x)$ should be a solution of the equation*

$$f(x) = \frac{d}{dx}\int_0^1 \frac{T_\nu(xu) - T_\nu(0)}{u}\, f(u)\, du + \frac{d}{dx}\int_1^\infty \frac{T_\nu(xu)}{u}\, f(u)\, du$$

is that it should be of the form (8.3), *where the integral is summable $(C, 1)$ almost everywhere and $\chi(t)$ satisfies* (8.1) *and* (8.2).

References.

1. S. Bochner, *Vorlesungen über Fouriersche Integrale* (Leipzig, 1932).
2. M. Fréchet, "Sur les ensembles abstraits", *Annales de l'Ecole Normale* (3), 38 (1921), 341–388.
3. G. H. Hardy and E. C. Titchmarsh, "Self-reciprocal functions", *Quart. J. of Math.* (Oxford), 1 (1930), 196–231.
4. E. W. Hobson, *Functions of a real variable*, 2, 2nd ed. (Cambridge, 1926).
5. H. Lebesgue, "Sur les intégrales singulières", *Annales de la Fac. des Sci. de Toulouse* (3), 1 (1909), 25–128.
6. A. C. Offord, "On Fourier transforms", *Proc. London Math. Soc.* (2), 38 (1934), 197–216.
7. M. Plancherel, "Sur les formules d'inversion de Fourier et de Hankel", *Proc. London Math. Soc.* (2), 24 (1925), 62–70.
8. E. C. Titchmarsh, "Hankel transforms", *Proc. Camb. Phil. Soc.*, 21 (1922), 463–73.
9. ———— "A note on Hankel transforms", *Journal London Math. Soc.*, 1 (1926), 195–6.
10. G. N. Watson, *Theory of Bessel Functions* (Cambridge, 1922).
11. E. T. Whittaker and G. N. Watson, *Modern analysis*, 4th ed. (Cambridge, 1927).

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ON TRIPLE PLANES WHOSE BRANCH CURVES ARE OF ORDER FOURTEEN

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I have recently described* a method of treating multiple planes (having only ordinary singularities) whose section genus π is not less than the number of sheets n. In the present paper I propose to apply this argument, both on account of the interest of the results and as an illustration of the method, to the case $n = 3$, $\pi = 5$, or the triple plane with 14-ic branch curve. The formula which I have often quoted† gives as the number of cusps on the branch curve

$$3\kappa = 48 - 3p_a,$$

since, of course, where there are only three sheets the branch curve has no nodes of the kind which enter into the formula. The investigation is simplified by the fact that the characteristic series of the prime sections is complete, since a curve of genus higher than 1 cannot have on it a g_3^2. The investigation referred to shows that on the surface Φ, a projective model of the system $|C'|$ adjoint to the prime sections $|C|$ of the triple plane Σ, the involution which corresponds to the triads of companion points of Σ, *i.e.* the characteristic series of $|C|$, is traced by ∞^2 trisecant lines, among which there are ∞^2 ruled cubics, every two of which have a generator in common, and which trace on the surface the curves of the net $|C|$. The Grassmannian of these lines is the del Pezzo surface of order 9 (not necessarily normal); and if the system is such that s of the lines lie in a general prime, the locus generated by them has as its prime section the del Pezzo surface of order $9-s$ (normal), represented on a plane by cubics with s simple base points; and, further, if these s points

† P. Du Val, *Journal London Math. Soc.*, 10 (1935), 12.
† *Ibid.*

are so disposed in the plane as to impose s' conditions on cubics and s'' on conics to pass through them, then the surface Φ or Σ has

$$p_g = 6 - s'', \quad p_a = 6 - s'.$$

It is clear that the surface will be regular unless either there are six base points on a conic (giving $p_g = 1$, $p_a = 0$) or $s > 6$ (giving $p_g = 0$, $p_a < 0$). We shall, in fact, find that neither of these cases can arise, but they require to be examined.

A ruled cubic on which a canonical curve of genus 5 lies cannot be a cone, since then three sets of the g_3^1 traced by the generators would be a prime section of the curve, which, however, is only an octavic. Thus the locus generated by the line system cannot be the cone projecting a del Pezzo surface of order 9 or 8; and the only locus other than a cone whose prime section is either of those surfaces, namely that represented on [3] by all quadrics, is not generated by a line system. We must, therefore, exclude the values $s = 0$, 1, and begin with the consideration of $s = 2$, which we expect to give a regular surface of genus 4.

Case I. $s = 2$. *Regular surface of genus* 4.

There is only one locus whose prime section is the del Pezzo surface of order 7*; this is represented on [3] by quadrics through a point A; its generating line system is represented by the lines through A, and the ruled cubics among this by planes through A. Each ruled cubic is represented on the corresponding plane by conics through A, so that the corresponding curve of the net $|C|$ is represented on this plane by a quintic passing twice through A. The surface Φ is accordingly represented by a quintic with a double point at A, from which clearly it projects into the triple plane; it is easily verified from the ordinary polar theory that this projection branches along a curve of order 14 with thirty-six cusps.

The same result might have been obtained by considering that, since the canonical curve of genus 5 lying on a ruled cubic is the residual section of the ruled cubic by a cubic primal which also contains a generator, and since two of the ruled cubics on the V_3 generated by the line system have a generator in common, the section of V_3 by a cubic primal, residual to one of the ruled cubics, is a surface of the kind which we are trying to construct; unless (as may happen for large values of s, when the whole figure is in fairly low dimensions) two ruled cubics have something more in common than one generator. This method will be useful in the sequel.

* F. Enriques, *Math. Annalen*, 46 (1895), 177.

The general quintic surface in [3] is the canonical regular surface of genus 4 and linear genus 6*. The presence of a node does not affect these invariants.

Case II. $s = 3$. *Regular surface of genus* 3.

I have pointed out lately† that the only two *general* types of locus having the sextic del Pezzo surface as prime section are those represented by cubic surfaces through a twisted cubic curve and having a node in a fixed point of it, and by cubics through three skew lines. Of these the latter is generated by three line systems each having as its Grassmannian the super-normal octavic del Pezzo surface, and is accordingly of no use to us; the former, on the other hand, has two systems of generators, each having as its Grassmannian the nonic del Pezzo surface. In each line system there are ∞^2 ruled cubics, the two systems of ruled cubics being residual with regard to prime sections. It follows that the residual section of the locus by a cubic primal which meets it further in a ruled cubic of the first system, say (α), is a surface Φ such as we are seeking; the net $|C|$ is traced by the ruled cubics (α), so that the canonical system $|C'-C|$ is traced by those of the other system, (β) say. If in the representation on [3] we take (α) to correspond to the quadrics through the base curve, (β) to the planes through the base node, the surface is represented by a septimic going twice through the base curve and five times through the base node. The canonical model of the surface is its projection from this point, a double plane with an octavic branch curve ($p_g = p_a = 3$, $p^{(1)} = 3$). Otherwise, since the traces of the two systems of ruled cubics on each other are conics, it is easily seen that the ruled cubics (β) trace on the surface a net of normal hyperelliptic septimics of genus 3, whose characteristic series is the semicanonical g_2^1. The involution whose model is the canonical double plane is, in fact, traced on the surface by one of the line systems of V_3 in exactly the same way as that whose model is the triple plane Σ is traced by the other system.

Case III. $s = 4$. *Regular surface of genus* 2.

The general V_3 whose prime section is the quintic del Pezzo surface‡, is represented on [3] by cubic surfaces through a rational quartic curve, and has one line system (corresponding to the chords of the base curve) of which three go through each point of the locus, and whose Grassmannian

* F. Enriques and G. Campedella, *Teoria delle superficie algebriche* (1932), 315.

† P. Du Val, *Journal London Math. Soc.*, 8 (1933), 206.

‡ J. A. Todd., *Proc. London Math. Soc.* (2), 30 (1929), 537.

is not the del Pezzo surface, but a surface represented on a plane by quintics without base points. If, however, the base curve in the representation is allowed to degenerate into a line and a cubic with one common point, then the line system breaks up into two, one represented by the common transversals of the line and the cubic, the other by chords of the cubic. The first of these generates the locus doubly (*i.e.* two lines of the system pass through each point), the second generates it simply. Every quadric through the base cubic curve represents a ruled cubic whose generators are of the second system (it can readily be seen that the exception provided by the quadric through both base curves is illusory), so that the Grassmannian of this system is the del Pezzo surface, as we require, and this is the most general quintic locus having such a system.

There is a pencil of quadrics on the locus, represented by the planes through the base line. There is also a node, represented by the first neighbourhood of the common point of the two base curves; this is a base point both for the net of ruled cubics and for the pencil of quadrics (which are residual with respect to prime sections, and which trace on each other conics through the node); the two systems of generating planes of the tangent cone at the node are tangent planes to ruled cubics and quadrics respectively. The locus is, in fact, case (ii) in Todd's list* of the special types of V_3^5.

The surface Φ is the section of this by a cubic primal, residual to a ruled cubic. The canonical system is traced on it by the primes through a ruled cubic, *i.e.* by the pencil of quadrics; and is thus a pencil of elliptic quartics. These do not have the node of V_3^5 (which is a simple point of the surface Φ) as a base point, so that the surface cannot be transformed into one on which the canonical series is a pencil of cubics. The surface clearly belongs to the known type of regular surfaces having $p_g = p_a = 2$, $p^{(1)} = 1$, the minimum order that can be given to the curves of the canonical system being in this case 4.

Case IV. $s = 5$. *Regular surface of genus* 1.

A V_3 whose prime section is the quartic del Pezzo surface is the intersection of two quadrics in [5]. If it has on it a ruled cubic, this lies in a unique prime which meets the locus residually in a plane. Conversely, if there is a plane on the locus, the ∞^2 primes through it meet the locus residually in a net of ruled cubics, any two of which have (as is easily seen) a line in common; in fact, any [3] through the plane meets the pencil of quadrics through the locus residually in a pencil of planes, the locus itself

* *Loc. cit.*

consequently in the base line of this pencil. The locus is accordingly, in this case, generated by a linc system of the required type. On the unique plane of the locus the ruled cubics trace a net of conics, of grade 1, and hence having three base points, nodes of the locus. It is, in fact, clear that this case of the intersection of two quadrics in [5] is just that which is obtained as the projection of the general V_3^5 from a point of itself; a plane arises from the neighbourhood of the projecting point, and three nodes in the plane from the three lines on the locus through the projecting point.

Now the surface Φ, being the residual section of V_3^4 by a cubic through a ruled cubic, is coresidual to a quadric section together with the plane on the locus; but since a prime section, together with the plane, is what corresponds to a prime section of V_3^5, it follows that Φ corresponds to the section, Ψ say, of V_3^5 by a quadric through the projecting point. Now, since Φ is the projection of Ψ from a point, and Σ is the projection of Φ from the plane on V_3^4, the two projections can be carried out in a single process. Ψ is the general quadric section of V_3^5*, and Σ is the projection of Ψ from the tangent [3] to V_3^5 at that point from which we have regarded V_3^5 as being projected into V_3^4. This [3] touches Ψ at the point in question, and cuts it in three further points, since it contains the three lines on V_3^5 through the point of contact, and each of these meets Ψ (which is a quadric section) in one further point. The projection thus has the effect of imposing on the prime sections of Ψ one double and three simple base points, all with the imposition of only four conditions.

It is perhaps worth commenting on the fact that, in constructing the system adjoint to the prime sections of Σ (which is done by adding the exceptional curves), what is directly obtained is not the standard model Ψ free from exceptional curves, but Φ, which has one exceptional line, lying in the unique plane of V_3^4. This is due to the fact that the addition of an exceptional curve to the prime sections of a surface reduces the order of the curve by unity.; so that the exceptional curves of Σ, which are three lines and a conic, appear on Φ as three points and a line.

Case V. $s = 6$. *Rational surface.*

If the six base points in the plane on which the prime sections of a V_3 are to be mapped do not lie on a conic, we get a regular surface of genus 0; we shall find, in fact, that it is a rational surface.

V_3 is a cubic primal in [4]; on this, if there is so much as a single ruled cubic, there are two nets of them, residual with respect to quadric sections,

* I have pointed out elsewhere [*Rendiconti Lincei* (6), 15 (1932), 345] that this is the general surface whose prime sections are canonical curves of genus 6.

the generators of each net forming a system of lines whose Grassmannian is what we want it to be. The primal is in this case determinantal, and has six nodes which are base points of both nets of ruled cubics, the lines of either system which pass through a node generating a cone which is a cubic of the corresponding net; the lines which join the six nodes by pairs thus belong to both systems. The locus can be represented on [3] by cubic surfaces through two twisted cubic curves which have five common points; the line systems are represented by the chords of the two base curves, and the nets of ruled cubics by the nets of quadrics through these curves.

The surface Φ is accordingly the section of the determinantal cubic by another cubic, residual to a ruled cubic surface. This is a well-known surface*, sometimes known as that of Bordiga; it is rational, being represented on a plane by quartics through ten points, and its equations are the vanishing of all cubic minors in a matrix of three rows and four columns whose elements are linear. It has thus through it a linear ∞^3 of determinantal cubics (which are easily found to be all the cubic primals containing the surface) of which the locus V_3 with which we began is one; these meet by pairs (residually to the surface) in ∞^4 ruled cubics, which trace on the Bordiga surface canonical curves of genus 5; the cubic primals meet by threes in trisecants of the Bordiga surface.

If the cubic primals (with the hexads of points of the surface which are their sets of nodes) are made to correspond to the ∞^3 planes of [3], then the ruled cubics (with the curves which they trace on the surface) correspond to the ∞^4 lines of [3], and the trisecants (with the triads of points which they trace on the surface) to the ∞^3 points of [3]. We have, accordingly, on the Bordiga surface ∞^3 triads, ∞^3 hexads, and ∞^4 canonical curves of genus 5, corresponding to the points, planes, and lines respectively in ordinary space. We can take as "Grassmannian" of the system of curves the quadric in [5], just as we do for lines; and we conclude that the curve system is not linear, but is contained in a linear system of freedom 5. It is easy to see, when the Bordiga surface is represented on a plane by quartics with ten base points, that a canonical curve of genus 5 on it is represented by a septimic going twice through each of the base points; and of these there are, in fact, ∞^5. In this linear system is a quadratic subsystem of curves each having a g_3^1 (which imposes one condition on a curve of genus 5) and consequently traced on the surface by a ruled cubic. A pencil of curves through a hexad belongs entirely to the subsystem, and has a triad for residual base points; similarly a pencil through a triad

* Both this surface and the determinantal cubic are described by F. P. White, *Proc. Camb. Phil. Soc.*, 21 (1922), 216.

has a hexad for residual base points. The two systems of planes on the quadric in [5] give two types of net, falling wholly within the subsystem; and remembering the relation of the two systems of planes to points and planes in [3], we see that one type of net will have a triad for its base points, the other type a hexad; so that a hexad only imposes three conditions on the curves of the system. It is a net of this latter kind whose projective model is the triple plane Σ.

The relation of the triads and hexads to the two systems of planes on the quadric can be clarified by remarking that if, instead of taking the "Grassmannian" of the system of octavic curve, *i.e.* making the individual curves correspond to points in [5], we make them correspond to primes, the quadric, being a self-dual structure, is unchanged; it then appears that the projective model, Ξ say, of the whole system of curves lies on the quadric, the planes of one system being trisecant and those of the other six-secant to the surface. This means, incidentally, that the surface Ξ is the Grassmannian of a (3, 6) congruence in [3]*. By projecting from a six-secant plane we obtain the triple plane Σ.

Case VI. $s = s' = 6,\ s'' = 5.$ (*Impossible.*)

It is now necessary to examine the possibility that the trisecants of Φ may have a Grassmannian represented on a plane in such a way that the points representing the lines which lie in a general prime are six points on a conic; this would lead, by the general formula, to a surface having

$$p_g = 1, \quad p_a = 0.$$

It is clear what is the locus generated by the line system; for cubics through six points on a conic represent a cubic surface with a node, and a V_3 having this as general prime section is a cubic primal in [4] with a double line. This has clearly a line system of the kind required, since every plane through the double line meets the primal further in a line, and every prime through the double line meets the primal in a ruled cubic. By projecting from a point of the double line we represent the V_3 on [3], by means of cubic monoids through a rational sextic curve which passes four times through the double base point of the monoids. The $(n-1)$-ple base point of a system of n-ic monoids, whether curves or surfaces, we may conveniently

* I find since writing this that the surface Ξ has been considered from this point of view by G. Fano, *Mem. della R. Acc. di Scienze di Torina* (2), 51 (1902), 1; and by J. G. Semple, *Proc. London Math. Soc.* (2), 35 (1933), 294. But as they do not say just what I have said, I let it stand.

call the vertex. The lines through the vertex represent the lines on the locus, and the planes through it the ruled cubics.

On each of these planes the curve representing a canonical curve of genus 5 must be sextic, passing three times through the vertex and twice through the two further traces of the base curve. It follows that the surface Φ tracing such a curve on the general ruled cubic of the net must be represented by a sextic, passing twice through the base curve and having a triple point at the vertex. This, however, is clearly impossible, since the tangent cone at the vertex must be cubic with four double generators in the tangents to the four branches of the base curve. Thus no triple plane exists corresponding to this line system.

Case VII. $s - s' = 7$, $s'' = 6$. (*Impossible.*)

If $s = 7$, then automatically $s'' = 6$, since any set of conditions on a conic which prevents it from existing at all is equivalent to six independent conditions. We should expect accordingly a surface (if any) of genera $p_g = 0, p_a = -1$, which might be an elliptic ruled surface. The line system generates a locus whose prime section is represented by cubics with seven base points, and is thus a double plane with quartic branch curve. The locus is thus a double [3] branching along a quartic surface, so that the line system is a congruence of order 2 in [3]. It is of class 7, since the order of the Grassmannian, 9, is the sum of the order and class of the congruence.

Consider a curve C^{11} in [3], of order 11 and genus 3, passing seven times (its tangents there being general in direction) through a point A. It is easily seen to be the section of a quartic cone of vertex A by a quartic monoid of the same vertex, residual to five generators; and its canonical series is traced by planes through A. Consider the system of quartic monoids of vertex A containing the curve C^{11}; it is easily found to be of grade 2 and freedom 3, so that it represents a double [3]; the lines through A, other than those meeting C^{11}, represent a line system generating the double [3] simply, *i.e.* a congruence of order 2; and since, of the twelve lines through A which lie on any monoid of the system, five meet C^{11}, the other seven represent lines of the congruence, *i.e.* the latter is of class 7. It can, in fact, be easily verified that when the general monoid of the system is represented on a plane by quartics with twelve simple base points, the trace on it of another monoid with the same vertex is represented by a septimic passing simply through the base points; so that when we subtract the base curve C^{11} (represented by a quartic through five of the base points) we obtain as representative of the characteristic system the net of cubics through the remaining seven, which is what we want.

It does not appear easy to prove that the congruence generating the double [3] thus represented is the most general of its order and class; from the simplicity of the representation here found, however, it appears somewhat improbable that it is a special case of something else; and as we shall now proceed to prove, the locus thus represented has the same arrangement of singularities as in the general case.

The branch surface of the (2, 7) congruence is a quartic symmetroid with an eleventh node*. As is known†, the tangent cone to a symmetroid from one of its ten nodes breaks up into two cubic cones, whose nine common generators are the lines joining that node to the other nine. If the surface has an eleventh node, one of these two cubic cones has a double generator passing through the extra node, and is accordingly rational. The generators of this cone are the lines of the congruence through the node, and the cone is one of the ruled cubics in the congruence. The ten rational cubic cones so arising are, of course, the only ruled cubics in the congruence which are cones; the rest have double directrices passing through the eleventh node, and there is just one ruled cubic having a given line through this point as double line. The tangent cone to the locus from the eleventh node is a sextic with ten double generators, each joining it to one of the other ten nodes; this is the only ruled sextic in the congruence which is a cone. Now the condition for the ruled cubic represented by a plane through the vertex A to be a cone is that the four simple base points in it are collinear, *i.e.* that the plane is the join of A to a quadrisecant line of the base curve C^{11}; and the condition for the ruled sextic represented by a quadric cone of vertex A to be a cone is that the eight simple base points in it are coplanar, *i.e.* that the quadric cone projects from A an eight-secant conic of the base curve. We therefore require to prove that C^{11} has ten quadrisecant lines and one eight-secant conic. The latter is easy; for the complete series, including the prime sections of C^{11}, is a g^8_{11}, which can just be made to contain a bicanonical set, or the eight points in which C^{11} is met by a quadric cone of vertex A, together with a fixed residual set of three points. These three points on C^{11} can be joined by a plane which will meet the curve further in a bicanonical set, that is in eight points of the curve which lie on a conic. As for the quadrisecant lines, there is a formula due to Welchman‡ which gives their number as 10. It is thus clear that the scheme of singularities on the double [3] represented by this system of monoids is the same as on

* C. M. Jessop, *The line complex* (1903), Ch. XV.

† C. M. Jessop, *Quartic surfaces* (1916), Art. 101.

‡ W. G. Welchman, *Proc. Camb. Phil. Soc.*, 28 (1932), 206. (But see the note below.)

that generated by the general (2, 7) congruence. We can accordingly proceed, with little fear of error, on the assumption that what we have here is the general case; though it must be noted that this is a weak spot in the proof.

If now we try to construct the representative of the surface Φ, the argument is the same as in the previous case. It must trace on each of the planes which represent ruled cubics a septimic curve going four times through A and twice through each of the four simple base points; that is, it must be a septimic surface with a fourfold point at A, and having C^{11} as double curve, which is clearly impossible since the tangent cone at A must be quartic and have seven double generators, touching the seven branches of C^{11} there. We conclude that there is no triple plane with 14-ic branch curve which has $p_a = -1$.

Case VIII. $s = 9,\ s' = 8,\ s'' = 6.$ (*Impossible.*)

A system of lines in a plane whose Grassmannian is a (non-normal) del Pezzo surface of order 9 (*i.e.* a ninefold plane suitably branching) is the ninefold plane of lines, *i.e.* every line in the plane counts as nine distinct lines of the system, save the tangents of a certain curve (dual to the branch curve of the ninefold plane) which count as seven distinct and two coincident lines, while its inflexional tangents count as six distinct and three coincident lines. The lines lying in a prime are, of course, nine in number ($s = 9$), and are represented on the plane on which the Grassmannian is mapped by nine associated points, giving $s' = 8$ and again, of course, $s'' = 6$. This might be expected to give us a surface Φ having $p_g = 0$, $p_a = -2$, that is one referable to a ruled surface of genus 2. We shall find, however, that such a surface is impossible.

If a multiple plane is equivalent to a ruled surface of genus p, then the generators of the ruled surface appear on the plane as curves of order n say, which trace on the prime sections of the multiple plane an involution γ_n^1 of genus p. We have accordingly to consider the possibility that the prime section of Σ, a curve of genus 5 having a g_3^1, shall have also a γ_n^1 of genus 2. Now if a curve of genus π has a γ_n^1 of genus p with δ united points, then a canonical set of sets in the γ_n^1, with the set of united points, forms a canonical set on the curve, so that

$$\delta = 2\pi - 2 - n(2p-2)^*. \qquad \text{(i)}$$

Putting $\pi = 5,\ p = 2$, we see that the only possibilities are $n = 2,\ \delta = 4$, or $n = 3,\ \delta = 2$, or $n = 4,\ \delta = 0$. We have thus to prove that the presence on

* G. Castelnuovo, *Rend. Acc. Lincei* (4), 7 (1891), 294.

a curve of genus 5 of a γ_2^1, γ_3^1, or γ_4^1 of genus 2 is incompatible with that of a g_3^1.

Now if a curve of order ν and genus π has a γ_n^1 of genus p, the $[n-1]$'s which join the sets generate a scroll of order N, where

$$\nu-\pi = N-1-n(p-1)^*. \qquad \text{(ii)}$$

Thus, in the present case, we have to consider a ruled surface of order 6, a scroll of planes of order 7, or a developable of [3]'s which at first would seem to be of order 8, in each case of genus 2, on which the canonical curve of genus 5 may lie. The first of these is familiar†; it is the section of a quadric cone (with point vertex) in [4] by a quartic primal, residual to two planes of one system, having further a double line in a plane of this system. The traces of the planes of this system are quartics passing twice through the vertex of the cone (which is a double point of the surface) and are directrices of the surface; one such quartic consists of the double line with the two generators through the double point. The planes of the other system trace canonical pairs of generators, meeting on the double line. The canonical curve of genus 5 on the ruled surface meets each generator twice, and each of the directrix quartics four times, the four double points of its γ_2^1 being on one of the quartics. From the vertex of the quadric cone on which the ruled surface lies it projects into a double quadric branching along six generators of one system; and the curve projects into a quartic section of the quadric, with five double points (apparent, of course, on the two sheeted surface). On the other hand, if the canonical curve of genus 5 having a g_3^1 is projected from the vertex of any quadric cone containing it, the ruled cubic on which it lies becomes a quadric, and the curve projects into one which meets every generator of one system of the quadric in three points, and every generator of the other in five. It is clear that the two curves thus defined can have no common special case, since in either case if the quadric becomes a cone the figure is quite degenerate.

The septimic scroll of planes of genus 2 has a fivefold directrix plane, since a canonical pair of planes forms part of a prime section, and the canonical series of pairs of planes forms the variable part of a pencil of prime sections, whose fixed part is accordingly a fivefold plane. It is easily verified that the rest of the singular surface of the scroll is a double ruled cubic, meeting the fivefold plane in a conic; its generators are the intersections of the canonical pairs of planes, and its directrix is the minimum directrix of the scroll; but this is not to our purpose. The intersection of

* C. Segre, *Encyc. Math. Wiss.* III, C. 7, 955.

† W. L. Edge, *Ruled surfaces* (1931), 253.

the scroll with a ruled cubic is a curve of order 21, with three fivefold points in the fivefold plane; consider the possibility that this curve may be constituted in part by the canonical curve of genus 5, which meets each plane of the scroll in three points and the fivefold plane in two points (the united points of the γ_3^1). The residual part will be of order 13 and will meet the fivefold plane in a fivefold and two fourfold points; moreover, it will not meet the general plane of the scroll at all, *i.e.* it will lie entirely in a finite number of planes of the scroll. Now the only plane curves on a ruled cubic are lines and conics; and the maximum number of these which can be got to pass through three points are four lines (a generator through each of the points and the directrix through one of them) and three conics (those joining the points by pairs); these only make up a curve of order 10; from which we conclude that the canonical curve of genus 5 lying on the scroll cannot lie also on a ruled cubic.

With regard to the possibility of a γ_4^1 of genus 2, it is as easy to treat the general case of a $\gamma_{\pi-1}^1$ of genus 2 on the canonical curve of genus π. It is without united points by the formula (i) above; and a canonical pair of its sets makes a canonical set on the curve; there is a $g_{2\pi-2}^1$ in the canonical series composed of such pairs of sets. This means that the developable of primes joining the sets of the $\gamma_{\pi-1}^1$ consists of each prime of a pencil doubled, six of them, of course, being united, or that the scroll generates double $[\pi-1]$ branching along six primes of a pencil. But now from the formula (ii) we see that the virtual genus of the curve is not π but $3\pi-4$, which can only be reduced to π by the presence of $2\pi-4$ double points or their equivalent. But a canonical curve which has any singular points is hyperelliptic, and consists of a double normal rational $(\pi-1)$-ic with $2\pi+2$ branch points. Further, on a double space a double curve is to be regarded as having a double point effective in reducing genus wherever it touches the branch locus. We consequently find for our canonical curve in this case a double $(\pi-1)$-ic, whose intersections, with the six branch primes, are made up of $2\pi-4$ contacts and $2\pi+2$ proper branch points. To return to the case $\pi=5$, we note that the curve, being hyperelliptic, cannot have a g_3^1.

We are thus able to conclude that a triple plane with 14-ic branch curve cannot be birationally equivalent to a ruled surface of genus 2.

Case IX. $p_g=0,\ p_a=-3$. (*Impossible.*)

A priori, since a curve of genus 5 can have on it a γ_2^1 of genus 3, without united points, we might suppose that our triple plane might be generated by an ∞^1 of conics of genus 3, *i.e.* might be referable to a ruled surface of genus 3.

Since in this case the system $|C'|$ would only be a pencil, the surface Φ would be a line on which the mapping of Σ would be $(\infty, 1)$ and in which all the trisecants would coincide. The consideration of the trisecants is thus quite useless in this case. We can, however, easily prove that such a surface as we contemplate is impossible by showing, along the same lines as in the last section, that the presence of a γ_2^1 of genus 3 on a curve of genus 5 is incompatible with the presence of a g_3^1.

The joins of the pairs in the γ_2^1 on the canonical model of the curve generate an octavic ruled surface*, having on it one plane quartic as directrix, and one fourfold linear directrix, the four generators through any point of the latter belonging to a definite *semibicanonical* g_4^1 on the construct of genus 3. The tetrads of generators through any two points of the fourfold line, together with the curve of genus 5, form a quadric section of the surface; and the fourfold line and the plane of the directrix quartic are pole and polar with regard to every one of these quadrics, which are all the quadrics through the curve. It follows from this that if any quadric through the curve is a cone, its vertex must lie either on the fourfold line or in the plane of the directrix quartic. This cannot be true of all the quadrics through a ruled cubic, so that the curve here considered cannot be on a ruled cubic, cannot have a g_3^1, and cannot represent a prime section of a triple plane.

We conclude that a triple plane with 14-ic branch curve is necessarily one of the cases we have called *I*, *II*, *III*, *IV*, *V*. The number of cusps on the branch curve is 36, 39, 42, 45, and 48 in these cases respectively.

Note.

The formula which I have quoted from Welchman for the number of quadrisecants of a curve in space which has a multiple point may be held to be not very firmly established, since it has not been verified over a wide field, and the degeneration method on which it is based is by many regarded with suspicion. It may be as well, therefore, to give an independent verification of the number 10 in the case here considered, which (though somewhat complicated in form) really illustrates a method which with a little ingenuity can be applied to a good many problems of this type, as well as providing a strong additional argument in favour of Welchman's formula, by showing that it gives the correct result in a case quite other than any to which it has hitherto been applied.

* W. G. Welchman, "Special scrolls and involutions in canonical curves", *Proc. London Math. Soc.* (In the press.)

The prime sections of C^{11}, since this is of genus 3, belong to a complete g_{11}^8; let the sets of this be mapped by points of [8], and consider the locus representing sets which break up into a canonical set and a set of the residual g_7^4 (which is the complete series determined by the set of seven points which coincide in the sevenfold point of C^{11}). This locus is generated simply by ∞^2 [4]'s, each corresponding to the combination of a fixed canonical set with a variable set of g_7^4, and also by ∞^4 planes, each corresponding to the combination of a fixed heptad and a variable canonical set. This means that the locus is a case of what Room* calls $)|3, 5|, [8](,$ and is a projection of $(|3, 5|_1, [14])$ from a suitably chosen [5]. This is so chosen that a particular $(|3, 3|_1, [8])$ on the locus goes into a double cubic symmetroid in [5], whose points represent sets consisting of two canonical sets and a fixed triad; the points of the Veronese surface which is double on this symmetroid are pinch points of the whole locus and represent sets containing a canonical set counted twice; and points of [5] off the symmetroid represent sets containing a bicanonical set with the fixed triad; but all this does not concern us. The plane sections of the actual C^{11} in [3] are represented by a [3] in [8], containing a plane of the locus, since every canonical set, with the seven places at the multiple point, is a plane section. Consider the residual section of the locus by this [3]; it consists of lines each lying in a generating [4], since if the [3] contains a point of any [4], outside the plane, it contains the line joining that point to the point in which the plane meets the [4]. Each such line meets in a point any $)|3, 4|, [7]($ of the locus which traces on each generating [4] a [3] not meeting the plane. But $)|3, 4|, [7]($ is of order 10. It is, in fact, a projection of $)|3, 4|, [11](= (|3, 4|_1, [11])$ whose surface section $(|3, 4|_1, [8])$ is represented on a plane by quartics through six points. Hence the number of lines is 10; and since each such line clearly represents a pencil of plane sections of C^{11} containing a fixed canonical set, this is the number of quadrisecant lines to the curve.

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* T. G. Room, *Proc. Lond. Math. Soc.* (2), 36 (1933–4), 1. (In describing the locus I have, of course, used Room's notation of reversed brackets; but in view of its inelegance it is to be hoped that some more satisfactory notation will be devised for future work on this subject.)

ASYMPTOTIC GENERALIZATIONS OF WARING'S THEOREM*

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Introduction.

For every integral exponent $k \geqslant 3$ it is proposed to determine what number s of variables and what sets of positive integral coefficients

* A dissertation submitted for the degree of Ph.D. in the University of Chicago. The writer wishes to acknowledge his indebtedness to Prof. L. E. Dickson, who supervised his work as a graduate and who suggested this investigation.

$a_1, \ldots, a_s$ ensure a solution of the diophantine equation

$$n = \sum_{\nu=1}^{s} a_\nu h_\nu^k \quad (h_\nu \geqslant 0)$$

for every integer $n > b_1$, where b_1 depends only on k, s, and max a_ν.

For $a_1 = \ldots = a_s = 1$, the problem reduces to the classical Waring problem in asymptotic form. In this case the existence for every k of a sufficient s was proved by Hilbert* and the explicit value $s \geqslant (k-2)2^{k-1}+5$ established by Hardy and Littlewood†. Our first aim will be to prove that this same value of s, if coupled with the congruential conditions of Theorem 2, is likewise sufficient in the more general problem. We shall then develop in the sections on primitivity and base sets important tests for determining when the hypotheses of Theorem 2 are satisfied. Finally we shall exhibit the results obtainable for the specific cases $k = 3$, 4, and 5.

The method of proof throughout the chain of lemmas culminating in Theorems 1 and 2 follows as closely as possible that of Landau's presentation of the Hardy-Littlewood theory in his *Vorlesungen über Zahlentheorie* (hereafter quoted as Landau), erster Band, sechster Teil, Kapitel 1-2. In fact, Theorems 1 and 2 correspond respectively to what Landau calls the First and Second Hardy-Littlewood theorems. The principal ideas in the sections on primitivity, base sets, and derived coefficients have been developed by the writer.

Notation.

The notation for the four series of positive constants c_i, C_i, b_i, B_i denotes functional dependence as follows:

$$c(k, \max a_\nu), \quad C(k, \epsilon, \max a_\nu),$$

$$b(k, s, \max a_\nu), \quad B(k, s, \epsilon, \max a_\nu).$$

$$a = 1/k, \quad K = 2^{k-1}, \quad \kappa = 1 - 1/K.$$

$$f_\nu(x) = \sum_{h=0}^{\infty} x^{a_\nu h^k}.$$

$$r(n) = \frac{1}{2\pi i} \int \prod_{\nu=1}^{s} f_\nu(x) \frac{dx}{x^{n+1}}.$$

Major arcs have $q \leqslant n^\alpha$.

Minor arcs have $n^\alpha < q \leqslant n^{1-\alpha}$.

$$S_{\nu\rho}(x) = \sum_{h} \rho^{a_\nu h^k},$$

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* D. Hilbert, *Göttinger Nachrichten* (1909), 17; *Math. Annalen*, 67 (1909), 281.

† G. H. Hardy and J. E. Littlewood, *Math. Zeitschrift*, 12 (1922), 161.

where ρ is a primitive q-th root of unity and h ranges over a complete set of residues mod q.

$$A_0(q) = q^{-s} \sum_{\rho(q)} \prod_{\nu=1}^{s} S_{\nu\rho}\rho^{-n}.$$

$$\mathfrak{S}_0 = \sum_{q=1}^{\infty} A_0(q).$$

$$\psi_{\nu\rho}(x) = \frac{\Gamma(1+a)}{a_\nu{}^a} \frac{S_{\nu\rho}}{q} \frac{1}{(1-X)^a},$$

$$\chi_{0p} = 1 + A_0(p) + \ldots = \sum_{l=0}^{\infty} A_0(p^l),\ p \text{ prime}.$$

$$(a, b) = \text{g.c.d. } (a, b).$$

For $N_{sp}(p^l, n)$ and for P, see § 9.

CHAPTER I.

The first Hardy-Littlewood theorem extended.

1. *The integral expression for $r(n)$.* Let x be a complex variable and let

$$f_\nu(x) = \sum_{h=0}^{\infty} x^{a_\nu h^k} \quad (\nu = 1, \ldots, s).$$

The functions $f_\nu(x)$ and $\prod_{\nu=1}^{s} f_\nu(x)$ are regular for $|x| < 1$. If

$$r(n) = r(n, k, s, a_1, \ldots, a_s)$$

denotes the number of solutions in non-negative integers of

$$n = \sum_{\nu=1}^{s} a_\nu h_\nu{}^k, \tag{1}$$

then

$$\prod_{\nu=1}^{s} f_\nu(x) = \sum_{n=0}^{\infty} r(n)x^n,$$

and, by Cauchy's integral formula,

$$r(n) = \frac{1}{2\pi i} \int \prod_{\nu=1}^{s} f_\nu(x) \frac{dx}{x^{n+1}}, \tag{2}$$

where the integral is taken in the positive direction round the circle $|x| = e^{-1/n}$. We seek sufficient conditions to be imposed on s and the coefficients a_ν to ensure that $r(n) > 0$ for every integer n sufficiently large. Henceforth let $n \geqslant 2$.

2. *Major and minor arcs on the circle of integration.* With Hardy-Littlewood and Landau let $\alpha = c_8$ satisfy $0 < \alpha < a$, its precise value being

fixed by (31′) below. Consider on the interval $0 \leqslant l/q \leqslant 1$ those Farey fractions l/q which have $q \leqslant n^{1-\alpha}$. The corresponding $\rho = e^{2\pi i l/q}$ are then all primitive q-th roots of unity of index $q \leqslant n^{1-\alpha}$ with 1 occurring twice. Construct for each neighbouring pair, l/q and l'/q', the point $M = (l+l')/(q+q')$ and subdivide the circle of integration into subarcs with end-points $x_M = e^{-1/n} e^{2\pi i M}$. Each subarc, since it contains exactly one of the points $e^{-1/n} e^{2\pi i l/q}$, is associated with a certain q-th root of unity ρ. If this $q \leqslant n^{\alpha}$, the subarc is a *major arc*; if $n^{\alpha} < q \leqslant n^{1-\alpha}$, it is a *minor arc*. The equations of both of these arcs may be written*

$$x = \rho X, \quad X = e^{-y}, \quad y = (1/n) - \theta i, \quad -\theta_1 \leqslant \theta \leqslant \theta_2, \tag{3}$$

where

$$\frac{\pi}{qn^{1-\alpha}} \leqslant \theta_1 \leqslant \frac{2\pi}{qn^{1-\alpha}}, \qquad \frac{\pi}{qn^{1-\alpha}} \leqslant \theta_2 \leqslant \frac{2\pi}{qn^{1-\alpha}}; \tag{4}$$

$$\theta_1 \leqslant \pi, \quad \theta_2 \leqslant \pi.$$

Landau proves in Satz 258 that, on every subarc,

$$|1-X| \geqslant c\sqrt{(n^{-2}+\theta^2)} > c\,|\theta|, \quad \frac{|1-X|}{1-|X|} \leqslant 1 + n|\theta|, \tag{5}$$

where $c > 0$ is an absolute constant; that, on every minor arc,

$$\frac{|1-X|}{1-|X|} < 8; \tag{6}$$

and that, on every major arc,

$$\frac{|1-X|}{1-|X|} < 8n^{\alpha}/q. \tag{7}$$

For every n there will be a Farey subdivision and for each subdivision and on each subarc we shall find approximating functions $\psi_{\nu\rho}(x)$ for the $f_\nu(x)$. Since these require different treatments on major and on minor arcs, we rewrite (2) in the form

$$r(n) = \frac{1}{2\pi i}\left(\sum_{\substack{\text{Major}\\ \text{arcs}}} \int_{Mj} \prod_{\nu=1}^{s} f_\nu(x) \frac{dx}{x^{n+1}} + \sum_{\substack{\text{Minor}\\ \text{arcs}}} \int_{Mi} \prod_{\nu=1}^{s} f_\nu(x) \frac{dx}{x^{n+1}} \right). \tag{8}$$

3. *Lemmas for S, $S_{\nu\rho}$, $A_0(q)$, and $\mathfrak{S}_0$.* Before introducing the $\psi_{\nu\rho}(x)$ we must establish a chain of lemmas.

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* For these equations see Landau, 244; for details of the Farey subdivision, *ibid.*, 243–244, 98–100.

LEMMA 1. *Let ρ be any primitive q-th root of unity, $m>0$, let r be an integer, and let*

$$S(a_\nu, \rho)=\sum_{h=r+1}^{r+m}\rho^{a_\nu h^k}.$$

Then, for every $\epsilon>0$,

$$|S(a_\nu, \rho)|^K<C_{18}m^\epsilon q^\epsilon(m^{K-1}+m^K/q+m^{K-k}q).$$

For $a_\nu=1$ the lemma becomes Landau's Satz 267. Denote by C'_{18} the value of C_{18} which serves here.

For $a_\nu>1$ let $g=(a_\nu, q)$. Note that $1\leqslant g\leqslant a_\nu\leqslant\max a_\nu$ and that $\rho^{a_\nu}=\rho'$, a primitive (q/g)-th root of unity. Hence, by the above case,

$$\begin{aligned}|S(a_\nu, \rho)|^K=|S(1, \rho')|^K&<C'_{18}m^\epsilon(q/g)^\epsilon(m^{K-1}+m^Kg/q+m^{K-k}q/g)\\&\leqslant C'_{18}m^\epsilon q^\epsilon(m^{K-1}+m^Kg/q+m^{K-k}q)\\&\leqslant C'_{18}g\,m^\epsilon q^\epsilon(m^{K-1}+m^K/q+m^{K-k}q)\\&\leqslant C_{18}m^\epsilon q^\epsilon(m^{K-1}+m^K/q+m^{K-k}q)\end{aligned}$$

if $C'_{18}\max a_\nu=C_{18}$.

We obtain proofs of Lemmas 2–5 below by making verbal changes in the proofs of their analogues, Satz 268–Satz 271 in Landau; we replace S_ρ by $S_{\nu\rho}$, $A(q)$ by $A_0(q)$, $\mathfrak{S}$ by $\mathfrak{S}_0$, use Lemma 1 instead of Satz 267, and note that here the constants depend on $\max a_\nu$.

LEMMA 2. *If, in the hypothesis of Lemma 1, $m\leqslant q$, then*

$$|S(a_\nu, \rho)|<C_2q^{\kappa+\epsilon}.$$

LEMMA 3. *Let*

$$S_{\nu\rho}(x)=\sum_h\rho^{a_\nu h^k},$$

where ρ is a primitive q-th root of unity and h ranges over a complete set of residues mod q. *Then, for every $\epsilon>0$,*

$$|S_{\nu\rho}|<C_2q^{\kappa+\epsilon}.$$

LEMMA 4. *If*

$$A_0(q)=q^{-s}\sum_{\rho\,(q)}\prod_{\nu=1}^{s}S_{\nu\rho}\rho^{-n},$$

then, for every integral $q>0$, every $\epsilon>0$, and every n,

$$|A_0(q)|<B_1q^{1-(s/K)+\epsilon}.$$

LEMMA 5. *If $s>2K$, the singular series*

$$\mathfrak{S}_0=\sum_{q=1}^{\infty}A_0(q)$$

converges absolutely for every n.

4. *Two lemmas for* $f_\nu(x)$.

LEMMA 6. *If* $\omega = [n^{a+\frac{1}{2}\epsilon}]$, *then, on the entire circle* $|x| = e^{-1/n}$, *and for every* $\epsilon > 0$,

$$\left| f_\nu(x) - \sum_{h=0}^{\omega} x^{a_\nu h^k} \right| < C_{33}.$$

The proof is as follows:

$$\left| f_\nu(x) - \sum_{h=0}^{\omega} x^{a_\nu h^k} \right| = \left| \sum_{h=\omega+1}^{\infty} x^{a_\nu h^k} \right| \leqslant \sum_{h=\omega+1}^{\infty} |x|^{a_\nu h^k}$$

$$= \sum_{h=\omega+1}^{\infty} (e^{-1/n})^{a_\nu h^k} \leqslant \sum_{h=\omega+1}^{\infty} e^{-h^k/n} < C_{33},$$

the last inequality being obtained in the course of the proof of Satz 274 in Landau.

LEMMA 7. *Let* ρ *be an arbitrary root of unity,* $j \geqslant 0$ *an integer,* $x = \rho X$, $\omega = [n^{a+\frac{1}{2}\epsilon}]$, *and*

$$\tau_\nu(j) = \sum_{h=0}^{[j^n/a_\nu a]} \rho^{a_\nu h^k} = \sum_{0 \leqslant a_\nu h^k \leqslant j}^{h \geqslant 0} \rho^{a_\nu h^k};$$

then, on the entire circle $x = e^{-1/n}$,

$$f_\nu(x) = (1-X) \sum_{j=0}^{\infty} \tau_\nu(j) X^j, \tag{9}$$

and

$$\left| f_\nu(x) - (1-X) \sum_{j=0}^{a_\nu \omega^k} \tau_\nu(j) X^j - \tau_\nu(a_\nu \omega^k) X^{a_\nu \omega^k + 1} \right| < C_{33}. \tag{10}$$

If $\tau_\nu(-1)$ is defined as zero, we have

$$f_\nu(x) = \sum_{h=0}^{\infty} \rho^{a_\nu h^k} X^{a_\nu h^k} = \sum_{j=0}^{\infty} [\tau_\nu(j) - \tau_\nu(j-1)] X^j,$$

$$f_\nu(x) = \sum_{j=0}^{\infty} \tau_\nu(j)(X^j - X^{j+1}) = (1-X) \sum_{j=0}^{\infty} \tau_\nu(j) X^j,$$

which proves equation (9).

Secondly,

$$\sum_{h=0}^{\omega} x^{a_\nu h^k} = \sum_{h=0}^{\omega} \rho^{a_\nu h^k} X^{a_\nu h^k} = \sum_{j=0}^{a_\nu \omega^k} [\tau_\nu(j) - \tau_\nu(j-1)] X^j$$

$$= \sum_{j=0}^{a_\nu \omega^k} \tau_\nu(j)(X^j - X^{j+1}) + \tau_\nu(a_\nu \omega^k) X^{a_\nu \omega^k + 1}$$

$$= (1-X) \sum_{j=0}^{a_\nu \omega^k} \tau_\nu(j) X^j + \tau_\nu(a_\nu \omega^k) X^{a_\nu \omega^k + 1}.$$

Substitution of this value in Lemma 6 gives equation (10).

5. *On minor arcs, $f_\nu(x)$ approximates zero.*

LEMMA 8. *On all minor arcs, for every $\epsilon > 0$,*

$$|f_\nu(x)| < C_4 n^{a-(a/K)+\epsilon}.$$

The proof is a straightforward generalization of the proof of Satz 276 in Landau: use $m = [j^a/a_\nu{}^a]+1$ instead of $m = [j^a]+1$, Lemma 1 instead of Satz 267, the $\tau_\nu(j)$ of Lemma 7 instead of $\tau(j)$, $0 \leqslant j \leqslant a_\nu \omega^k$ instead of $0 \leqslant j \leqslant \omega^k$, and (10) instead of (302).

6. *On major arcs, $f_\nu(x)$ approximates $\psi_{\nu\rho}(x)$.* For use in the proof of Lemma 11 we quote Lemmas 9 and 10 from Landau, where they are proved as Satz 277 and Satz 273, respectively.

LEMMA 9. *For $\beta > 0$ and integral $j > 0$,*

$$\left| \frac{\Gamma(\beta+1+j)}{j!} - j^\beta \right| < \gamma(\beta) j^{\beta-1},$$

where $\gamma = \gamma(\beta)$ depends only on β and not on j.

LEMMA 10. *Let ρ be a primitive q-th root of unity,*

$$q \leqslant n^a, \quad \epsilon > 0, \quad 1 \leqslant m \leqslant n^{a+\epsilon}, \quad \theta \text{ real},$$

$$\frac{1}{n}\left(\frac{n^a}{q}\right)^{K/(K+1)} \leqslant |\theta| \leqslant \frac{\pi}{k!\, q m^{k-1}},$$

$$S = \sum_{h=1}^{m} (\rho e^{\theta i})^{h^k};$$

then

$$|S| < C_{25} n^{aK/(K+1)+\epsilon}.$$

LEMMA 11. *On major arcs, for every $\epsilon > 0$,*

$$|f_\nu(x) - \psi_{\nu\rho}(x)| < C_5 n^{aK/(K+1)+\epsilon},$$

where $\psi_{\nu\rho}(x)$ is defined in equation (12) below.

The proof falls into two cases if we put $(a_\nu, q) = d$, $q = dQ$, and divide the arc $\rho X = \rho e^{-1/n} e^{i\theta}$, $-\theta_1 \leqslant \theta \leqslant \theta_2$, into two parts according as

$$\text{(I)} \quad a_\nu |\theta| < \frac{1}{n}\left(\frac{n^a}{Q}\right)^{K/(K+1)},$$

$$\text{(II)} \quad a_\nu |\theta| \geqslant \frac{1}{n}\left(\frac{n^a}{Q}\right)^{K/(K+1)}.$$

Case I. By Lemma 7,

$$f_\nu(x) = (1-X)\sum_{j=0}^{\infty}\tau_\nu(j)X^j, \quad \tau_\nu(j) = \sum_{h=0}^{[j^a/a_\nu{}^a]}\rho^{a_\nu h^k}. \tag{9}$$

For every $j \geqslant 0$, $\tau_\nu(j)$ falls into $[(j/a_\nu)^a/q]$ partial sums of q terms each plus $[j^a/a_\nu{}^a]+1-[j^a/a_\nu{}^a q]q \leqslant q$ consecutive end-terms. Each of the partial sums is equal to $S_{\nu\rho}$. Lemma 2 applied to the end-terms yields

$$|\tau_\nu(j)-[j^a/a_\nu{}^a q]\,S_{\nu\rho}| < C_2 q^{\kappa+\epsilon},$$

$$|\tau_\nu(j)-(j^a/a_\nu{}^a q)\,S_{\nu\rho}| < C_2 q^{\kappa+\epsilon}+|S_{\nu\rho}|.$$

Since Lemma 3 ensures that $|S_{\nu\rho}| < C_2 q^{\kappa+\epsilon}$, this becomes

$$|\tau_\nu(j)-(j^a/a_\nu{}^a q)\,S_{\nu\rho}| < 2C_2 q^{\kappa+\epsilon} < 2C_2 q^\kappa n^\epsilon. \tag{11}$$

By definition,

$$\psi_{\nu\rho}(x) = (1-X)\frac{\Gamma(1+a)}{a_\nu{}^a}\frac{S_{\nu\rho}}{q}(1-X)^{-a-1} = (1-X)\sum_{j=0}^{\infty}v_\nu(j)X^j, \tag{12}$$

where

$$v_\nu(j) = \frac{\Gamma(1+a)}{a_\nu{}^a}\frac{S_{\nu\rho}}{q}\binom{-a-1}{j}(-1)^j = \frac{S_{\nu\rho}}{a_\nu{}^a q}\Gamma(1+a)\frac{(a+1)\ldots(a+j)}{j\,!},$$

$$v_\nu(j) = \frac{S_{\nu\rho}}{a_\nu{}^a q}\frac{\Gamma(a+1+j)}{j\,!}. \tag{13}$$

By Lemma 9, for $j > 0$,

$$\left|\frac{\Gamma(a+1+j)}{j\,!}-j^a\right| < c_{20}j^{a-1} \leqslant c_{20};$$

hence, for $j \geqslant 0$,

$$\left|\frac{\Gamma(a+1+j)}{j\,!}-j^a\right| < c_{21}. \tag{14}$$

By (13), (14), and the fact that $|S_{\nu\rho}| \leqslant q$, we have, for $j \geqslant 0$,

$$\left|v_\nu(j)-\frac{j^a S_{\nu\rho}}{a_\nu{}^a q}\right| = \frac{1}{a_\nu{}^a}\left|\frac{S_{\nu\rho}}{q}\right|\cdot\left|\frac{\Gamma(a+1+j)}{j\,!}-j^a\right| < \frac{c_{21}}{a_\nu{}^a} \leqslant c_{21} < c_{21}q^\kappa n^\epsilon. \tag{15}$$

Now (11) and (15) give, for $j \geqslant 0$,

$$|\tau_\nu(j)-v_\nu(j)| < C_{39}q^\kappa n^\epsilon. \tag{16}$$

From (9), (12), (16), and the second equation of (5), it follows that

$$(17)\quad |f_\nu(x)-\psi_{\nu\rho}(x)|$$

$$= \left|(1-X)\sum_{j=0}^{\infty}[\tau_\nu(j)-v_\nu(j)]X^j\right| < |1-X|\,C_{39}\,q^\kappa n^\epsilon \sum_{j=0}^{\infty}|X|^j$$

$$= C_{39}q^\kappa n^\epsilon \frac{|1-X|}{1-|X|} \leqslant C_{39}q^\kappa n^\epsilon(1+n|\theta|) \leqslant C_{39}q^\kappa n^\epsilon(1+na_\nu|\theta|).$$

Since $Q \leqslant q \leqslant n^\alpha < n^\alpha$, condition (I) gives

$$1+na_\nu|\theta| < 1+\left(\frac{n^\alpha}{Q}\right)^{K/(K+1)} < 2\left(\frac{n^\alpha}{Q}\right)^{K/(K+1)}.$$

If we insert this in (17), substitute dQ for q, and use the relations $d \leqslant a_\nu \leqslant \max a_\nu$ and $Q \geqslant 1$, we obtain

$$|f_\nu(x)-\psi_{\nu\rho}(x)| < C_{39}q^\kappa n^\epsilon 2\left(\frac{n^\alpha}{Q}\right)^{K/(K+1)} = 2C_{39}n^{\alpha K/(K+1)+\epsilon}d^\kappa Q^\kappa Q^{-K/(K+1)}$$

$$\leqslant 2C_{39}n^{\alpha K/(K+1)+\epsilon}(\max a_\nu)^\kappa Q^{\kappa-K/(K+1)} \leqslant C_{40}n^{\alpha K/(K+1)+\epsilon},$$

since the exponent of Q is negative.

Case II. Without loss of generality we may take $\epsilon < (a-\alpha)/2k$, $\omega = [n^{\alpha+\frac{1}{2}\epsilon}]$, and redefine

$$\tau_\nu(j) = \sum_{h\geqslant 0}^{0\leqslant a_\nu h^k \leqslant j} \rho^{a_\nu h^k} e^{i\theta a_\nu h^k} = \sum_{h=0}^{[j^\alpha/a_\nu{}^\alpha]} \rho^{a_\nu h^k} e^{i\theta a_\nu h^k}.$$

If $a_\nu > j$,

$$(18)\qquad \tau_\nu(j) = 1.$$

If $a_\nu \leqslant j \leqslant a_\nu\omega^k$, then $1 \leqslant j^\alpha/a_\nu{}^\alpha \leqslant \omega$. Since $\rho^{a_\nu} = \rho_1$ is a primitive Q-*th* root of unity,

$$\tau_\nu(j)-1 = \sum_{h=1}^{[j^\alpha/a_\nu{}^\alpha]} \rho^{a_\nu h^k} e^{ia_\nu\theta h^k} = \sum_{h=1}^{[j^\alpha/a_\nu{}^\alpha]} (\rho_1 e^{a_\nu\theta i})^{h^k},$$

an expression which may be taken as the S of Lemma 10 with q replaced by Q, ρ by ρ_1, θ by $\theta' = a_\nu\theta$, m by $m' = [j^\alpha/a_\nu{}^\alpha]$. We verify the hypotheses of Lemma 10, finding that $Q \leqslant q \leqslant n^\alpha$, $1 \leqslant m' \leqslant j^\alpha/a_\nu{}^\alpha \leqslant \omega \leqslant n^{\alpha+\frac{1}{2}\epsilon} < n^{\alpha+\epsilon}$, and, by Case II,

$$\frac{1}{n}\left(\frac{n^\alpha}{Q}\right)^{K/(K+1)} \leqslant |a_\nu\theta| = |\theta'|.$$

The only hypothesis yet unverified is

$$|a_\nu \theta| \leqslant \frac{\pi}{k!\, Qm^{k-1}}.$$

This holds if $n > c_{24} = (k!\, 2 \max a_\nu)^{2/(a-\alpha)}$; for, with our ϵ, $n > c_{24}$ implies that $n^{a-\alpha-k\epsilon} > k!\, 2 \max a_\nu$, and this coupled with $|\theta| a_\nu < 2\pi a_\nu/(qn^{1-\alpha})$, which follows from (4), ensures that

$$\frac{\pi}{k!\, Qm^{k-1}} \geqslant \frac{\pi}{k!\, qm^{k-1}} \geqslant \frac{\pi}{k!\, qn^{(k-1)\alpha} n^{(k-1)\epsilon}} > \frac{\pi}{k!\, qn^{1-\alpha} n^{k\epsilon}} = \frac{2\pi a_\nu}{qn^{1-\alpha}} \frac{n^{a-\alpha-k\epsilon}}{k!\, 2a_\nu} > a_\nu |\theta|.$$

Hence, by Lemma 10, for $n > c_{24}$ and $a_\nu \leqslant j \leqslant a_\nu \omega^k$,

$$|\tau_\nu(j) - 1| < C_{25} n^{aK/(K+1)+\epsilon}. \tag{19}$$

For the subcase $n \leqslant c_{24}$, the proof of Lemma 11 is immediate:

$$\begin{aligned} |f_\nu(x) - \psi_{\nu\rho}(x)| &\leqslant |f_\nu(x)| + |\psi_{\nu\rho}(x)| \\ &\leqslant \sum_{h=0}^{\infty} e^{-a_\nu h^k/c_{24}} + a_\nu^{-\alpha}\, \Gamma(1+\alpha)(1-e^{-1/c_{24}})^{-\alpha} \\ &\leqslant c_{\nu 25} \leqslant \max c_{\nu 25} < C_{41} n^{aK/(K+1)+\epsilon}. \end{aligned} \tag{20}$$

For $n > c_{24}$ and $0 \leqslant j \leqslant a_\nu \omega^k$, we first derive, from (18) and (19),

$$|\tau_\nu(j)| < C_{42} n^{aK/(K+1)+\epsilon}. \tag{21}$$

Now by Lemma 6, with $R = e^{-1/n} = |x|$, we have

$$|f_\nu(x)| < C_{33} + \left| \sum_{h=0}^{\omega} x^{a_\nu h^k} \right| = C_{33} + \left| \sum_{h=0}^{\omega} \rho^{a_\nu h^k} e^{i a_\nu \theta h^k} R^{a_\nu h^k} \right|. \tag{22}$$

If we define $\tau_\nu(-1)$ as zero, we have

$$\begin{aligned} \sum_{h=0}^{\omega} \rho^{a_\nu h^k} e^{i a_\nu \theta h^k} R^{a_\nu h^k} &= \sum_{j=0}^{a_\nu \omega^k} [\tau_\nu(j) - \tau_\nu(j-1)]\, R^j \\ &= (1-R) \sum_{j=0}^{a_\nu \omega^k} \tau_\nu(j)\, R^j + \tau_\nu(a_\nu \omega^k)\, R^{a_\nu \omega^k + 1}; \end{aligned}$$

so that, by (21),

$$\begin{aligned} &\left| \sum_{h=0}^{\omega} \rho^{a_\nu h^k} e^{i a_\nu \theta h^k} R^{a_\nu h^k} \right| \\ &\quad < C_{42} n^{aK/(K+1)+\epsilon} \left\{ (1-R) \sum_{j=0}^{\infty} R^j + 1 \right\} < C_{43} n^{aK/(K+1)+\epsilon}. \end{aligned} \tag{23}$$

From (22) and (23) it follows that, for $n > c_{24}$,

$$|f_\nu(x)| < C_{44} n^{aK/(K+1)+\epsilon}. \tag{24}$$

By equation (5), $|1-X| \geqslant c|\theta|$. Hence Lemma 3 gives

$$(25)\quad |\psi_{\nu\rho}(x)| = \frac{\Gamma(1+a)}{a_\nu{}^a}\left|\frac{S_{\nu\rho}}{q}\right|\frac{1}{|1-X|^a} < \frac{\Gamma(1+a)}{1}\, C_2 q^{-1/K} q^{\epsilon}\, \frac{1}{(c|\theta|)^a}$$

$$< C'_{45} q^{-1/K} n^{\epsilon} \left\{ a_\nu n \left(\frac{Q}{n^a}\right)^{K/(K+1)} \right\}^a$$

$$\leqslant C'_{45} (\max a_\nu)^a q^{-1/K} n^{\epsilon} \left\{ n \left(\frac{Q}{n^a}\right)^{K/(K+1)} \right\}^a$$

$$< C''_{45} n^{aK/(K+1)+\epsilon},$$

where the last inequality is proved by Landau, 268.

Taken together, (24) and (25) yield, for $n > c_{24}$,

$$|f_\nu(x) - \psi_{\nu\rho}(x)| < C'_{44} n^{aK/(K+1)+\epsilon},$$

which with (20) completes the proof of Case II. Lemma 11 follows with $C_5 = C_{40} + C'_{44}$.

7. *The first Hardy-Littlewood theorem extended.* In order to avoid breaking the proof of Theorem 1, we first quote three lemmas from Landau and prove the supplementary Lemma 15.

LEMMA 12. (Landau, Satz 223.) *If $r > 0$ and if*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is regular for $|x| \leqslant r$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{\phi i})|^2 d\phi = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

LEMMA 13. (Landau, Satz 224.) *If $0 < r < 1$ and $\epsilon > 0$, then*

$$\sum_{m=1}^{\infty} m^{\epsilon} r^m < \frac{A}{(1-r)^{1+\epsilon}},$$

where A is positive and depends only on ϵ.

LEMMA 14. (Landau, Satz 262–1). *The number of solutions $R(m) = R(m, k)$ of the diophantine equation*

$$h_1{}^k + h_2{}^k = m \quad (h_1 \geqslant 0,\ h_2 \geqslant 0)$$

satisfies, for every $m > 0$ and every $\epsilon > 0$, the inequality

$$R(m) < C_{10} m^{\epsilon}.$$

LEMMA 15. (1) *The number of solutions* $R_\nu(m) = R_\nu(m, a_\nu, k)$ *of the diophantine equation*

$$a_\nu(h_1{}^k + h_2{}^k) = m \quad (h_1 \geqslant 0,\ h_2 \geqslant 0)$$

satisfies, for every $m > 0$ *and every* $\epsilon > 0$,

$$R_\nu(m) < C_{10}\, m^\epsilon.$$

(2) *For every* $m > 0$,

$$\sum_{j=0}^{m} R_\nu(j) \leqslant 4m^{2a}.$$

(3) *For* $m > 0$,

$$W_\nu(m) = \sum_{j=0}^{m} R_\nu{}^2(j) < C_{11}\, m^{2a+\epsilon}.$$

If $m \not\equiv 0 \pmod{a_\nu}$, $R_\nu(m) = 0$. If $m = m_\nu a_\nu$, then we have

$$R_\nu(m) = R(m_\nu) < C_{10}\, m_\nu{}^\epsilon \leqslant C_{10}\, m^\epsilon$$

by Lemma 14, proving (1).

Since j in (2) does not exceed m, $h_1 \leqslant m^a/a_\nu{}^a \leqslant m^a$ and $h_2 \leqslant m^a/a_\nu{}^a \leqslant m^a$. Hence the maximum number of values each can take is at most $1+m^a \leqslant 2m^a$, and the maximum number of pairs is at most $4m^{2a}$. This proves (2).

From (1) and (2), for $m > 0$, relation (3) follows thus:

$$\sum_{j=0}^{m} R_\nu{}^2(j) \leqslant \max_{0 \leqslant j \leqslant m} R_\nu(j) \sum_{j=0}^{m} R_\nu(j) < C_{14}\, m^\epsilon 4m^{2a} = C_{11}\, m^{2a+\epsilon}.$$

We are now ready to construct the proof of Theorem 1, the first to involve explicitly the $r(n)$ of equations (2) and (8) about which our investigation began. Henceforth, unless otherwise indicated, let Π indicate the product for $\nu = 1, \ldots, s$.

THEOREM 1. *For* $s > (k-2)\,K+4$,

$$\left| r(n) - \frac{\Gamma^s(1+a)}{\Gamma(sa)\,\Pi\, a_\nu{}^a}\, n^{sa-1}\, \mathfrak{S}_0 \right| < b_7\, n^{sa-1-c_{27}}. \tag{26}$$

This theorem is an extension for general positive integral coefficients of what Landau calls the first Hardy-Littlewood theorem.

Since, for $x = e^{-1/n} e^{i\theta}$, $|dx/x^{n+1}| = e\,|d\theta|$, equation (8) implies that

$$\left| r(n) - \frac{1}{2\pi i} \sum_{Mj} \int_{Mj} \Pi\, f_\nu(x)\, \frac{dx}{x^{n+1}} \right| = \left| \sum_{Mi} \int_{Mi} \Pi\, f_\nu(x)\, \frac{dx}{x^{n+1}} \right| \tag{27}$$

$$\leqslant \frac{e}{2\pi} \sum_{Mi} \int_{Mi} |\Pi\, f_\nu(x)|\, d\theta.$$

On minor arcs, ϵ/s instead of ϵ in Lemma 8 gives, for all values of ν,

$$|f_\nu(x)| < B_3 n^{\alpha-\alpha/K+\epsilon/s};$$

$$\int_{Mi}\Big|\prod_{\nu=1}^{s} f_\nu(x)\Big|d\theta \leqslant \Big(\max_{Mi}\Big|\prod_{\nu=5}^{s} f_\nu(x)\Big|\Big)\int_{Mi}\Big|\prod_{\nu=1}^{4} f_\nu(x)\Big|d\theta$$

$$< B_3^{s-4} n^{(s-4)(\alpha-\alpha/K)+\epsilon}\int_{Mi}\Big|\prod_{\nu=1}^{4} f_\nu(x)\Big|d\theta,$$

$$(28)\quad \sum_{Mi}\int_{Mi}\Big|\prod_{\nu=1}^{s} f_\nu(x)\Big|d\theta \leqslant B_7 n^{(s-4)(\alpha-\alpha/K)+\epsilon}\int_{-\pi}^{\pi}\Big|\prod_{\nu=1}^{4} f_\nu(e^{-1/n}e^{i\phi})\Big|d\phi$$

$$\leqslant B_7 n^{(s-4)(\alpha-\alpha/K)+\epsilon}\int_{-\pi}^{\pi}\sum_{\nu=1}^{4}|f_\nu^4(e^{-1/n}e^{i\phi})|\,d\phi.$$

For $R_\nu(m)$, as in Lemma 15,

$$f_\nu^2(x) = \sum_{m=0}^{\infty} R_\nu(m)\,x^m,$$

to which Lemma 12 is applicable with $r = e^{-1/n}$, $f(x)$ replaced by $f_\nu^2(x)$, $a_n x^n$ by $R_\nu(m)\,x^m$, and $W_\nu(m)$ as in Lemma 15, (3). Hence, if

$$W_\nu(0) = R_\nu^2(0) = 1,$$

$$\int_{-\pi}^{\pi}|f_\nu^4(e^{-1/n}e^{i\phi})|\,d\phi = 2\pi\sum_{m=0}^{\infty} R_\nu^2(m)e^{-2m/n}$$

$$= 2\pi\sum_{m=0}^{\infty}\{W_\nu(m)-W_\nu(m-1)\}e^{-2m/n}$$

$$= 2\pi\sum_{m=0}^{\infty} W_\nu(m)[e^{-2m/n}-e^{-2(m+1)/n}]$$

$$= 2\pi(1-e^{-2/n})\sum_{m=0}^{\infty} W_\nu(m)e^{-2m/n}.$$

By Lemma 15, for $m>0$, $W_\nu(m) < C_{11}m^{2\alpha+\epsilon}$. Hence, by Lemma 13, with ϵ replaced by $2\alpha+\epsilon$, r by $e^{-2/n}$, and A by C_{46},

$$(29)\quad \int_{-\pi}^{\pi}|f_\nu^4(e^{-1/n}e^{i\phi})\,d\phi < 2\pi(1-e^{-2/n})\{1+\sum_{m=1}^{\infty} C_{11}m^{2\alpha+\epsilon}(e^{-2/n})^m\}$$

$$< 2\pi(1-e^{-2/n})\{1+C_{46}(1-e^{-2/n})^{-1-2\alpha-\epsilon}\}$$

$$< C_{47}n^{2\alpha+\epsilon}.$$

From (29) and (28),

$$(30)\qquad \sum_{Mi}\int_{Mi}\Big|\prod_{\nu=1}^{s} f_\nu(x)\Big|d\theta < B_7\,4C_{47}\,n^{(s-4)(\alpha-\alpha/K)+2\alpha+2\epsilon}.$$

If we write $\frac{1}{2}\epsilon$ instead of ϵ and put $B_6 = 2eB_7C_{47}/\pi$, the inequality (30) reduces (27) to

$$(31) \qquad \left| r(n) - \frac{1}{2\pi i} \sum_{Mj} \int_{Mj} \Pi f_\nu(x) \frac{dx}{x^{n+1}} \right| < B_6 n^{(s-4)(a-a/K)+2a+\epsilon}.$$

Throughout the rest of this section let $s > (k-2)K+4$; for $k=3$ let $a = 17/60$; and for $k \geqslant 4$ let a satisfy

$$(31') \qquad (1-2a)K/a+4 = (k-2)K+4.5.$$

Then, as shown in Landau, 271 and 246, for a suitable $\epsilon = c_{29}$,

$$(s-4)(a-a/K)+2a+\epsilon \leqslant sa-1-c_{29},$$

so that (31) becomes

$$(32) \qquad \left| r(n) - \frac{1}{2\pi i} \sum_{Mj} \int_{Mj} \Pi f_\nu(x) \frac{dx}{x^{n+1}} \right| < b_8 n^{sa-1-c_{29}}.$$

Preparatory to replacing $f_\nu(x)$ by $\psi_{\nu\rho}(x)$, we examine

$$(33) \qquad \left| \sum_{Mj} \int_{Mj} \Pi f_\nu(x) \frac{dx}{x^{n+1}} - \sum_{Mj} \int_{Mj} \Pi \psi_{\nu\rho}(x) \frac{dx}{x^{n+1}} \right| \leqslant e \sum_{Mj} \int_{Mj} |\Pi f_\nu(x) - \Pi \psi_{\nu\rho}(x)| \, d\theta.$$

On major arcs Lemma 11, with ϵ/s instead of ϵ, gives, for every $\epsilon > 0$,

$$(34) \qquad |f_\nu(x) - \psi_{\nu\rho}(x)| < B_8 n^{aK/(K+1)+\epsilon/s} = \Psi.$$

By an identity obtained from Taylor's expansion and by (34),

$$(35) \qquad |\Pi f_\nu(x) - \Pi \psi_{\nu\rho}(x)| = \left| \sum_{\substack{\nu=1 \\ s \text{ terms}}}^{s} (f_\nu - \psi_\nu)\psi_\nu^{-1} \Pi \psi_j + \sum_{\substack{\nu \\ \binom{s}{2} \text{ terms}}} \sum_{\mu} (f_\nu - \psi_\nu)(f_\mu - \psi_\mu)\psi_\nu^{-1}\psi_\mu^{-1} \Pi \psi_j + \ldots \right|$$

$$< \Psi \sum_\nu |\psi_\nu^{-1} \Pi \psi_j| + \Psi^2 \sum_\nu \sum_\mu |\psi_\nu^{-1}\psi_\mu^{-1} \Pi \psi_j| + \ldots + \Psi^s,$$

where all products are taken for ν or j from 1 to s. Let T_ν be equal to the last member of (35) with every $\psi_{j\rho} \neq \psi_{\nu\rho}$ replaced by $\psi_{\nu\rho}$; *i.e.*

$$T_\nu = \Psi \left\{ s|\psi_\nu|^{s-1} + \binom{s}{2} \Psi |\psi_\nu|^{s-2} + \ldots + \binom{s}{s-1} \Psi^{s-2} |\psi_\nu| + \Psi^{s-1} \right\},$$

$$(36) \qquad T_\nu \leqslant \Psi 2^s \max(\Psi^{s-1}, |\psi_\nu|^{s-1}) \leqslant 2^s \{\Psi^s + \Psi |\psi_\nu|^{s-1}\},$$

since T_ν has 2^s terms. By (35) and (36), then (34),

$$|\Pi f_\nu(x) - \Pi \psi_{\nu\rho}(x)| < \Sigma T_\nu \leqslant s2^s\{B_8{}^s n^{aKs/(K+1)+\epsilon} + B_8 n^{aK/(K+1)+\epsilon}.|\psi_{\nu\rho}|^{s-1}\},$$

$$(37)\quad \sum_{Mj}\int_{Mj}|\Pi f_\nu(x) - \Pi \psi_{\nu\rho}(x)|\,d\theta$$

$$< B_9 n^{aKs/(K+1)+\epsilon}\sum_{Mj}\int_{Mj} d\theta + B_9 n^{aK/(K+1)+\epsilon}\sum_{Mj}\int_{Mj}|\psi_{\nu\rho}(x)|^{s-1}d\theta.$$

By Lemma 3, with $\epsilon/(s-1)$ instead of ϵ, and by the first of equations (5),

$$(38)\quad |\psi_{\nu\rho}(x)| = \frac{\Gamma(1+a)}{a_\nu{}^a}\left|\frac{S_{\nu\rho}}{q}\right||1-X|^{-a} < B_{10}q^{-1/K}q^{\epsilon/(s-1)}(n^{-2}+\theta^2)^{-\frac{1}{2}a},$$

which is the same upper bound as that found by Landau, 272, for $|\psi_\rho(x)|$. Hence, by the argument given there and on 273, for $s > (k-2)K+4$ and for a suitable $\epsilon = c_{30}$,

$$(39)\qquad \sum_{Mj}\int_{Mj}|\Pi f_\nu(x) - \Pi \psi_{\nu\rho}(x)|\,d\theta < b_9 n^{sa-1-c_{31}},$$

and, by (32) and (33),

$$(40)\qquad \left|r(n) - \frac{1}{2\pi i}\sum_{Mj}\int_{Mj}\Pi\psi_{\nu\rho}(x)\frac{dx}{x^{n+1}}\right| < b_{10}n^{sa-1-c_{32}}.$$

The rest of the proof of Theorem 1 is closely parallel to the work of Landau. If in the *Zweiter Schritt* we use Lemma 3 instead of (272), we conclude that the error introduced by integrating in (40) around the entire circle C instead of merely along major arcs is expressed, for $s > (k-2)K+4 > 2K$ and for an appropriate $\epsilon = c_{33}$, by

$$(41)\qquad \left|\sum_{Mj}\int_{Mj}\Pi\psi_{\nu\rho}(x)\frac{dx}{x^{n+1}} - \sum_{Mj}\int_{C}\Pi\psi_{\nu\rho}(x)\frac{dx}{x^{n+1}}\right| < b_{11}n^{sa-1-c_{34}};$$

so that, by (40) and (41),

$$(42)\qquad \left|r(n) - \frac{1}{2\pi i}\sum_{Mj}\int_{C}\Pi\psi_{\nu\rho}(x)\frac{dx}{x^{n+1}}\right| < b_{12}n^{sa-1-c_{35}}.$$

To evaluate the integrals of (42), we write

$$\Pi\psi_{\nu\rho}(x) = \frac{\Gamma^s(1+a)}{\Pi a_\nu{}^a}\frac{\Pi S_{\nu\rho}}{q^s}(1-\rho^{-1}x)^{-sa}$$

$$= \frac{\Gamma^s(1+a)}{\Pi a_\nu{}^a}\frac{\Pi S_{\nu\rho}}{q^s}\sum_{m=0}^{\infty}\frac{sa\ldots(sa+m-1)}{m!}\rho^{-m}x^m;$$

whence

$$\frac{1}{2\pi i}\int_C \Pi\,\psi_{\nu\rho}(x)\,\frac{dx}{x^{n+1}}$$

$$= \text{coef. of } x^n \text{ above} = \frac{\Gamma^s(1+a)}{\Pi\,a_\nu{}^a}\,\frac{\Pi\,S_{\nu\rho}}{q^s}\,\frac{sa\,\ldots\,(sa+n-1)}{n!}\,\rho^{-1}$$

$$= \frac{\Gamma^s(1+a)}{\Gamma(sa)\,\Pi\,a_\nu{}^a}\,\frac{\Pi\,S_{\nu\rho}}{q^s}\,\frac{\Gamma(sa+n)}{n!}\,\rho^{-n}.$$

Hence, if we use the notation

$$A_0(q) = q^{-s}\sum_{\rho\,(q)}\Pi\,S_{\nu\rho}\rho^{-n},$$

and remember that on major arcs $q \leqslant n^a$, (42) becomes

$$(43)\qquad \left| r(n) - \frac{\Gamma(sa+n)}{n!}\left(\frac{\Gamma^s(1+a)}{\Gamma(sa)\,\Pi\,a_\nu{}^a}\sum_{q\leqslant n^a} A_0(q)\right)\right| < b_{12}\,n^{sa-1-c_{35}}.$$

Next we propose to replace $\Gamma(sa+n)/n!$ in (43) by n^{sa-1}. By Lemma 9, with $\beta = sa-1$, we have

$$\left|\frac{\Gamma(sa+n)}{n!} - n^{sa-1}\right| < b_{13}\,n^{sa-2};$$

and hence, by Lemma 5 (*a*),

$$(44)\qquad \left|\frac{\Gamma(sa+n)}{n!}\left(\frac{\Gamma^s(1+a)}{\Gamma(sa)\,\Pi\,a_\nu{}^a}\sum_{q\leqslant n^a} A_0(q)\right) - n^{sa-1}\left(\frac{\Gamma^s(1+a)}{\Gamma(sa)\,\Pi\,a_\nu{}^a}\sum_{q\leqslant n^a} A_0(q)\right)\right|$$

$$< b_{14}\,n^{sa-2}\sum_{q=1}^{\infty} q^{-1-1/2K} = b_{15}\,n^{sa-2}.$$

Taken together, (43) and (44) yield

$$(45)\qquad \left| r(n) - n^{sa-1}\,\frac{\Gamma^s(1+a)}{\Gamma(sa)\,\Pi\,a_\nu{}^a}\sum_{q\leqslant n^a} A_0(q)\right| < b_{16}\,n^{sa-1-c_{36}}.$$

We now replace the sum in (45) by

$$\mathfrak{S}_0 = \sum_{q=1}^{\infty} A_0(q).$$

By Lemma 5 (*a*),

$$\left|\sum_{q>n^a} A_0(q)\right| \leqslant \sum_{q>n^a} b_6\,q^{-1-1/2K} < b_{17}\,n^{-c_{37}};$$

so that (45) becomes

$$\left| r(n) - \frac{\Gamma^s(1+a)}{\Gamma(sa)\,\Pi\, a_\nu{}^a}\, n^{sa-1}\, \mathfrak{S}_0 \right| < b_7\, n^{sa-1-c_{27}}, \tag{26}$$

completing the proof of Theorem 1.

The importance of Theorem 1 depends upon a positive sign for $\mathfrak{S}_0$, as will become evident at the end of Chapter II.

CHAPTER II.

The second Hardy-Littlewood theorem extended.

The aim of this chapter, which culminates in Theorem 2, is to find sufficient conditions to be imposed on s and on the coefficients a_ν to ensure that $\mathfrak{S}_0 > 0$.

8. *Expression of $\mathfrak{S}_0$ as a function of the χ_{0p}.* If we define

$$\chi_{0p} = 1 + A_0(p) + A_0(p^2) + \ldots = \sum_{l=0}^{\infty} A_0(p^l),\ p \text{ prime},$$

then a straightforward generalization of the proofs of Satz 280–Satz 284 in Landau, with $\mathfrak{S}$ replaced by $\mathfrak{S}_0$, $A(q)$ by $A_0(q)$, S_ρ by $S_{\nu\rho}$, and χ_p by χ_{0p}, establishes

LEMMA 16. *If for a fixed s the singular series $\mathfrak{S}_0$ converges absolutely for every n, then the series χ_{0p} converges absolutely for every n and*

$$\mathfrak{S}_0 = \prod_p \chi_{0p}.$$

9. *A lower bound for χ_{0p} for all p's.* Consider, for integral $l \geqslant 0$, the congruence

$$\sum_{\nu=1}^{s} a_\nu h_\nu{}^k \equiv n \pmod{p^l}, \quad 0 \leqslant h_\nu < p^l. \tag{46}$$

We read $p \dagger q$ as "p not a divisor of q" and define

$M_0(p^l, n)$ = total number of solutions of (46);

$N_0(p^l, n)$ = number of solutions of (46) with $p \dagger$ g.c.d. (h_ν);

$N_{sp}(p^l, n)$ = number of *special solutions* of (46); *i.e., the number of solutions having* $p \dagger$ g.c.d. $(a_\nu h_\nu)$.

With these definitions the methods of Landau's Satz 285 and Satz 286, applied to $A_0(p^l)$ and $M_0(p^l)$ instead of to $A(p^l)$ and $M(p^l)$, yield an easy proof of the following Lemma.

LEMMA 17. *For* $l \geqslant 0$,

$$\sum_{\lambda=0}^{l} A_0(p^\lambda) = p^{-(s-1)l} M_0(p^l).$$

We define four new symbols:

$$\Theta = \text{multiplicity of } p \text{ in } k.$$

$$k_0 = k/p^\Theta.$$

$$\gamma = \Theta+1 \text{ if } p > 2.$$

$$= \Theta+2 \text{ if } p = 2.$$

$$P = p^\gamma.$$

It follows, Landau, 282, that in all instances $\gamma \leqslant k$. Moreover, we have

LEMMA 18. (Landau, Satz 290.) *Let* $l \geqslant \gamma$; *let* x, y, z *be integers for which*

$$x = y + zp^{l-\Theta-1}.$$

Then

$$x^k \equiv y^k + k_0 y^{k-1} z p^{l-1} \pmod{p^l}.$$

LEMMA 19. *For* $l > \gamma$,

$$N_{sp}(p^l) = p^{s-1} N_{sp}(p^{l-1}).$$

Recall that $N_{sp}(p^l)$ denotes the number of solutions of

$$\sum_{\nu=1}^{s} a_\nu h_\nu^k \equiv n \pmod{p^l}, \quad 0 \leqslant h_\nu < p^l, \quad p + \text{g.c.d. } (a_\nu h_\nu). \tag{47}$$

The admissible h_ν are given without duplication by

$$h_\nu = y_\nu + z_\nu p^{l-\Theta-1}, \quad 0 \leqslant y_\nu < p^{l-\Theta-1}, \quad 0 = z_\nu < p^{\Theta+1}, \quad p + \text{g.c.d. } (a_\nu h_\nu). \tag{48}$$

By Lemma 18, then,

$$h_\nu^k \equiv y_\nu^k + k_0 y_\nu^{k-1} z_\nu p^{l-1} \pmod{p^l},$$

so that (47) becomes

$$\sum_{\nu=1}^{s} a_\nu (y_\nu^k + k_0 y_\nu^{k-1} z_\nu p^{l-1}) \equiv n \pmod{p^l}$$

with the conditions (48); and this in turn is equivalent to the pair of congruences

$$\sum_{\nu=1}^{s} a_\nu y_\nu^k \equiv n \pmod{p^{l-1}}, \quad 0 \leqslant y_\nu < p^{l-\Theta-1}, \quad p + \text{g.c.d. } (a_\nu y_\nu), \tag{49}$$

$$k_0 \sum_{\nu=1}^{s} a_\nu y_\nu^{k-1} z_\nu \equiv \left(n - \sum_\nu a_\nu y_\nu^k\right) \Big/ p^{l-1} \pmod{p}, \quad 0 \leqslant z_\nu < p^{\Theta+1}. \tag{50}$$

Congruence (49) would have $N_{sp}(p^{l-1})$ special solutions for y_ν ranging from zero to p^{l-1}; but by Lemma 18, taken modulo p^{l-1}, the congruence

$$y \equiv u \pmod{p^{l-\Theta-1}}$$

implies that

$$y^k \equiv u^k \pmod{p^{l-1}};$$

so that residual sets modulo $p^{l-\Theta-1}$ satisfy (49). Hence (49) actually has $p^{-\Theta s} N_{sp}(p^{l-1})$ special solutions.

For every special solution of (49) there are $p^{(\Theta+1)s-1}$ sets of z_ν satisfying (50); for p cannot divide every product $a_\nu y_\nu$. If, for example, $p+a_1 y_1$, we assign arbitrarily $z_2, \ldots, z_s$ (in $p^{\Theta+1}$ ways for each z, in $p^{(\Theta+1)(s-1)}$ ways for the set $z_2, \ldots, z_s$), and z_1 is then uniquely determined mod p (since $p+k_0 a_1 y_1{}^k$) by (50). To this determination of $z_1 \pmod p$ there correspond p^Θ values on $0 \leqslant z_1 < p^{\Theta+1}$, and hence, for each special solution of (49), there are $p^{(\Theta+1)(s-1)} p^\Theta = p^{(\Theta+1)s-1}$ special solutions of (50). Since there are $p^{-\Theta s} N_{sp}(p^{l-1})$ of these, we conclude that

$$N_{sp}(p^l) = p^{-\Theta s} N_{sp}(p^{l-1}) p^{(\Theta+1)s-1} = p^{s-1} N_{sp}(p^{l-1}).$$

When $p > \max a_\nu$, primitive and special solutions coincide, giving, as a corollary to Lemma 19, the following

LEMMA 20. *For $l > \gamma$ and $p > \max a_\nu$,*

$$N_0(p^l) = p^{s-1} N_0(p^{l-1}).$$

The next two lemmas are obvious for $l = \gamma$. For $l > \gamma$ they follow from Lemmas 19 and 20 applied $l-\gamma$ times.

LEMMA 21. *For $l \geqslant \gamma$,*

$$N_{sp}(p^l) = p^{(l-\gamma)(s-1)} N_{sp}(P).$$

LEMMA 22. *For $l \geqslant \gamma$ and for $p > \max a_\nu$,*

$$N_0(p^l) = p^{(l-\gamma)(s-1)} N_0(P).$$

LEMMA 23. *Let the multiplicity of p^k in n, $n \neq 0$, be β, i.e. $n = (p^k)^\beta p^\sigma$, where $0 \leqslant \sigma < k$; and let*

$$l_0 = \max(\beta k + \sigma + 1,\ \beta k + \gamma).$$

Then, if $p > \max a_\nu$,

$$A_0(p^l) = 0 \quad \text{for} \quad l > l_0, \tag{51}$$

and

$$\chi_{0p} = P^{1-s} N_0(P, 0) \sum_{\alpha=0}^{\beta-1} p^{\alpha(k-s)} + p^{\beta(k-s)} P^{1-s} N_0(P, n/p^{\beta k}). \tag{52}$$

The proof is strictly parallel to that of Satz 293, Landau, with A_0, M_0, N_0, and χ_{0p} used instead of A, M, N, and χ_p, and Lemmas 17 and 22 instead of Satz 286 and Satz 292, respectively.

In particular, if in Lemma 23 we have $p+2kn$, then equations (51) and $l_0 = 1$ imply that

$$\chi_{0p} = 1 + A_0(p). \tag{53}$$

Lemma 23 finds its chief application in Section 10. Lemma 21 enables us to prove the final lemma of this section:

LEMMA 24. *If, for every* n,

$$N_{sp}(P, n) > 0,$$

then, for every n,

$$\chi_{0p} \geqslant P^{1-s} = b(p).$$

Proof. By Lemma 21, for $l \geqslant \gamma$,

$$p^{-(s-1)l} N_{sp}(p^l) = p^{-\gamma(s-1)} N_{sp}(P) = P^{1-s} N_{sp}(P). \tag{54}$$

By definition, Lemma 17, and (54),

$$\chi_{0p} = \lim_{l=\infty} \sum_{\lambda=0}^{l} A_0(p^\lambda) = \lim_{l=\infty} p^{-(s-1)l} M_0(p^l)$$
$$\geqslant \varliminf_{l=\infty} p^{-(s-1)l} N_{sp}(p^l) = \varliminf_{l=\infty} P^{1-s} N_{sp}(P),$$

which cannot be less than P^{1-s} if the congruence (46) has a special solution (mod P) for every n.

10. *A lower bound for* χ_{0p} *for large* p*'s.* Most of the proofs of this section are at once reduced to cases in Landau by the hypothesis

$$p > \max a_\nu, \tag{H}$$

which implies that $(p, a_\nu) = 1$; so that, if ρ_1 is a primitive q-th root of unity, so is $\rho_1^{a_\nu} = \rho_2$, and we have $S_{\nu\rho_1} = S_{\rho_2}$. Where no other difficulty than this is involved, the only proof offered is a reference to the analogue in Landau.

LEMMA 25. *If* $p > \max a_\nu$, $q = p^l$, $p+k$, $2 \leqslant l \leqslant k$, *then*

$$S_{\nu\rho} = p^{l-1}.$$

The proof reduces by (H) to Satz 307, Landau.

LEMMA 26. *For* $p > \max a_\nu$ *and* $q = p$,

$$|S_{\nu\rho}| \leqslant (k-1)\sqrt{p}.$$

See (H) and Satz 311, Landau.

LEMMA 27. *For* $p > \max a_\nu$, $q = p^l$, $l > k$,

$$S_{\nu\rho} = p^{k-1} S_{\nu\rho^{p^k}}.$$

See (H) and Satz 312, Landau.

LEMMA 28. *Let* $T_{\nu\rho} = S_{\nu\rho}/q^{1-a}$. *Then, for* $p > \max a_\nu$, $q = p^l$, $l > k$,

$$T_{\nu\rho} = T_{\nu\rho^{p^k}}.$$

See (H) and Satz 313, Landau.

LEMMA 29. *For* $q = p^l$ *and* $l \geqslant 1$,

$$|T_{\nu\rho}| \leqslant 1 \quad \textit{for } p > c_{38},$$

$$|T_{\nu\rho}| \leqslant c_{39} \quad \textit{for all values of } p.$$

See the proof of Satz 314, Landau, using Lemma 28 instead of Satz 313. Corresponding to the cases (1), (2), and (3), take

(1) $p > k + \max a_\nu$, $2 \leqslant l \leqslant k$, Lemma 25 for Satz 307.

(2) $p > k + \max a_\nu$, $l = 1$, Lemma 27 for Satz 311.

(3) $p \leqslant k + \max a_\nu$, with $c_{41} = k + \max a_\nu$.

LEMMA 30. *For all values of* q,

$$|T_{\nu\rho}| < c_{15} \quad \textit{and} \quad |S_{\nu\rho}| < c_{15} q^{1-a}.$$

See (H) and Satz 315, Landau, with Lemma 29 instead of Satz 314.

LEMMA 31. $$|A_0(q)| < b_{19} q^{1-sa}.$$

The proof is by Lemma 30; for

$$|A_0(q)| \leqslant \sum_{\rho(q)} \left| \prod_{\nu=1}^{s} S_{\nu\rho} \right| q^{-s} < q(c_{15} q^{-a})^s = b_{19} q^{1-sa}.$$

By Lemma 31 it follows that, for $s > 2k$, the singular series

$$\mathfrak{S}_0 = \sum_{q=1}^{\infty} A_0(q)$$

converges absolutely, and hence, by Lemma 16,

$$\mathfrak{S}_0 = \prod_p \chi_{0p} \tag{55}$$

is valid for $s > 2k$. Since we consider only $k > 2$, $s > (k-2)K + 4$ implies that $s > 2k$, and (55) holds for the s of Theorem 1.

LEMMA 32. $$|A_0(p)| < b_{21}\, p^{1-\frac{1}{2}s}.$$

The proof has two cases. For $p \leqslant \max a_\nu$,

$$|A_0(p)| \leqslant p^{-s} \sum_{\rho(p)} \left| \prod_{\nu=1}^{s} S_{\nu\rho}\rho^{-n} \right| \leqslant p^{-s}(p-1)p^s \cdot 1 = p-1 < \max a_\nu.$$

For $p > \max a_\nu$ Lemma 26 gives

$$|A_0(p)| = \left| \sum_{\rho(p)} p^{-s} \prod_{\nu=1}^{s} S_{\nu\rho}\rho^{-n} \right| < p[p^{-1}(k-1)\sqrt{p}]^s = b_{22}\, p^{1-\frac{1}{2}s}.$$

LEMMA 33. *For $p > k+\max a_\nu$ and $p+n$,*

$$|\chi_{0p}-1| = |A_0(p)| < b_{21}\, p^{1-\frac{1}{2}s}.$$

Since $p+2kn$, the equality is implied by (53) and the inequality by Lemma 32.

LEMMA 34. *For $p > k+\max a_\nu$ and p^k+n,*

$$|\chi_{0p}-1| < b_{25} p^{1-\frac{1}{2}s}.$$

The proof is parallel to that of Satz 321, Landau, with χ_{0p}, $A_0(p)$, $S_{\nu\rho}$ for χ_p, $A(p)$, S_ρ, respectively, Lemma 23 for Satz 293, Lemma 33 for Satz 320, Lemma 32 for Satz 318, and Lemma 25 for Satz 307. The case $p \mid 2k$ is vacuous because we have $p > k+\max a_\nu$.

LEMMA 35. $$\chi_{0p} > 1-b_{28}\, p^{1-\frac{1}{2}s}.$$

The case p^k+n is provided for by Lemma 34. The case $p^k \mid n$ implies that $\beta > 0$. If, then,

(1) $p > k+\max a_\nu$, $\gamma = 1$, $P = p^\gamma = p$, and, by Lemma 23,

$$\chi_{0p}(n) \geqslant p^{1-s} N_0(p, 0) = p^{1-s} N_0(p, p) = \chi_{0p}(p) > 1-b_{25}\, p^{1-\frac{1}{2}s},$$

where the final inequality follows from Lemma 34.

(2) $p \leqslant k+\max a_\nu$, for an appropriate b_{29},

$$\chi_{0p} \geqslant 0 > 1-b_{29}\, p^{1-\frac{1}{2}s}.$$

As an immediate corollary, Lemma 35 yields

LEMMA 36. *For $s \geqslant 5$,*

$$\chi_{0p} > 1-b_{28}\, p^{-\frac{3}{2}}.$$

Lemma 36 in turn implies that

LEMMA 37. *For $s \geqslant 5$ and $p > b_{18}$,*

$$\chi_{0p} > 1-p^{-\frac{5}{4}}.$$

11. *The proof of Theorem* 2. We can now prove the extension of what Landau calls the second Hardy-Littlewood theorem.

THEOREM 2. *If* $s \geqslant (k-2)K+5$, *and if for every* n *there is a special solution of*

$$\sum_{\nu=1}^{s} a_\nu h_\nu^k \equiv n \pmod{P},$$

then $$\mathfrak{S}_0 > b_4.$$

Proof. By Lemmas 31, 24, and 37,

$$\mathfrak{S}_0 = \prod_p \chi_{0p} = \prod_{p \leqslant b_{18}} \chi_{0p} \prod_{p > b_{18}} \chi_{0p} > \prod_{p \leqslant b_{18}} b(p) \prod_{p > b_{18}} (1-p^{-5}) = b_4.$$

The hypotheses of Theorems 1 and 2 always yield a Waring theorem of the type which we seek, for Theorem 1 implies that

$$\lim_{n=\infty} \left| \frac{r(n)}{n^{sa-1}} - \frac{\Gamma^s(1+a)}{\Gamma(sa) \prod a_\nu^a} \mathfrak{S}_0 \right| = 0.$$

Hence, for $n > b_1$,

$$\left| \frac{r(n)}{n^{sa-1}} - \frac{\Gamma^s(1+a)}{\Gamma(sa) \prod a_\nu^a} \mathfrak{S}_0 \right| < \frac{b_4}{2} \frac{\Gamma^s(1+a)}{\Gamma(sa) \prod a_\nu^a},$$

$$\frac{r(n)}{n^{sa-1}} > \left| \frac{\Gamma^s(1+a)}{\Gamma(sa) \prod a_\nu^a} \left(\mathfrak{S}_0 - \frac{b_4}{2} \right) \right| > \frac{b_4}{2} \frac{\Gamma^s(1+a)}{\Gamma(sa) \prod a_\nu^a} = b_2,$$

i.e., for n sufficiently large, $r(n) > 0$. Our problem is thus reduced to finding when the second hypothesis of Theorem 2 is satisfied.

We summarize the implications of Theorems 1 and 2 in

THEOREM 2′. *Under the hypotheses of Theorem* 2 *there exists a positive constant* b_1, *depending only on* k, s, *and* $\max a_\nu$, *such that, for every integer* $n > b_1$, *there is a solution of the diophantine equation*

$$\sum_{\nu=1}^{s} a_\nu h_\nu^k = n \quad (h_\nu \geqslant 0).$$

CHAPTER III.

Sufficient conditions for Waring sets of coefficients.

Henceforth we shall keep $s \geqslant (k-2)K+5$, as required in Theorems 1 and 2, and seek sets of coefficients a_ν for which the congruence

(55′) $$\sum_{\nu=1}^{s} a_\nu h_\nu^k \equiv n \pmod{P}$$

has a special solution for every integer n and every prime p. A special solution, let us recall, must have at least one of the products $a_\nu h_\nu$ prime to p. Such a set of coefficients, since it always yields an asymptotic Waring theorem of our generalized type, will be called an *admissible set* or a *Waring set*.

12. *The primitivity conditions.* The residue class represented by $n_1 \pmod m$ is called *equivalent* to the residue class represented by n_2 if the congruence

$$x^k n_1 \equiv n_2 \pmod m$$

has a solution with $(x, m) = 1$. It is shown in Landau, 288, that equivalence is reflexive, symmetric, and transitive, and furthermore that, for $p > 2$, the number of sets of equivalent residue classes mod P is $1+r$, where

$$r = \frac{P-1}{p-1}(k, p-1), \tag{55''}$$

and that the set represented by zero contains no other residue class, but that every other set has $(p-1)/(k, p-1)$ incongruent equivalent residue classes.

LEMMA 38. *If, for a given prime p, the coefficients $a_1, \ldots, a_r$ are all prime to p, then, for every $n \not\equiv 0 \pmod P$, there is a special solution of*

$$\sum_{\nu=1}^{r} a_\nu h_\nu^k \equiv n \pmod P. \tag{56}$$

Note that it is sufficient to prove the lemma for a single representative from each of the r sets of equivalent residue classes. For if $(h_1, \ldots, h_r)$ is a special solution of (56) for $n = n_1$, and if n_2 is equivalent to n_1, then there is an x prime to p such that

$$n_2 \equiv x^k n_1 \equiv x^k \sum_{\nu=1}^{r} a_\nu h_\nu^k \equiv \sum_{\nu=1}^{r} a_\nu (x h_\nu)^k \pmod P, \tag{57}$$

thus exhibiting a special solution for $n = n_2$.

Let any set of equivalent classes containing an a_ν be called an *A-set*. If $n = a_j$, then $(0, 0, \ldots, h_j = 1, \ldots, 0, 0)$ is a special solution of (56). Hence, by (57), for every n in an A-set, there is a special solution of (56). If, therefore, every set is an A-set, the lemma is proved.

If t sets contain no a_ν (*blank sets*), then somewhere among the A-sets are distributed t extra a_ν. Choose in each A-set a particular a_ν as representative and label the t extra coefficients $e_1, e_2, \ldots, e_t$. For the

moment let a denote the a_ν which represents the A-set containing e_1. Since $(a, p) = 1$, each of the $P-1$ residue classes over which n ranges may be represented by a number of the form ma, $1 \leqslant m < P$. Now represent each set by the least multiple ma which it contains and arrange these in ascending order

$$m_1 a = 1 \cdot a, \quad m_2 a, \quad \ldots, \quad m_r a.$$

Let $m_j a$ represent the first blank set. Since e_1 is equivalent to a, there is an x prime to p for which $e_1 x^k \equiv a \pmod P$, and hence

$$m_j a - e_1 x^k \equiv m_j a - a \equiv (m_j - 1) a \pmod P.$$

Since $m_j \geqslant 2$, we have $a \leqslant (m_j-1)a < m_j a$; therefore $(m_j-1)a$ belongs to one of the $j-1$ preceding A-sets. Hence there is a special solution of the partial congruences

$$m_j a - e_1 x^k \equiv \sum_{a_\nu \neq e_1, \ldots, e_t} a_\nu h_\nu^k, \quad m_j a \equiv \sum_{a_\nu \neq e_2, \ldots, e_t} a_\nu h_\nu^k \pmod P.$$

If e_2 is in a different set from e_1, we represent the residue classes by multiples of a new a, rearrange the sets, and repeat the above argument. By using in succession the t extra coefficients, we deal with all the t blank sets.

If $p = 2$, then $\gamma = \Theta + 2$, $P = 2^{\Theta+2}$, $r = 2^{\Theta+2} - 1 =$ the number of residue classes over which n ranges $\geqslant$ the number of sets of equivalent residue classes. Hence the preceding proof is applicable, possibly with some leeway, to the case $p = 2$, as well as to $p > 2$.

LEMMA 39. *If, for a given prime p, the coefficients $a_1, \ldots, a_{r+1}$ are all prime to p, then for every n there is a special solution of*

$$\sum_{\nu=1}^{r+1} a_\nu h_\nu^k \equiv n \pmod P.$$

If $n \not\equiv 0 \pmod P$, the proof follows from Lemma 38. If $n \equiv 0 \pmod P$, then $n - a_{r+1} \not\equiv 0$, and again, by Lemma 38,

$$n \equiv (n - a_{r+1}) + a_{r+1} \equiv \sum_{\nu=1}^{r} a_\nu h_\nu^k + a_{r+1} \cdot 1^k.$$

Lemma 39 yields at once

THEOREM 3. *A set of coefficients a_ν is admissible if for every prime p at least $r+1$ of the a_ν are prime to p.*

Since, by the computations given by Landau on p. 291, r never exceeds $4k-1$, Theorem 3 gives

THEOREM 4. *A set of coefficients a_ν is admissible if for every prime p at least $4k$ of the a_ν are prime to p.*

COROLLARY 1. *By taking $4k$ of the a_ν equal to 1, the rest may be left arbitrary positive integers.*

COROLLARY 2. *By taking $4k+1$ of the a_ν relatively prime in pairs, the rest may be left arbitrary.*

Our theorems will gain in conciseness if we adopt the following definition: *A set a_ν is of primitivity m if for every prime p at least m of the a_ν are prime to p.* For example, Theorem 4 would then read "A set of coefficients a_ν is admissible if it is of primitivity $4k$".

The condition "primitivity $4k$" of Theorem 4 will be improved in the following sections for certain exponents k, but it should be noted that there are exponents for which no better sufficiency theorem, in terms only of the primitivity of the coefficients, can be obtained. For $k = 2^\lambda$, for example, the coefficients of the diophantine equation

$$\sum_{\nu=1}^{4\cdot 2^\lambda - 1} h_\nu^{2^\lambda} + \sum_{\mu = 4\cdot 2^\lambda}^{s} 2^{2^\lambda} h_\mu^{2^\lambda} = N$$

are of primitivity $4k-1$, and yet this equation has no solution in non-negative integers if $\lambda \geqslant 3$ and $N \equiv 4 \cdot 2 \pmod{2^{2^\lambda}}$. The proof rests on notes by A. Kempner, *Math. Annalen*, 72 (1912), 395.

13. *The case k an odd prime.* It is easy to verify that, if k is an odd prime,

(58) (*a*) for $p \neq k$, $r \leqslant k$,

(*b*) for $p = k$, $r = k+1$, $P = k^2$.

With these values of r we can improve the $4k$ of Theorem 4.

THEOREM 5. *If k is an odd prime, a set of coefficients a_ν is admissible if it is of primitivity $k+2$.*

This follows from (58) and Theorem 3.

THEOREM 6. *If k is an odd prime, a set of coefficients a_ν is admissible if it is of primitivity $k+1$ and if, for every n, $N_{sp}(k^2, n) > 0$.*

This is proved for (58a) by Theorem 3 and primitivity $k+1$; for (58*b*) by $N_{sp}(k^2, n) > 0$.

THEOREM 7. *If k is an odd prime, a set of coefficients a_ν is admissible if it is of primitivity $k+1$ and if it contains a subset with* g.c.d. $= 1$ *for which, with appropriate choice of signs, $\Sigma \pm a_\nu = 0$.*

The proof for $n \not\equiv 0$ is by (58) and Lemma 38. For $n \equiv 0$, note that k odd implies that, for all moduli P, $(-1)^k \equiv -1$ and $1^k \equiv 1$; hence $N_{sp}(P, 0) > 0$ by the last hypothesis of Theorem 7.

THEOREM 8. *If k is an odd prime, a set of coefficients a_ν is admissible if* (1) *it is of primitivity k, if* (2) *for every n, $N_{sp}(k^2, n) > 0$, and if* (3) *it contains a subset with* g.c.d. $= 1$ *for which, with appropriate choice of signs,* $\Sigma \pm a = 0$.

For $n \equiv 0$, the proof is as in Theorem 7. For $n \not\equiv 0$, it follows for (58a) from (1) and Lemma 38; for (58b), from (2).

For particular exponents better theorems are obtainable. Some of these are developed for $k = 3$, 4, and 5 in Chapter IV.

14. *Ordinary and special base sets.* We shall develop a method, particularly fruitful for odd exponents, of constructing certain admissible sets of coefficients not obtainable by Theorems 3–8.

We first define a *base set of order r* as a set $a_1, \ldots, a_r$ having

$$(59)\quad \begin{cases} (a)\ \text{for } k \text{ even}, \\ \qquad a_1 = 1, \quad a_j \leqslant a_{j+1} \leqslant 1 + \sum_{\nu=1}^{j} a_\nu, \\ (b)\ \text{for } k \text{ odd}, \\ \qquad a_1 = 1, \quad a_j \leqslant a_{j+1} \leqslant 1 + 2\sum_{\nu=1}^{j} a_\nu, \end{cases}$$

and a *special base set of order r* as having

$$(60)\quad \begin{cases} (a)\ \text{for } k \text{ even}, \\ \qquad a_1 = a_2 = 1, \quad a_j \leqslant a_{j+1} \leqslant \sum_{\nu=1}^{j} a_\nu \ \text{for } j \geqslant 2, \\ (b)\ \text{for } k \text{ odd}, \\ \qquad a_1 = a_2 = 1, \quad a_j \leqslant a_{j+1} \leqslant 1 + 2\sum_{\nu=1}^{j} a_\nu \ \text{for } j \geqslant 2. \end{cases}$$

These sets range respectively from

$$1, 1, \ldots, 1_r \ \text{to}\ 1, 2, 2^2, \ldots, 2^{r-1}.$$

$$1, 1, \ldots, 1_r \ \text{to}\ 1, 3, 3^2, \ldots, 3^{r-1}.$$

$$1, 1, \ldots, 1_r \ \text{to}\ 1, 1, 2, \ldots, 2^{r-2}.$$

$$1, 1, \ldots, 1_r \ \text{to}\ 1, 1, 5, 5.3, \ldots, 5.3^{r-3}.$$

From the above definitions follows Lemma 40 (41), obviously for k even and for k odd by the fact that -1 is then a k-ic residue mod P for every prime p.

LEMMA 40 (41). *If $a_1, \ldots, a_r$ is a (special) base set of order r, then, for $1 \leqslant n \leqslant a_j$, there is a (special) solution of the partial congruence*

$$n \equiv \sum_{\nu=1}^{j} a_\nu h_\nu^k \pmod{P}.$$

We can now prove the following double lemma.

LEMMA 42 (43). *If $a_1, \ldots, a_r$ is a (special) base set of order r, with r as in (55''), then, for every $n \equiv 0 \pmod{P}$, there is a (special) solution of*

$$n \equiv \sum_{\nu=1}^{r} a_\nu h_\nu^k \pmod{P}.$$

The proof of Lemma 42 (43) resembles that of Lemma 38. For $p > 2$, let each of the r sets of equivalent residue classes mod P over which n ranges be represented by the least positive residue which it contains and arrange these in ascending order

$$1 = n_1, \ n_2, \ \ldots, \ n_r.$$

As in Lemma 38, it is sufficient to make a proof for a single representative from each set. Since $n_1 = a_1 = 1$, there is a (special) solution of

$$n_1 \equiv a_1 h_1^k \pmod{P}$$

with $h_1 = 1$. If $n_2 \leqslant a_2$, there is a (special) solution of

$$n_2 \equiv \sum_{\nu=1}^{2} a_\nu h_\nu^k \pmod{P}$$

by Lemma 40 (41). If $n_2 > a_2$, then $0 < n_2 - a_2 < n_2$, so that $n_2 - a_2$ belongs to the first set and there are (special) solutions for

$$n_2 - a_2 \cdot 1^k \equiv a_1 h_1^k, \quad n_2 \equiv \sum_{\nu=1}^{2} a_\nu h_\nu^k \pmod{P}.$$

An induction completes the proof for $p > 2$.

For $p = 2$, the closing paragraph under Lemma 38 applies.

THEOREM 9. *If for a given prime p the coefficients a_ν contain an ordinary base set of order r, and if the remaining coefficients have* g.c.d. $= 1$, *then, for every n,*

$$N_{sp}(P, n) > 0.$$

Proof. For $p \nmid n$ there is a solution by Lemma 42. It is necessarily special. For $p \mid n$ one of the extra coefficients a_λ is prime to p, so that

$p+n-a_\lambda$. Hence, by Lemma 42, there are special solutions of

$$n-a_\lambda \cdot 1^k \equiv \sum_{\nu=1}^{r} a_\nu h_\nu^k, \quad n \equiv \sum_{\nu=1}^{s} a_\nu h_\nu^k \pmod{P}.$$

THEOREM 10. *If k is odd and if the coefficients a_ν contain a special base set of order r, then, for every n,*

$$N_{sp}(P, n) > 0.$$

The proof is by Lemma 43 for $n \not\equiv 0 \pmod{P}$. For $n \equiv 0 \pmod{P}$, since $a_1 = a_2 = 1$, there is a special solution with $h_1 = 1$, $h_2 = -1$, remaining h's zero.

THEOREM 11. *If k is even and if the coefficients a_ν contain a special base set of order $r+1$, then, for every n,*

$$N_{sp}(P, n) > 0.$$

Lemma 43 proves the case $n \not\equiv 0 \pmod{P}$. For $n \equiv 0$, note that, by dropping off a_1, any special base set of order $r+1$ becomes an ordinary base set of order r. By counting $a_1 = 1$ as one of the "remaining coefficients", Theorem 9 applies.

15. *Derived coefficients.* Whenever k is odd, it is possible to assign most of the coefficients and to determine the rest of them as functions of these. While the process is quite general and a description of it properly belongs here, it seems preferable to illustrate it instead for a special case. This is done in § 17.

CHAPTER IV.

Special cases.

It is interesting to inquire what the theorems of Chapter III yield when applied to cubes, fourth powers, and fifth powers.

16. *The case $k = 3$.* For $k = 3$, Theorem 1 requires $s \geqslant 9$; if $p = 3$, $r = 4$; if $p \neq 3$, $r \leqslant 3$. Hence the theorems of § 13 reduce to:

THEOREM 12. *For $k = 3$, any set of nine (or more) coefficients a_ν is admissible if it satisfies one of the following sets of conditions:*

(*a*) *Primitivity* 5. (By Theorem 5.)

(*b*) *Primitivity* 4; *and some subset of the* a_ν *has* g.c.d. $= 1$ *and satisfies* $\Sigma \pm a_\nu = 0$. (By Theorem 7.)

Example: 5, 7, 12, 13, 19, $+4$ arbitrary.

(*c*) *Primitivity* 4; *and for every* n *there is a special solution of* $n \equiv \sum_{\nu=1}^{9} a_\nu h_\nu^3 \pmod 9$. (By Theorem 6.)

Example: 29, 31, 37, 41, 53, $+4$ arbitrary.

(*d*) *Primitivity* 3; *for every* n *there is a special solution of*

$$n \equiv \sum_{\nu=1}^{9} a_\nu h_\nu^3 \pmod 9;$$

and some subset of the a_ν *has* g.c.d. $= 1$ *and satisfies* $\Sigma \pm a_\nu = 0$ (By Theorem 8.)

Example: 12, 19, 31, 53, $+5$ arbitrary.

By Theorems 9 and 10 when $p \neq 3$, and by verifying that $N_{sp}(9, n) > 0$ for every n when $p = 3$, we can prove

THEOREM 13. *Nine coefficients* a_ν *are admissible if their respective values are given by the following table*:

a_1, a_2, a_3	a_4, ..., a_9	Proof by
1 1 1	$a_4 \not\equiv 0 \pmod 9$, rest arbitrary	Theorem 10
1 1 2 1 1 3 1 1 4 1 1 5	arbitrary	Theorem 10
1 2 2 . . . 1 2 7 1 3 3 . . . 1 3 9	arbitrary except for g.c.d. $= 1$	Theorem 9

17. *Derived coefficients for* $k = 3$. Derived coefficients a_1, a_2, a_3 may be obtained as functions of arbitrarily assigned $a_4, \ldots, a_9$ by noting that ± 1 are both cubic residues for all moduli P and using any special base set as

follows to determine a_1, a_2, a_3. With the set 1, 1, 2, for example, consider the positive values of a_1, a_2, a_3 given by

$$\text{(A)} \quad a_1 = 1+(\pm a_4 \pm a_5 \pm a_6), \quad a_2 = 1+(\pm a_7 \pm a_8), \quad a_3 = 2+(\pm a_9)$$

when all combinations of $\pm$ signs are used. At least four, two, and one positive values are determined for a_1, a_2, and a_3, respectively. This gives eight sets a_1, a_2, a_3, each of which taken with the original $a_4, \ldots, a_9$ constitutes a Waring set. By permuting either the 1, 1, 2 or the $a_4, \ldots, a_9$, or both, new sets a_1, a_2, a_3 are found. Moreover, any of the special base sets 1, 1, 3; 1, 1, 4; or 1, 1, 5, or any other partition of $a_4, \ldots, a_9$ into three parentheses, will yield further derived values for a_1, a_2, a_3.

If the coefficients $a_4, \ldots, a_9$ are partitioned into four parentheses instead of three, a_1, a_2, a_3 may be determined as above from any three parentheses, while the a's in the fourth may be regarded as absolutely arbitrary coefficients. It is possible, for example, to assign three coefficients a_4, a_5, a_6, to determine three more a_1, a_2, a_3 by the equations

$$a_1 = 1+a_4, \quad a_2 = 1+a_5, \quad a_3 = 2 \pm a_6,$$

and to leave a_7, a_8, a_9 quite arbitrary.

THEOREM 14. *If to six of the nine coefficients any assigned values are given, the other three coefficients may be determined as functions of these in various ways by equations of the type* (A) *above.*

18. *The case* $k = 4$. For $k = 4$, Theorem 1 requires $s \geqslant 21$. Equation (55″) yields

$$r \leqslant 4 \text{ if } p \neq 2; \quad r = 15 \text{ if } p = 2.$$

Hence Theorems 3 and 4 reduce to

THEOREM 15. *For* $k = 4$, *a set of twenty-one* (*or more*) *coefficients* a_ν *is admissible*

(*a*) *if it is of primitivity* 16 (*by Theorem* 4),

(*b*) *or if it is of primitivity* 5 *and if for every* n *there is a special solution of* $n \equiv \sum_{\nu=1}^{s} a_\nu h_\nu^k \pmod{16}$ (*by Theorem* 3).

Example: 17, 49, 67, 37, 43, 71, +15 arbitrary.

THEOREM 16. *For* $k = 4$, *any set of twenty-one coefficients containing*

1, 1, 2, 4, 8

is admissible and the remaining sixteen coefficients may be regarded as arbitrary.

Proof. For $p = 2$ we have $P = 16$, and we verify that for every n there is a special solution of the partial congruence

$$h_1^4+h_2^4+2h_3^4+4h_4^4+8h_5^4 \equiv n \pmod{P}.$$

Since 1, 1, 2, 4, 8 is a special base set of order 5, and since for $p \neq 2$ we have $r \leqslant 4$, the case $p \neq 2$ is covered by Theorem 11.

The special base sets of order 5 are 1, 1, 1, 1, 1–4; 1, 1, 1, 2, 2–5; 1, 1, 1, 3, 3–6; 1, 1, 2, 2, 2–6; 1, 1, 2, 3, 3–7; 1, 1, 2, 4, 4–8. Except for the case $p = 2$ the proof of Theorem 16 applies if $a_1, \ldots, a_5$ are any of these sets instead of 1, 1, 2, 4, 8. We deal with the case $p = 2$ by specifying enough additional coefficients to ensure a special solution for every n of

$$n \equiv \sum_{\nu=1}^{s} a_\nu h_\nu^4 \pmod{16}.$$

If permutations are ignored, the admissible sets thus derived can be catalogued numerically in a table of about 1,300 entries. The following is a section of such a table. It is understood that in each entry enough arbitrary coefficients to make $s \geqslant 21$ must be supplied and that, beginning with a_6, each coefficient may be replaced by any positive integer to which it is congruent modulo 16.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	$a_{13} \ldots a_{20}\, a_{21}$
1	1	1	1	1	1	4	4	15				
					1	4	6					
					1	4	11					
					1	4	12	15				
					1	4	13	14				
					1	4	13	15	15			
					1	4	14	14	15			
					1	4	14	15	15	15		
					1	4	15	15	15	15	15	
					1	5	5					
					1	5	12					

Theorem 9 shows that the above method of construction yields another class of admissible sets if ordinary base sets ranging from 1, 2, 2, 2, 2 to 1, 2, 4, 8, 16 are used instead of special base sets and if the "arbitrary" coefficients are required to have g.c.d. = 1.

19. *The case* $k = 5$, $s \geqslant 53$. For $p = 5$, $r = 6$; for $p \neq 5$, $r \leqslant 5$.

THEOREM 17. *For* $k = 5$, *a set of fifty-three* (*or more*) *coefficients* a_ν *is admissible*

(*a*) *If it is of primitivity* 7.

Example: 8 *distinct primes*+45 *arbitrary*.

(*b*) *Or if it is of primitivity* 6 *and there is a special solution of* $n \equiv \sum_{\nu=1}^{s} a_\nu h_\nu^5 \pmod{25}$ *for every* n.

Example: 7 *distinct primes*+46 *arbitrary*.

(*c*) *Or if it is of primitivity* 6 *and if some subset of the* a_ν *has* g.c.d. $= 1$ *and satisfies* $\Sigma \pm a_\nu = 0$.

Example: 14, 17, 31, 37, 41, 43, 47, +46 *arbitrary*.

(*d*) *Or if it is of primitivity* 5 *and satisfies both* (*b*) *and* (*c*).

Examples: 14, 17, 31, 37, 41, 43, +47 *arbitrary*.
1, 1, 1, 1, 1, +48 *arbitrary*.

THEOREM 18. *Of the fifty-three coefficients forty-eight will be arbitrary if the first five are chosen as a special base set of order* 5.

There are 890 such sets, with the range 1, 1, 1, 1, 1–9; 1, 1, 1, 1, 2–11; ...; 1, 1, 5, 14, 14–43; 1, 1, 5, 15, 15–45. The theorem follows for $p \neq 5$ by Theorem 10, for $p = 5$ by verification.

THEOREM 19. *If the first five coefficients constitute an ordinary base set of order* 5, *the remaining forty-eight coefficients need have only* g.c.d. $= 1$.

After deleting special base sets, the ordinary base sets of order 5 range from 1, 2, 2, 2, 2 to 1, 3, 9, 27, 81. The existence of a special solution now follows for $p \neq 5$ by Theorem 9, for $p = 5$ by verification.

Note on universal theorems.

The theorems of this paper are proved only for large integers n, but it is interesting to ask which of them hold for all positive integers. If it were not for the forbidding size of the constant b_1, it would be possible for any particular problem to decide this by actual verification for $1 \leqslant n \leqslant b_1$. In any case, the greatest number of exceptions tend to occur for smaller integers. If we can verify that a proposed Waring theorem holds for an initial interval $1 \leqslant n \leqslant 1000$, there is a better chance of its being true than if we had inspected $1000000 \leqslant n \leqslant 1001000$.

Along with proofs of numerous universal generalizations of Waring's theorem, Prof. L. E. Dickson has given a number of empirical theorems proved for long initial stretches of positive integers*. For $k = 4$, for example, he shows that all positive integers $n \leqslant 4100$ are represented by the form $(1_{11}\, 8_1)$, *i.e.*, by a form with eleven coefficients 1 and one coefficient 8. If nine arbitrary coefficients are adjoined to make a total of twenty-one, this is still true, and Theorem 15*b* adds to the range of n the infinite interval $n > b_1$.

Other forms given by Prof. Dickson and of similar interest are:

for $k = 3$, (11223), (11224), (11234), and (1111114).

for $k = 4$, $(1_{17}\, 2_1)$, (1122337), (1122346), (11112238), (11112247), (11122228), (112234m) with $m = 6$, 7, 8, or 9.

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* L. E. Dickson, *American J. of Math.*, 49 (1927), 241; *American Math. Monthly*, 34 (1927), 177.

ON THE NUMBER OF PRIMES IN AN ARITHMETIC PROGRESSION

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Introduction.

It is well known that if $(k, l) = 1$ and $\pi(m;\ k, l)$ denotes the number of primes p satisfying $p \leqslant m$, $p \equiv l \pmod{k}$, then

$$\pi(m;\ k, l) = \frac{1}{\phi(k)} \int_2^m \frac{du}{\log u} + O(me^{-A\sqrt{\log m}}),$$

where A is a positive constant, and the constant implied in the O is dependent on k. The object of this paper is to find an asymptotic formula for $\pi(m;\ k, l)$ which holds uniformly in k. Such a problem has been investigated by Titchmarsh*, who uses one of Brun's elementary formulae to obtain the result

$$\pi(m;\ k, l) = O\left(\frac{m}{\phi(k)\log m}\right) \quad \text{if} \quad k \leqslant m^a, \quad 0 < a < 1.$$

In this paper I use a transcendental method to obtain a result which is more precise than the above formula in the restricted range $k < e^{A\sqrt{\log m}}$, where A is a positive constant. The proof is modelled on that of Landau's "Vorlesungen über Zahlentheorie" for the corresponding result in which k is fixed, and involves a consideration of L-functions with the modulus k variable. I have made considerable use of the work done by Titchmarsh on this subject.

I shall prove the following theorems. Throughout the paper A denotes a positive constant, not always the same one, independent of both m and k; $A_1, A_2, \ldots$ always denote the same constants.

* (1) E. C. Titchmarsh, *Rend. di Palermo*, 54 (1930), 414–429. This paper will hereafter be referred to as T. (2) E. C. Titchmarsh, *Rend. di Palermo*, 57 (1933), 478–9.

THEOREM 1. *Let σ_1 be the greatest real zero possessed by L-functions with modulus k, and let $(k, l) = 1$. Then*

$$(1) \qquad \pi(m;\ k, l) = \frac{1}{\phi(k)} \int_2^m \frac{du}{\log u} + O(me^{-A\sqrt{\log m}}) + O\left(\frac{m^{\sigma_1}}{\phi(k) \log m}\right).$$

It appears likely that σ_1, defined in Theorem 1, satisfies the inequality $\sigma_1 < 1 - A/\log k$, and if this hypothesis is correct the last term on the right-hand side of (1) is at most of the same order as the second term. We are not able to prove this hypothesis, but a theorem due to Landau shows that, in a certain sense which is made clear in the statement of Lemma 9, there cannot be too many exceptions to the hypothesis.

It is well known* that the $\phi(k)$ characters formed with modulus k can be divided into two classes, consisting of primitive and imprimitive characters. Further, each imprimitive character (mod k) is associated with a certain primitive character whose modulus is a divisor of k. If χ is an imprimitive character (mod k) and X the primitive character (mod k_1) with which χ is associated, then

$$(2) \qquad L(s, \chi) = \prod_p \{1 - X(p)p^{-s}\} L(s, X),$$

where the product is taken over all primes which divide k but not k_1. Equation (2) shows us that, apart from the zeros lying on the imaginary axis, the zeros of $L(s, \chi)$ and $L(s, X)$ are the same, and it follows that, in studying the real zeros of L-functions, we need consider only those functions which are formed with primitive characters.

An examination of the real zeros of L-functions formed with complex characters is found in Titchmarsh's paper. The second theorem deals with the real zeros of L-functions formed with real primitive characters.

THEOREM II. *If $k \geqslant 2$, and χ is a real primitive character* (mod k), *then $L(s, \chi)$ has no real zeros for*

$$\sigma > 1 - \frac{A_1}{k^{\frac{1}{2}} \log^2 k}.$$

It follows from (2) that this theorem also holds if χ is a real imprimitive character.

In § 5 I make an application of the main results of the paper to the problem of finding an asymptotic formula for the number of representations of an integer m as the sum of a prime and a quadratfrei number.

* Landau, *Primzahlen*, 1 (1909), 478–483.

I wish to express my thanks to Dr. Estermann for his valuable suggestions and criticism.

Notation. The formula $f(m, k) = O\{g(m, k)\}$ means that

$$|f(m, k)| < A|g(m, k)|$$

for all $m > A$.

We assume throughout that $k \geqslant 2$. $L(s, \chi)$ denotes a Dirichlet L-function defined by the equation

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s},$$

where the series on the right-hand side is convergent, and elsewhere by means of analytic continuation, and χ is a character of residues mod k.

1. LEMMA 1. *$L(s, \chi)$ has no zeros for $t \geqslant 3$, $\sigma > 1 - A_2/\log kt$.*

This is Lemma 3 of T.

LEMMA 2.

$$\left|\frac{L'}{L}(s, \chi)\right| < A_3 \log kt \quad \left(1 - \frac{A_4}{\log kt} \leqslant \sigma \leqslant 2, \quad t \geqslant 4\right).$$

This is Lemma 4 of T.

LEMMA 3. *If χ is not the principal character,*

$$|L(s, \chi)| < A_5 \log k \quad \left(1 - \frac{1}{\log k} \leqslant \sigma \leqslant 3, \quad |t| \leqslant 11\right)$$

$$< A_6 k^{\frac{1}{2}} \qquad \left(\tfrac{1}{2} \leqslant \sigma \leqslant 3, \quad |t| \leqslant 11\right).$$

On noting that $|L(s, \chi)| < A_7$ if $\sigma > 2$, the lemma follows immediately from Lemma 5 of T.

LEMMA 4. *If χ is not the principal character,*

$$|L'(s, \chi)| < A_8 \log^2 k \quad \left(1 - \frac{1}{2 \log k} \leqslant \sigma \leqslant 2, \quad |t| \leqslant 10\right)$$

$$< A_9 k^{\frac{1}{2}} \qquad \left(\tfrac{3}{4} \leqslant \sigma \leqslant 2, \quad |t| \leqslant 10\right).$$

The results are an immediate consequence of Cauchy's inequality $|f'(s_0)| < M/r$, where $f(s)$ is regular on and inside the circle $|s - s_0| \leqslant r$, and M is the upper bound of $|f(s)|$ on the circle.

Let $s_0 = \sigma_0 + it_0$, where $|t_0| \leqslant 10$. For the first result we take

$$\sigma_0 \geqslant 1 - 1/(2 \log k),$$

$r = 1/(2\log k)$, when $M \leqslant A_5 \log k$ by Lemma 3; the second result follows on taking $\sigma_0 \geqslant \frac{3}{4}$, $r = \frac{1}{4}$, and $M \leqslant A_6 k^{\frac{1}{2}}$.

LEMMA 5. *If χ is any complex character, $L(s, \chi)$ has no zeros for $\sigma \geqslant 1 - A_{10}/\log k$, $|t| \leqslant 5$.*

This is Lemma 6 of T.

LEMMA 6. *If χ is a real non-principal character, $L(s, \chi)$ has no zeros for $\sigma \geqslant 1 - A_{11}/\log k$, $0 < t \leqslant 5$.*

This is Lemma 8 of T. Following the suggestion of Titchmarsh, we replace the first part of the proof, due to Gronwall*, which states that $L(s, \chi)$ has no zero in the region $\sigma > 1 - A_{12}/\log k$, $0 < t < A_{13}/\log k$, by a proof depending upon Landau's Theorem 374.

Let $\beta + i\gamma$, where $\beta \geqslant 1 - 1/(c\log k)$, $0 < \gamma \leqslant 1/(c\log k)$, be a zero of $L(s, \chi)$ and let $s_0 = \sigma_0 + i\gamma$, $\sigma_0 = 1 + 1/(b\log k)$, where b and c are positive constants. Then $\beta - i\gamma$ is also a zero of $L(s, \chi)$ and both zeros lie within the circle $|s - s_0| \leqslant \frac{1}{4}$ if $c > A_{14}$, $b > A_{14}$. Also $L(s, \chi)$ is regular within the circle $|s - s_0| \leqslant \frac{1}{2}$, and, on using Lemma 3 and the inequality

$$\left| \frac{1}{L(s_0, \chi)} \right| \leqslant \sum_{n=1}^{\infty} n^{-\sigma_0} < \frac{2}{\sigma_0 - 1} = 2b\log k,$$

it is clear that

$$\left| \frac{L(s, \chi)}{L(s_0, \chi)} \right| < e^M \quad \text{if} \quad |s - s_0| \leqslant \tfrac{1}{2},$$

where $M = A_{15}\log k + \log b$.

We apply Landau's Theorem 374 in the form†

$$-R\frac{f'}{f}(s_0) < \frac{4M}{r} + \sum_{\rho} R\left(\frac{1}{\rho - s_0}\right),$$

where ρ runs through the roots of $f(s) = 0$ in the circle $|s - s_0| \leqslant \frac{1}{4}$, taking $f(s) = L(s, \chi)$, $r = \frac{1}{2}$. Clearly

$$\sum_{\rho} R\left(\frac{1}{\rho - s_0}\right) \leqslant R\left(\frac{1}{\beta + i\gamma - s_0}\right) + R\left(\frac{1}{\beta - i\gamma - s_0}\right)$$

$$= -\frac{1}{\sigma_0 - \beta} - \frac{\sigma_0 - \beta}{(\sigma_0 - \beta)^2 + 4\gamma^2},$$

* Gronwall, *Rend. di Palermo*, 35 (1913), 145–159.

† Landau, *Vorlesungen über Zahlentheorie*, 2 (1927), 17, formula (433).

and hence

$$-R\frac{L'}{L}(s_0,\chi)<8(A_{15}\log k+\log b)-\left[\frac{1}{\sigma_0-\beta}+\frac{\sigma_0-\beta}{(\sigma_0-\beta)^2+4\gamma^2}\right].$$

Also

$$R\frac{L'}{L}(s_0,\chi)<\left|\frac{\zeta'}{\zeta}(\sigma_0)\right|<\frac{5}{4(\sigma_0-1)}\quad\text{if}\quad b>A_{16}.$$

It follows that

$$\frac{1}{\sigma_0-\beta}+\frac{\sigma_0-\beta}{(\sigma_0-\beta)^2+4\gamma^2}<A_{17}\log k+8\log b+\frac{5}{4(\sigma_0-1)},\tag{3}$$

and hence

$$\begin{aligned}\frac{\sigma_0-\beta}{(\sigma_0-\beta)^2+4\gamma^2}&<A_{17}\log k+8\log b+\frac{1}{4(\sigma_0-1)}+\frac{1-\beta}{(\sigma_0-1)(\sigma_0-\beta)}\\&<A_{17}\log k+8\log b+\tfrac{1}{4}b\log k+\frac{b^2}{c}\log k\\&<\left(\tfrac{1}{2}b+\frac{b^2}{c}\right)\log k\quad\text{if}\quad b>A_{18}.\end{aligned}$$

Now

$$\sigma_0-\beta=1-\beta+\frac{1}{b\log k}\leqslant\left(\frac{1}{b}+\frac{1}{c}\right)\frac{1}{\log k},$$

and hence

$$1-\beta+\frac{1}{b\log k}<\left[\left(\frac{1}{b}+\frac{1}{c}\right)^2+\frac{4}{c^2}\right]\left[\tfrac{1}{2}b+\frac{b^2}{c}\right]\frac{1}{\log k},$$

which is impossible if we choose b and c so that

$$\frac{1}{b}\geqslant\left[\left(\frac{1}{b}+\frac{1}{c}\right)^2+\frac{4}{c^2}\right]\left[\tfrac{1}{2}b+\frac{b^2}{c}\right].$$

That b and c may be chosen to satisfy this inequality follows at once from the fact that, as $c\to\infty$,

$$\frac{1}{b}-\left[\left(\frac{1}{b}+\frac{1}{c}\right)^2+\frac{4}{c^2}\right]\left[\tfrac{1}{2}b+\frac{b^2}{c}\right]\to\frac{1}{2b}>0.$$

The result follows.

LEMMA 7. *If χ is a real non-principal character* (mod k), *then $L(\sigma,\chi)$ has at most one zero for $\sigma\geqslant 1-A_{19}/\log k$.*

Let β and β', where $\beta' \geqslant \beta$, be two real zeros of $L(s, \chi)$. We use the method employed in the proof of Lemma 6 and obtain the result

$$\frac{1}{\sigma_0-\beta}+\frac{1}{\sigma_0-\beta'} < A_{20}\log k+8\log b+\frac{5}{4(\sigma_0-1)},$$

which replaces (3).

Hence
$$\frac{2}{\sigma_0-\beta} < A_{20}\log k+8\log b+\tfrac{5}{4}\, b\log k$$
$$< \tfrac{3}{2}b\log k \quad \text{if} \quad b > A_{21}.$$

It follows that

$$\sigma_0-\beta > \frac{4}{3b\log k}, \quad \text{whence} \quad \beta < 1-\frac{1}{3b\log k}.$$

The result follows.

2. In this section we consider the real zeros of $L(s, \chi)$, where χ is a real primitive character, and prove Theorem II stated in the introduction.

We employ the following result*.

If $\chi(n)$ is a primitive character to modulus k, then

$$\sum_{n=1}^{k-1} \chi(n)\,\xi_k^{mn} = \epsilon(\chi)\,k^{\frac{1}{2}}\,\bar{\chi}(m),$$

where ξ_k is an abbreviation for $e^{2\pi i/k}$, and $\epsilon(\chi)$ is a constant of modulus unity, depending on χ. If $\chi(n)$ is also a real character, this formula becomes

$$\sum_{n=1}^{k-1} \chi(n)\,\xi_k^{mn} = \epsilon(\chi)\,k^{\frac{1}{2}}\,\chi(m). \tag{4}$$

We consider separately the two cases $\chi(-1) = -1$ and $\chi(-1) = 1$.

2.1. In this section $\chi(n)$ denotes a real primitive character for which $\chi(-1) = -1$.

We substitute the expression for $\chi(m)$ given in (4) in the formula defining $L(1, \chi)$, viz.

$$L(1, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m},$$

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* Landau, *Primzahlen*, 1 (1909), 483–494.

and obtain

$$(5)\qquad \epsilon(\chi)\, k^{\frac{1}{2}}\, L(1,\chi) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{k-1} \chi(n)\, \xi_k^{mn}$$

$$= \sum_{n=1}^{k-1} \chi(n) \sum_{m=1}^{\infty} \frac{1}{m}\, \xi_k^{mn} = -\sum_{n=1}^{k-1} \chi(n) \log(1-\xi_k{}^n),$$

where throughout this section and the next $\log x$ denotes the principal value of the logarithm. Now

$$\chi(n) = \chi(n-k) = \chi(k-n)\chi(-1) = -\chi(k-n),$$

so that

$$\sum_{n=1}^{k-1} \chi(n) \log(1-\xi_k{}^n) = -\sum_{n=1}^{k-1} \chi(n) \log(1-\xi_k^{k-n})$$

$$= -\sum_{n=1}^{k-1} \chi(n) \log(1-\xi_k^{-n}),$$

and hence we may write (5) in the form

$$-2\epsilon(\chi)\, k^{\frac{1}{2}}\, L(1,\chi) = \sum_{n=1}^{k-1} \chi(n) \left[\log(1-\xi_k{}^n) - \log(1-\xi_k^{-n})\right]$$

$$= \sum_{n=1}^{k-1} \chi(n)\, i\left(\frac{2\pi n}{k} - \pi\right) = \frac{2\pi i}{k} \sum_{n=1}^{k-1} n\chi(n),$$

since $\sum_{n=1}^{k-1} \chi(n) = 0$.

Put $S = \sum_{n=1}^{k-1} n\chi(n)$. Then

$$(6)\qquad -\epsilon(\chi)\, k^{\frac{1}{2}}\, L(1,\chi) = \pi i k^{-1} S,$$

and, since $L(1,\chi) \neq 0$, it follows that $S \neq 0$.

If r is any number satisfying $(r, k) = 1$, the numbers nr, where n takes all values of a complete set of residues, form a complete set of residues, and hence

$$S \equiv \sum_{n=1}^{k} nr\,\chi(nr) \pmod{k}.$$

Now

$$\sum_{n=1}^{k} nr\,\chi(nr) = r\chi(r) \sum_{n=1}^{k} n\chi(n) = r\chi(r)\, S,$$

and hence

$$S \equiv r\chi(r)\, S \pmod{k},$$

or

$$(7)\qquad S[1-r\chi(r)] \equiv 0 \pmod{k}.$$

Let $k = 2^a k_0$, where k_0 is odd. Then, if we choose r to satisfy the simultaneous congruences

$$r \equiv 3 \pmod{2^a}, \qquad r \equiv 2 \pmod{k_0},$$

it is clear that $(r, k) = 1$. Further,

$$1 - r\chi(r) \equiv 4 \text{ or } -2 \pmod{2^a},$$

$$1 - r\chi(r) \equiv 3 \text{ or } -1 \pmod{k_0},$$

from which it follows that the greatest common divisor of $\{1 - r\chi(r)\}$ and k is a divisor of 12. Hence, by (7), we see that $12S/k$ is an integer, and therefore $|S| \geqslant \frac{1}{12}k$.

It now follows from (6) that

$$|L(1, \chi)| \geqslant \tfrac{1}{12}\pi k^{-\frac{1}{2}}. \tag{8}$$

2.2. In this section $\chi(n)$ denotes a real primitive character for which $\chi(-1) = 1$.

It is easily seen that, in this case,

$$-k^{\frac{1}{2}}\epsilon(\chi)\, L(1, \chi) = \sum_{n=1}^{k-1} \chi(n) \log(1 - \xi_k{}^n).$$

Let

$$\alpha = \prod_{1 \leqslant n \leqslant k-1} (1 - \xi_k{}^n)^{\chi(n)}. \tag{9}$$

Then we may write

$$\alpha = \prod_{1 \leqslant n \leqslant \frac{1}{2}(k-1)} (1 - \xi_k{}^n)^{\chi(n)} \prod_{\frac{1}{2}k \leqslant n \leqslant k-1} (1 - \xi_k{}^n)^{\chi(n)},$$

and, on transforming the last product by writing $k - n$ for n and noting that $\chi(k-n) = \chi(n)$, we obtain

$$\alpha = \prod_{1 \leqslant n \leqslant \frac{1}{2}(k-1)} \{(1 - \xi_k{}^n)(1 - \bar{\xi}_k{}^n)\}^{\chi(n)}$$

$$= \prod_{1 \leqslant n \leqslant \frac{1}{2}(k-1)} \left(4 \sin^2 \frac{\pi n}{k}\right)^{\chi(n)}.$$

It follows that α is a positive number. Also it is easily seen that

$$\log \alpha = \sum_{n=1}^{k-1} \chi(n) \log(1 - \xi_k{}^n),$$

and hence

$$-k^{\frac{1}{2}}\epsilon(\chi)\, L(1, \chi) = \log \alpha. \tag{10}$$

If $(n, k) = 1$, the congruence $nn' \equiv 1 \pmod{k}$ is soluble, and hence

$$1-\xi_k = 1-\xi_k^{nn'} = (1-\xi_k^{n})\,\phi_1(\xi_k),$$

where $\phi_1(x) = \sum_{r=0}^{n'-1} x^{rn}$, which is a polynomial in x with integer coefficients. Also, if r is any integer,

$$1-\xi_k^{r} = (1-\xi_k^{rn})\,\phi_1(\xi_k^{r}).$$

Also

$$1-\xi_k^{n} = (1-\xi_k)\,\phi_2(\xi_k),$$

$$1-\xi_k^{rn} = (1-\xi_k^{r})\,\phi_2(\xi_k^{r}),$$

where $\phi_2(x) = \sum_{r=0}^{n-1} x^{r}$, which is another polynomial in x with integer coefficients.

It follows that

$$\begin{aligned}\prod_{1\leqslant n\leqslant k-1}(1-\xi_k^{nr})^{\chi(n)} &= \prod_{\substack{1\leqslant n\leqslant k-1\\ \chi(n)=1}}(1-\xi_k^{nr}) \prod_{\substack{1\leqslant n\leqslant k-1\\ \chi(n)=-1}}(1-\xi^{nr})^{-1}\\ &= \prod_{\substack{1\leqslant n\leqslant k-1\\ \chi(n)=1}}\{(1-\xi_k^{r})\,\phi_2(\xi_k^{r})\} \prod_{\substack{1\leqslant n\leqslant k-1\\ \chi(n)=-1}}\{(1-\xi_k^{r})^{-1}\phi_1(\xi_k^{r})\}\\ &= \prod_{\substack{1\leqslant n\leqslant k-1\\ \chi(n)=1}}\phi_2(\xi_k^{r}) \prod_{\substack{1\leqslant n\leqslant k-1\\ \chi(n)=-1}}\phi_1(\xi_k^{r}),\end{aligned}$$

since the number of values of n for which $\chi(n) = 1$ is equal to the number of values of n for which $\chi(n) = -1$. Hence

$$\prod_{1\leqslant n\leqslant k-1}(1-\xi_k^{nr})^{\chi(n)} = g_1(\xi_k^{r}),$$

where

$$g_1(x) = \prod_{\substack{1\leqslant n\leqslant k-1\\ \chi(n)=1}}\phi_2(x) \prod_{\substack{1\leqslant n\leqslant k-1\\ \chi(n)=-1}}\phi_1(x),$$

which is a polynomial in x.

Also, if $(r, k) = 1$,

$$\begin{aligned}\prod_{1\leqslant n\leqslant k-1}(1-\xi_k^{nr})^{\chi(n)} &= \left[\prod_{1\leqslant n\leqslant k-1}(1-\xi_k^{nr})^{\chi(nr)}\right]^{\chi(r)}\\ &= \left[\prod_{1\leqslant n\leqslant k-1}(1-\xi_k^{n})^{\chi(n)}\right]^{\chi(r)} = \alpha^{\chi(r)}.\end{aligned}$$

Hence $g_1(\xi_k^{r}) = \alpha^{\chi(r)}$. Similarly $g_2(\xi_k^{r}) = \alpha^{-\chi(r)}$,

where $$g_2(x) = \prod_{\substack{1 \leqslant n \leqslant k-1 \\ \chi(n)=1}} \phi_1(x) \prod_{\substack{1 \leqslant n \leqslant k-1 \\ \chi(n)=-1}} \phi_2(x),$$

which is also a polynomial in x.

It follows that

$$g_1(\xi_k{}^r)+g_2(\xi_k{}^r) = \alpha^{\chi(r)}+\alpha^{-\chi(r)} = \alpha+\alpha^{-1},$$

since $\chi(r) = \pm 1$. On writing $g(x) = g_1(x)+g_2(x)$, we have

$$g(\xi_k{}^r) = \alpha+\alpha^{-1}, \tag{11}$$

where $g(x)$ is a polynomial in x with integer coefficients.

Now it is known that $f(x) = \prod_{\rho}(x-\rho)$, where ρ runs through the $\phi(k)$ primitive k-th roots of unity, is a polynomial in x of degree $\phi(k)$ with integer coefficients. We may put

$$g(x) = f(x)\,q(x)+t(x),$$

where $t(x)$ is a polynomial in x (with integer coefficients) of degree less than $\phi(k)$. Also, on putting $x = \xi_k{}^r$, where $(r, k) = 1$, we have $g(\xi_k{}^r) = t(\xi_k{}^r)$, since $f(\xi_k{}^r) = 0$, and hence, by (11),

$$t(\xi_k{}^r) = \alpha+\alpha^{-1}$$

for $\phi(k)$ values of r, namely all those for which $r < k$, $(r, k) = 1$, and, since $t(x)$ is a polynomial of degree less than $\phi(k)$, we have $t(x) \equiv \alpha+\alpha^{-1}$, from which it follows that $\alpha+\alpha^{-1}$ is an integer. Now α is positive and, on using the result $L(1, \chi) \neq 0$, it follows from (10) that $\alpha \neq 1$. Hence $\alpha+\alpha^{-1} \geqslant 3$, which shows that $\alpha \geqslant \frac{1}{2}(3+\sqrt{5})$ or $\alpha \leqslant \frac{1}{2}(3-\sqrt{5})$, and in both cases $|\log \alpha| \geqslant \log[\frac{1}{2}(3+\sqrt{5})]$. It now follows from (10) that

$$|L(1, \chi)| > A_{22}\,k^{-\frac{1}{2}}. \tag{12}$$

2.3. Formulae (8) and (12) show that, if χ is a real primitive character,

$$|L(1, \chi)| > A_{23}\,k^{-\frac{1}{2}}. \tag{13}$$

By Lemma 4,

$$|L'(\sigma, \chi)| < A_8 \log^2 k \quad \text{if} \quad 1-\frac{1}{2\log k} \leqslant \sigma \leqslant 2,$$

and it follows that, in this domain, we have the following inequality :

$$|L(1, \chi)-L(\sigma, \chi)| = \left|\int_{\sigma}^{1} L'(s, \chi)\,ds\right| < |1-\sigma|\,A_8 \log^2 k.$$

This inequality, with (13), leads at once to the result that there is an absolute constant A_1 such that $L(\sigma, \chi) \neq 0$ for $\sigma > 1 - A_1/(\sqrt{k} \log^2 k)$.

3. LEMMA 8. *If $\xi \geqslant 2$, then, of all the L-functions formed with real primitive characters to moduli $k \leqslant \xi$, there is at most one function which has a real zero for $\sigma > 1 - A_{24}/\log \xi$.*

We first prove the following theorem, which is due to Landau*.

If ϑ_1 is the greatest real zero of an L-function corresponding to a real primitive character (mod k_1) and ϑ_2 of a similar L-function (mod k_2), where k_1 may be equal to k_2, but the two characters must be distinct, then

$$1 - \min(\vartheta_1, \vartheta_2) > A_{25}/\log k_1 k_2.$$

If $X_1(n)$ is a real primitive character (mod k_1) and $X_2(n)$ a real primitive character (mod k_2), we define characters $\chi_1(n)$, $\chi_2(n)$ to modulus $k_1 k_2$ by the rules:

$$\chi_r(n) = \left\{ \begin{array}{ll} X_r(n) & \text{if } (n, k_1 k_2) = 1 \\ 0 & \text{if } (n, k_1 k_2) > 1 \end{array} \right\} \quad (r = 1, 2).$$

It is easy to show† that $\chi_1(n)$ and $\chi_2(n)$ are different characters (mod $k_1 k_2$). Let $\chi_0(n)$ be the principal character (mod $k_1 k_2$) and $\chi_3(n) = \chi_1(n)\chi_2(n)$. Then

$$\sum_{\nu=0}^{3} -\frac{L'}{L}(s, \chi_\nu) = \sum_{p, m} \frac{\log p}{p^{ms}} \sum_{\nu=0}^{3} \chi_\nu(p^m) > 0 \quad (s > 1). \tag{14}$$

We obtain inequalities for the terms occurring on the left-hand side of (14). First we have

$$-\frac{L'}{L}(s, \chi_0) = -\sum_{p | k_1 k_2} \frac{\log p}{p^s - 1} - \frac{\zeta'}{\zeta}(s) < -\frac{\zeta'}{\zeta}(s) \tag{15}$$

$$< \frac{1}{s-1} + A_{26}.$$

Next, if χ denotes a real non-principal character (mod $k_1 k_2$) and $s_0 = 1 + 1/(b \log k_1 k_2)$, $L(s, \chi)$ is regular within the circle $|s - s_0| \leqslant \frac{1}{2}$

* Landau, *Göttinger Nachrichten*, 1918, 285–295. We shall simplify the details of Landau's proof by a use of Theorem 374 of Landau's *Vorlesungen über Zahlentheorie*, 2, 17.

† The result is obvious in the case $k_1 = k_2$. For the proof when $k_1 \neq k_2$ see, for example, Landau, *ibid.*, 279, foot-note.

where it satisfies the inequality*

$$\left| \frac{L(s, \chi)}{L(s_0, \chi)} \right| < e^M,$$

where $M = A_{27} \log k_1 k_2 + \log b$.

On applying Landau's Theorem 374 as in Lemma 6, we obtain the result

$$-\frac{L'}{L}(s_0, \chi) < 8(A_{27} \log k_1 k_2 + \log b) + \sum_\rho R\left(\frac{1}{\rho - s_0}\right). \tag{16}$$

Hence

$$-\frac{L'}{L}(s_0, \chi_3) < 8(A_{27} \log k_1 k_2 + \log b). \tag{17}$$

It is clear from (2) that the zeros of $L(s, \chi_r)$ which do not lie on the imaginary axis are those of $L(s, X_r)$, so that, if ϑ_1 is the greatest real zero of $L(s, X_1)$, and ϑ_2 the greatest real zero of $L(s, X_2)$ (we may suppose that both zeros lie within the circle $|s - s_0| \leqslant \frac{1}{4}$), we have at once, from (16),

$$-\frac{L'}{L}(s_0, \chi_r) < 8(A_{27} \log k_1 k_2 + \log b) - \frac{1}{s_0 - \vartheta_r} \quad (r = 1, 2). \tag{18}$$

On putting $s = s_0$ in (14), and using (15), (17), and (18), we obtain the result

$$b \log k_1 k_2 + A_{26} + 24(A_{27} \log k_1 k_2 + \log b) - \frac{1}{s_0 - \vartheta_1} - \frac{1}{s_0 - \vartheta_2} > 0,$$

which may be written in the form

$$\frac{1}{s_0 - \vartheta_1} + \frac{1}{s_0 - \vartheta_2} < \tfrac{3}{2} b \log k_1 k_2 \quad \text{if} \quad b > A_{28}.$$

If now $\vartheta = \min(\vartheta_1, \vartheta_2)$, we have

$$\frac{2}{s_0 - \vartheta} < \tfrac{3}{2} b \log k_1 k_2,$$

whence

$$1 - \vartheta > \frac{4}{3b \log k_1 k_2} - \frac{1}{b \log k_1 k_2} = \frac{1}{3b \log k_1 k_2}.$$

The theorem is thus proved, and the application to the proof of Lemma 8 is immediate.

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* See proof of Lemma 6.

Of all the L-functions formed with real primitive characters to moduli $k \leqslant \xi$, let $L_1(s)$, with modulus k_1, be that with the greatest real zero. Then, for any real zero ϑ of any other L-function (mod k_2), we have

$$1-\vartheta > \frac{A_{25}}{\log k_1 k_2} \geqslant \frac{A_{25}}{2\log \xi}.$$

LEMMA 9. *If $\xi \geqslant 2$, then, of all the L-functions formed with primitive characters to moduli $k \leqslant \xi$, there is at most one function which has a real zero for $\sigma > 1-A_{29}/\log \xi$. Also, if such a function exists, it is formed with a real non-principal character.*

We consider the real zeros of L-functions formed with primitive characters to moduli k, where $1 \leqslant k \leqslant \xi$.

In the case $k = 1$, there is a unique primitive character, and it is clear that we may choose A_{30} so that the associated L-function, which is $\zeta(s)$, has no real zeros for $\sigma > 1-A_{30}/\log \xi$.

If χ is a complex character to modulus $k \leqslant \xi$, then, by Lemma 5, $L(s, \chi)$ has no real zeros for $\sigma \geqslant 1-A_{10}/\log \xi$.

There remains for consideration the case of L-functions formed with real primitive characters to moduli k, where $2 \leqslant k \leqslant \xi$, and, on using Lemma 8, Lemma 9 follows immediately on taking

$$A_{29} = \min (A_{30}, A_{10}, A_{24}).$$

On using (2) we deduce from Lemma 9 the following result.

If $\xi \geqslant 2$, and if there are L-functions with moduli $k \leqslant \xi$ which possess a real zero for $\sigma > 1-A_{29}/\log \xi$, then such functions are formed with characters which are associated with a unique real primitive character.

LEMMA 10. *If χ is a non-principal character* (mod k), *and $L(s, \chi)$ has no zeros for $\sigma \geqslant 1-\delta$, $|t| \leqslant 5$, then*

$$\left|\frac{L'}{L}(s, \chi)\right| < \frac{A_{31}}{\delta} + A_{31}\log k \quad \left(\sigma \geqslant 1-\frac{\delta}{4},\ |t| \leqslant 4\right).$$

This is Lemma 10 of T.

LEMMA 11. *If χ is a real non-principal character* (mod k), *then $L(s, \chi)$ has at most a simple zero in the domain $\sigma \geqslant 1-A_{32}/\log k$, $|t| \leqslant 5$. If such a zero exists, let us denote it by β. Then*

$$\left|\frac{L'}{L}(s, \chi) - \frac{1}{s-\beta}\right| < A_{33}\log^3 k \quad \left(\sigma \geqslant 1-\frac{A_{32}}{4\log k},\ |t| \leqslant 4\right).$$

The first result follows from Lemmas 6 and 7 on taking

$$A_{32} = \min(A_{11}, A_{19}).$$

To prove the second result, let

$$f(s) = \log\left\{\frac{L(s,\chi)}{s-\beta}\right\}, \quad s_0 = \tfrac{3}{2}+it_0, \quad |t_0| \leqslant 4.$$

Then $f(s)$ is regular for $|s-s_0| \leqslant \frac{1}{2}+\frac{A_{32}}{2\log k}$, and $f(s_0) = O(1)$. Also

$$L(s,\chi)-L(\beta,\chi) = \int_\beta^s L'(u,\chi)\,du,$$

and hence, since $L(\beta,\chi) = 0$,

$$\left|\frac{L(s,\chi)}{s-\beta}\right| = \left|\frac{1}{s-\beta}\int_\beta^s L'(u,\chi)\,du\right| < A_9 k^{\frac{1}{2}} < k^{A_{34}}$$

by Lemma 4. It follows that $Rf(s) < A_{34}\log k$, and the theorem follows on applying Landau's Theorem 225 to

$$f(s) \quad \text{with} \quad r = \tfrac{1}{2}+A_{32}/(2\log k), \quad \rho = \tfrac{1}{2}+A_{32}/(4\log k).$$

LEMMA 12. *If χ_0 is the principal character* (mod k),

$$\left|\frac{L'}{L}(s,\chi_0)+\frac{1}{s-1}\right| < A_{35}\log k \quad (\tfrac{3}{4} \leqslant \sigma \leqslant 2, \ |t| \leqslant 6).$$

This is Lemma 7 of T.

4. In this section we obtain an asymptotic formula for $\pi(m;\ k, l)$ which holds uniformly in k. Until (32) we shall assume that

$$k \leqslant e^{\sqrt{\log m}}. \tag{19}$$

4.1. It is well known* that

$$-\frac{L'}{L}(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)\Lambda(n)}{n^s} \quad (\sigma > 1), \tag{20}$$

where

$$\Lambda(n) = \sum_{\substack{p,\,l\\ p^l=n}} \log p.$$

* Landau, *Vorlesungen über Zahlentheorie*, 1, 205.

Let

$$\theta(m) = \sum_{n=1}^{m} \Lambda(n)\chi(n), \tag{21}$$

and put

$$y = m+\tfrac{1}{2}, \quad a = 1+1/\log y, \quad T = e^{\sqrt{\log y}}. \tag{22}$$

On using a theorem of Landau* we have the following results:

$$\left|\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{y^s}{s}\chi(n)\Lambda(n)n^{-s}\,ds - \chi(n)\Lambda(n)\right| \leqslant \frac{1}{T}\frac{y^a n^{-a}|\chi(n)\Lambda(n)|}{|\log y-\log n|} \quad \text{if} \quad n<y;$$

$$\left|\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{y^s}{s}\chi(n)\Lambda(n)n^{-s}\,ds\right| \leqslant \frac{1}{T}\frac{y^a n^{-a}|\chi(n)\Lambda(n)|}{|\log y-\log n|} \quad \text{if} \quad n>y.$$

It follows immediately from these and (20) and (21) that

$$\left|\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{y^s}{s}\left(-\frac{L'}{L}(s,\chi)\right)ds - \theta(m)\right| \leqslant \frac{1}{T}\sum_{n=1}^{\infty}\frac{\Lambda(n)y^a n^{-a}}{|\log y-\log n|} = e^{-\sqrt{\log y}}(S_1+S_2), \tag{23}$$

where

$$S_1 = \sum_{\substack{n\\ \frac{1}{2}y\leqslant n\leqslant 2y}}\frac{\Lambda(n)y^a n^{-a}}{|\log y-\log n|},$$

$$S_2 = \sum_{\substack{n\\ n<\frac{1}{2}y}}\frac{\Lambda(n)y^a n^{-a}}{|\log y-\log n|} + \sum_{\substack{n\\ n>2y}}\frac{\Lambda(n)y^a n^{-a}}{|\log y-\log n|}.$$

It is necessary to evaluate S_1 and S_2.

$$S_1 \leqslant A_{36}\sum_{\substack{n\\ \frac{1}{2}y\leqslant n\leqslant 2y}}\frac{\log n}{|\log y-\log n|} \leqslant A_{36}\log 2y \sum_{\substack{n\\ \frac{1}{2}y\leqslant n\leqslant 2y}}\frac{1}{|\log\{1+(n-y)/y\}|}$$

$$\leqslant A_{36}\log 2y \sum_{\substack{n\\ \frac{1}{2}y\leqslant n\leqslant 2y}} A_{37}\left|\frac{y}{n-y}\right| = O(y\log^2 y).$$

Next consider S_2. Since $|\log y-\log n|>\log 2$ if $n<\frac{1}{2}y$ or $n>2y$, we have

$$S_2 < A_{38}y\sum_{n=1}^{\infty}\Lambda(n)n^{-a} = A_{38}y\,O\left(\frac{1}{a-1}\right) = O(y\log y).$$

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* Landau, *Primzahlen*, 1, 342.

It thus follows from (22) and (23) that

$$\left|\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{y^s}{s}\left(-\frac{L'}{L}(s,\chi)\right)ds-\theta(m)\right| = O(me^{-A\sqrt{\log m}}). \tag{24}$$

4.2. If χ_0 is the principal character (mod k), the zeros of $L(s, \chi_0)$ for $\sigma \neq 0$ are those of $\zeta(s)$, and hence we may choose A_{39} so that $L(s, \chi_0)$ has no zeros for $\sigma \geqslant 1-A_{39}/\log k$, $|t| \leqslant 5$. Thus, on taking

$$A_{40} = \tfrac{1}{4}\min(A_{39}, A_{10}, A_{11}, A_{32}, A_{29}),$$

it is clear from Lemmas 5 and 6 that the only possible zeros of $L(s, \chi)$ for $\sigma \geqslant 1-4A_{40}/\log k$, $|t| \leqslant 5$, occur when χ is a real non-principal character, when there may be zeros on the real axis, and for each such χ, by Lemma 11, there is at most a simple zero β in the domain. Further, we deduce from Lemma 9 and (2) that of all the L-functions to modulus k, where $k \geqslant 2$, there is at most one function which has a real zero for $\sigma \geqslant 1-4A_{40}/\log k$. If such a function exists, let us denote it by $L(s, \chi_1)$; then the only zero of $L(s, \chi_1)$ in the domain $\sigma \geqslant 1-4A_{40}/\log k$, $|t| \leqslant 5$, is a simple real zero β.

To evaluate the integral in (24), we move the contour to the following curve Γ_1:

$$\sigma = 1-\frac{A_4}{\log|kt|}, \quad T \geqslant |t| > 4;$$

$$\sigma = 1-\frac{A_{40}}{\log k}, \quad |t| \leqslant 4,$$

joining up by straight lines.

It is clear from the choice of A_{40} and Lemma 1 that within and on the contour formed by the old and new paths of integration the integrand is regular unless (1) $\chi = \chi_0$, the principal character, when the integrand has a simple pole at $s = 1$; or (2) $\chi = \chi_1$, when three cases arise according as

$$\beta \geqslant 1-3A_{40}/(4\log k),\ 1-5A_{40}/(4\log k) < \beta < 1-3A_{40}/(4\log k),$$

or

$$\beta \leqslant 1-5A_{40}/(4\log k);$$

in the first case the integrand has a simple pole at $s = \beta$; in the second case we indent Γ_1 by the right-hand portion γ of the circle $|s-\beta| = A_{40}/(2\log k)$, and call the new path of integration Γ_2; in the third case the integrand is regular.

Hence, on writing

$$I_r = \frac{1}{2\pi i}\int_{\Gamma_r}\frac{y^s}{s}\left(-\frac{L'}{L}(s,\chi)\right)ds \quad (r = 1, 2),$$

and using (24), we have the following result:

$$(25)\quad \theta(m)=\begin{cases} I_1+m+O(me^{-A\sqrt{\log m}}) & \text{if } \chi=\chi_0; \\ I_1+O(m^{\beta})+O(me^{-A\sqrt{\log m}}) & \text{if } \chi=\chi_1 \text{ and } \beta\geqslant 1-3A_{40}/(4\log k); \\ I_2+O(me^{-A\sqrt{\log m}}) & \text{if } \chi=\chi_1 \text{ and } 1-5A_{40}/(4\log k)<\beta<1-3A_{40}/(4\log k); \\ I_1+O(me^{-A\sqrt{\log m}}) & \text{if } \chi=\chi_1 \text{ and } \beta\leqslant 1-5A_{40}/(4\log k); \\ I_1+O(me^{-A\sqrt{\log m}}) & \text{if } \chi \text{ is any other character.} \end{cases}$$

We now calculate the order of I_1. First it is clear from Lemma 2 that

$$\left|\int_{a-iT}^{b-iT}+\int_{b+iT}^{a+iT}\frac{y^s}{s}\frac{L'}{L}(s,\chi)\,ds\right|=O\left(\frac{y}{T}\log kT\right)=O(me^{-A\sqrt{\log m}}),$$

on using (19) and (22), where $b=1-A_4/\log kT$.

Again, by an application of Lemma 2, we see that the order of the integrand on the curved portion of Γ_1 is

$$O\left(\frac{y^b}{|s|}\log kT\right)=O\left(me^{-A\sqrt{\log m}}\frac{1}{|s|}\right),$$

on using (19) and (22). The corresponding integral is of order

$$O(me^{-A\sqrt{\log m}})\int_4^T\frac{1}{|s|}\left|\frac{ds}{dt}\right|dt=O(me^{-A\sqrt{\log m}})\int_4^T\frac{dt}{t}$$
$$=O(me^{-A\sqrt{\log m}}\log T)=O(me^{-A\sqrt{\log m}}).$$

On the remaining portion of the path, which consists of three straight lines, we have the following results:

If $\chi=\chi_0$, $\left|\frac{L'}{L}(s,\chi)\right|<A\log k$ (Lemma 12); if $\chi=\chi_1$ and $\beta\geqslant 1-\frac{3A_{40}}{4\log k}$, or $\beta\leqslant 1-\frac{5A_{40}}{4\log k}$, $\left|\frac{L'}{L}(s,\chi)\right|<A\log^3 k$ [on the line $\sigma=1-A_{40}/\log k$, apply Lemma 11; on the lines $t=\pm 4$, apply Lemma 11 if $A_{40}/\log k>A_4/\log(4k)$, and otherwise Lemma 2]; if χ is any other character,

$$\left|\frac{L'}{L}(s,\chi)\right|<A\log k$$

[on the line $\sigma=1-A_{40}/\log k$, apply Lemma 10; on the lines $t=\pm 4$, apply Lemma 10 if $A_{40}/\log k>A_4/\log(4k)$, and otherwise Lemma 2].

Thus the order of the integrand on these straight lines is

$$O(y^{1-A/\log k}\log^3 k) = O(me^{-A\sqrt{\log m}})$$

by (19) and (22), and the corresponding integral has the same order.

It is now clear that we may write $I_1 = O(me^{-A\sqrt{\log m}})$ in (25).

We next consider I_2. Apart from the term $\int_\gamma \frac{y^s}{s}\frac{L'}{L}(s,\chi)\,ds$, the calculation of the order of I_2 is similar to that of I_1.

By Lemma 11, it is clear that, on γ, $\frac{L'}{L}(s,\chi) = O(\log^3 k)$, and hence

$$\int_\gamma \frac{y^s}{s}\frac{L'}{L}(s,\chi)\,ds = O\left(y^{1-A_{40}/(4\log k)}\log^3 k\,\frac{1}{\log k}\right) = O(me^{-A\sqrt{\log m}}).$$

Hence $I_2 = O(me^{-A\sqrt{\log m}})$.

We now replace I_1 and I_2 in (25) by $O(me^{-A\sqrt{\log m}})$, and further simplify this formula by using the fact that $\beta = \sigma_1$ if $\sigma_1 \geqslant 1 - 4A_{40}/\log k$, where σ_1 has been defined in the introduction. We obtain

$$\theta(m) = \begin{cases} m + O(me^{-A\sqrt{\log m}}) & \text{if } \chi = \chi_0; \\ O(me^{-A\sqrt{\log m}}) + O(m^{\sigma_1}) & \text{if } \chi = \chi_1; \\ O(me^{-A\sqrt{\log m}}) & \text{if } \chi \text{ is any other character.} \end{cases} \tag{26}$$

4.3. We define

$$\tau(m) = \sum_{\substack{n=1 \\ n \equiv l \,(\mathrm{mod}\, k)}}^{m} \Lambda(n). \tag{27}$$

Then

$$\tau(m) = \frac{1}{h}\sum_{n=1}^{m}\Lambda(n)\sum_\chi \frac{1}{\chi(l)}\chi(n),$$

where $h = \phi(k)$, and the last summation is taken over all characters χ with modulus k. On inverting the order of summation and using (21), we have

$$\tau(m) = \frac{1}{h}\sum_\chi \frac{1}{\chi(l)}\sum_{n=1}^{m}\Lambda(n)\chi(n) = \frac{1}{h}\sum_\chi \frac{1}{\chi(l)}\theta(m),$$

and hence, (by 26),

$$\tau(m) = \frac{m}{h} + O\left(\frac{m^{\sigma_1}}{h}\right) + O(me^{-A\sqrt{\log m}}). \tag{28}$$

Next we define $\psi(m)$ by the equation

$$\psi(m) = \sum_{\substack{n=2 \\ n \equiv l \pmod k}}^{m} \frac{\Lambda(n)}{\log n}. \tag{29}$$

Then

$$\begin{aligned}\psi(m) &= \sum_{\substack{p,\ v \\ p^v \leqslant m,\ p^v \equiv l \pmod k}} v^{-1} \qquad (30)\\ &= \pi(m;\ k,\ l) + \sum_{2 \leqslant v \leqslant \log m/\log 2} v^{-1} \sum_{\substack{p \\ p \leqslant m^{1/v},\ p^v \equiv l \pmod k}} 1 \\ &= \pi(m;\ k,\ l) + O\Big(\sum_{v \leqslant 2 \log m} \sqrt{m} \Big) \\ &= \pi(m;\ k,\ l) + O(\sqrt{m} \log m).\end{aligned}$$

On writing

$$\tau(n) = \frac{n}{h} + \Delta(n), \tag{31}$$

it is obvious from (29) and (27) that

$$\begin{aligned}\psi(m) &= \sum_{n=2}^{m} \frac{\tau(n) - \tau(n-1)}{\log n} = \sum_{n=2}^{m} \frac{1}{\log n} \left\{ \frac{1}{h} + \Delta(n) - \Delta(n-1) \right\} \\ &= \frac{1}{h} \sum_{n=2}^{m} \frac{1}{\log n} + \frac{\Delta(m)}{\log m} - \frac{\Delta(1)}{\log 2} + \sum_{n=2}^{m-1} \Delta(n) \left\{ \frac{1}{\log n} - \frac{1}{\log (n+1)} \right\} \\ &= \frac{1}{h} \sum_{n=2}^{m} \frac{1}{\log n} + \frac{\Delta(m)}{\log m} + O(1) + \Sigma_1 + \Sigma_2,\end{aligned}$$

where

$$\Sigma_1 = \sum_{n=2}^{[m^{\frac{1}{2}}]} \Delta(n) \left\{ \frac{1}{\log n} - \frac{1}{\log (n+1)} \right\} = O\Big(\sum_{n=2}^{[m^{\frac{1}{2}}]} n \Big) = O(m^{\frac{1}{2}})$$

by (28) and (31);

$$\begin{aligned}\Sigma_2 &= \sum_{n=[m^{\frac{1}{2}}]+1}^{m-1} \Delta(n) \left\{ \frac{1}{\log n} - \frac{1}{\log (n+1)} \right\} \\ &= \left[O\left(\frac{m^{\sigma_1}}{h} \right) + O(m e^{-A\sqrt{\log m}}) \right] \sum_{n=[m^{\frac{1}{2}}]+1}^{m-1} \left\{ \frac{1}{\log n} - \frac{1}{\log (n+1)} \right\} \\ &= \left[O\left(\frac{m^{\sigma_1}}{h} \right) + O(m e^{-A\sqrt{\log m}}) \right] \frac{1}{\log m}.\end{aligned}$$

It follows that

$$\psi(m) = \frac{1}{h} \sum_{n=2}^{m} \frac{1}{\log n} + O\left(\frac{m^{\sigma_1}}{h \log m}\right) + O(me^{-A\sqrt{\log m}})$$

and on using (30) we can write this equation in the form

$$\pi(m;\ k,\ l) = \frac{1}{h} \int_2^m \frac{du}{\log u} + O\left(\frac{m^{\sigma_1}}{h \log m}\right) + O(me^{-A\sqrt{\log m}}). \tag{32}$$

This formula has been proved for values of k satisfying the inequality $k \leqslant e^{\sqrt{\log m}}$. If $k > e^{\sqrt{\log m}}$, it is clear from the elementary results

$$\pi(m;\ k,\ l) < 1 + mk^{-1} < 1 + me^{-\sqrt{\log m}},$$

$$\frac{1}{h} \int_2^m \frac{du}{\log u} = O\left(\frac{m}{h \log m}\right) = O\left(\frac{m}{\sqrt{k}}\right) = O(me^{-\frac{1}{2}\sqrt{\log m}})$$

that the formula (32) still holds, and hence Theorem I is proved.

For the purpose of the next paragraph we shall require some results which are a direct consequence of Theorems I, II, and Lemma 9, but which have not been stated explicitly in the introduction.

Suppose that σ_2, the greatest real zero possessed by L-functions with moduli $k \leqslant e^{\sqrt{\log m}}$, satisfies the inequality $\sigma_2 \leqslant 1 - A_{29}/\sqrt{\log m}$. Then, if $k \leqslant e^{\sqrt{\log m}}$,

$$O\left(\frac{m^{\sigma_1}}{\phi(k) \log m}\right) = O(m^{1-A/\sqrt{\log m}}) = O(me^{-A\sqrt{\log m}});$$

if $k > e^{\sqrt{\log m}}$,

$$O\left(\frac{m^{\sigma_1}}{\phi(k) \log m}\right) = O\left(\frac{m}{\sqrt{k}}\right) = O(me^{-A\sqrt{\log m}}).$$

Hence, by (1),

$$\pi(m;\ k,\ l) = \frac{1}{\phi(k)} \int_2^m \frac{du}{\log u} + O(me^{-A\sqrt{\log m}}). \tag{33}$$

If, on the other hand, the above hypothesis is not true, then, by Lemma 9 and (2), all L-functions with moduli $k \leqslant e^{\sqrt{\log m}}$ which possess a real zero for $\sigma > 1 - A_{29}/\sqrt{\log m}$ are formed with characters which are associated with a unique, real, primitive, and non-principal character; let k_1 be the modulus of this character. Then we have, from (1) and Theorem II,

$$\pi(m;\ k,\ l) = \frac{1}{\phi(k)} \int_2^m \frac{du}{\log u} + O(me^{-A\sqrt{\log m}}) \tag{34}$$

if k_1+k and $k \leqslant e^{\sqrt{\log m}}$;

$$(35) \qquad \pi(m;\ k,\ l) = \frac{1}{\phi(k)} \int_2^m \frac{du}{\log u} + O(me^{-A\sqrt{\log m}}) + O\left(\frac{1}{\phi(k)\log m}\, m^{1-A/(\sqrt{k_1}\log^2 k_1)}\right)$$

if $k_1 | k$ and $k \leqslant e^{\sqrt{\log m}}$.

It is clear that the results (34) and (35) continue to hold if the restriction $k \leqslant e^{\sqrt{\log m}}$ is removed. Finally, the result

$$(36) \qquad \pi(m;\ k,\ l) = \frac{1}{\phi(k)} \int_2^m \frac{du}{\log u} + O(me^{-A\sqrt{\log m}}) + O\left(\frac{1}{\phi(k)\log m}\, m^{1-A/(\sqrt{k}\log^2 k)}\right)$$

holds for all values of k.

5. It is the object of this section to make an application of the main results of the paper to prove the following theorem.

THEOREM III. *Let $Q(m)$ denote the number of representations of an integer m as the sum of a prime and a quadratfrei number. Then*

$$Q(m) = c\left\{\int_2^m \frac{du}{\log u}\right\} \prod_{p|m}\left(\frac{p^2-p}{p^2-p-1}\right) + O\left\{\frac{m}{\log^5 m}(\log\log m)^8 \log\log\log m\right\},$$

where

$$c = \prod_p [1-(p^2-p)^{-1}].$$

The problem of finding an asymptotic formula for $Q(m)$ has been considered by Estermann*, who proves the result

$$Q(m) \sim \frac{cm}{\log m} \prod_{p|m}\left(\frac{p^2-p}{p^2-p-1}\right).$$

It is clear that the result of Theorem III implies that

$$Q(m) = \frac{cm}{\log m} \prod_{p|m}\left(\frac{p^2-p}{p^2-p-1}\right) + O\left(\frac{m}{\log^2 m}\right).$$

On using the result

$$\sum_{\substack{n,s \\ n^2 s = h}} \mu(n) = \begin{cases} 1 & \text{if } h \text{ is quadratfrei,} \\ 0 & \text{otherwise,} \end{cases}$$

* T. Estermann, *Journal London Math. Soc.*, 6 (1931), 219–221.

we may write $Q(m)$ in the form

$$Q(m) = \sum_{\substack{n,\ s,\ p,\\ n^2 s+p=m}} \mu(n) = \sum_{n<\sqrt{m}} \mu(n) \sum_{\substack{p\\ p\equiv m \pmod{n^2}\\ p\leqslant m-n^2}} 1$$

$$= \sum_{n<\sqrt{m}} \mu(n)\{\pi(m;\ n^2,\ m)+O(1)\} = \sum_{n<\sqrt{m}} \mu(n)\,\pi(m;\ n^2,\ m)+O(\sqrt{m}).$$

Let

$$Q_1 = \sum_{\substack{n\\ n<\sqrt{m},\ (n,\ m)=1}} \mu(n)\,\pi(m;\ n^2,\ m).$$

Then, since $\pi(m;\ n^2,\ m) \leqslant 1$ if $(n,\ m) > 1$, it follows that

$$Q(m) = Q_1+O(\sqrt{m}). \tag{37}$$

Next, we define Q_2 by the formula

$$Q_2 = \left\{\int_2^m \frac{du}{\log u}\right\} \sum_{\substack{n\\ n<\sqrt{m},\ (n,\ m)=1}} \frac{\mu(n)}{\phi(n^2)}. \tag{38}$$

Then

$$|Q_1-Q_2| < Q_3+Q_4, \tag{39}$$

where

$$Q_3 = \sum_{n<\log^5 m} \left|\pi(m;\ n^2,\ m)-\frac{1}{\phi(n^2)}\int_2^m \frac{du}{\log u}\right|;$$

$$Q_4 = \sum_{\log^5 m\leqslant n<\sqrt{m}} \left|\pi(m;\ n^2,\ m)-\frac{1}{\phi(n^2)}\int_2^m \frac{du}{\log u}\right|.$$

Q_4 is evaluated in an elementary manner. It is obvious that $\pi(m;\ n^2,\ m) < 1+mn^{-2}$, and hence

$$Q_4 = O\left\{\sum_{\log^5 m\leqslant n<\sqrt{m}} (1+mn^{-2})\right\}+O\left(\frac{m}{\log m}\right)\sum_{n\geqslant\log^5 m} \frac{1}{\phi(n^2)} \tag{40}$$

$$= O\left(\frac{m}{\log^5 m}\right)+O\left(\frac{m}{\log m}\right)\sum_{n\geqslant\log^5 m} n^{-\frac{3}{5}}$$

$$= O\left(\frac{m}{\log^5 m}\right).$$

Next consider Q_3. If (33) holds, we have

$$Q_3 = O(me^{-A\sqrt{\log m}}) \sum_{n<\log^5 m} 1 = O(me^{-A\sqrt{\log m}}). \tag{41}$$

Otherwise we use (34) and (35) and obtain

$$(42)\quad Q_3 = O(me^{-A\sqrt{\log m}}) \sum_{n<\log^5 m} 1 + O\left(\frac{m}{\log m}\right) \sum_{\substack{n \\ n<\log^5 m \\ n^2\equiv 0 (\mathrm{mod}\, k_1)}} \frac{1}{\phi(n^2)} e^{-A\log m/(\sqrt{k_1}\log^2 k_1)}$$

$$= O(me^{-A\sqrt{\log m}}) + O\left(\frac{m}{\log m} e^{-A\log m/(\sqrt{k_1}\log^2 k_1)}\right) \sum_{\substack{n \\ n<\log^5 m \\ n^2\equiv 0 (\mathrm{mod}\, k_1)}} \frac{1}{\phi(n^2)},$$

where k_1 depends upon m and satisfies the inequality $k_1 \leqslant e^{\sqrt{\log m}}$.

5.2. The number k_1 is the modulus of a real primitive character and we now investigate the possible values of k_1.

First it is clear that the condition $\chi(mK+1) = 0$ or 1 for every integer m, where $K \mid k$ and $K < k$, is a necessary condition that a character $\chi(n)$ to modulus k should be an imprimitive character. The condition is also sufficient. For if $(n_1, k) = 1$, we can choose n so that $nn_1 \equiv 1 \pmod{k}$; then it is clear that $(n, k) = 1$, $\chi(nn_1) = 1$, and $nn_1 \equiv 1 \pmod{K}$. Now suppose that $n_1 \equiv n_2 \pmod{K}$, where $(n_2, k) = 1$. Then $nn_2 \equiv nn_1 \equiv 1 \pmod{K}$, from which we have, on using the condition $\chi(mK+1) = 1$ if $mK+1$ is prime to k, $\chi(nn_2) = \chi(nn_1) = 1$, and hence $\chi(n_1) = \chi(n_2)$, which shows that χ is an imprimitive character (mod k).

Now if χ is a real character, $\chi(x^2) = [\chi(x)]^2 = 1$ or 0, and we deduce the following result:

If $mK+1$ is a quadratic residue (mod k) for all values of m, where $K \mid k$ and $K < k$, then there is no real primitive character (mod k).

This result enables us to prove the following theorem:

If k is the modulus of a real primitive character, then $k/(4, k)$ is quadratfrei.

Proof. It is well known* that if $(n, p) = 1$ and n is a quadratic residue to modulus p^l, where p is an odd prime and $l \geqslant 1$, then n is also a quadratic residue to modulus p^{l+1}; if $(n, 2) = 1$ and n is a quadratic residue to modulus 2^l, where $l \geqslant 3$, then n is also a quadratic residue to modulus 2^{l+1}. Also†, if $(a, b) = 1$ and n is a quadratic residue to moduli a and b, then n is a quadratic residue to modulus ab.

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* Landau, *Vorlesungen über Zahlentheorie*, 1, 45, Satz 87.

† Landau, *ibid.*, 1, 34, Satz 71.

Let $k = 2^l k_1$, where k_1 is odd, and suppose that k_1 is not quadratfrei; then in the canonical decomposition of k_1 there is a factor p^λ, where $\lambda > 1$. It is clear that the number 1 is a quadratic residue to moduli $p^{\lambda-1}$ and k/p^λ. Hence, for all integer values of m, $1+mk/p$ is a quadratic residue to moduli $p^{\lambda-1}$ and k/p^λ. It follows that $1+mk/p$ is a quadratic residue to moduli p^λ and k/p^λ and hence to modulus k. Hence there is no real primitive character $(\mathrm{mod}\, k)$ unless k_1 is quadratfrei.

Again, if $l > 3$, a similar argument shows that there is no real primitive character $(\mathrm{mod}\, k)$.

It follows that, if k is the modulus of a real primitive character, $k = k_1$, $2k_1$, $4k_1$, or $8k_1$, where k_1 is odd and quadratfrei.

In the case $k = 2k_1$, $\phi(k) = \phi(k_1)$ and it is easy to see that all the characters $(\mathrm{mod}\, k)$ are imprimitive. Hence the theorem is proved.

It may be shown that in the cases $k = k_1$ or $4k_1$, there is precisely one real primitive character; in the case $k = 8k_1$, there are two real primitive characters.

5.3. We are now able to evaluate the series occurring on the right-hand side of (42).

We use the result*

$$\frac{1}{\phi(x)} = O\left(\frac{\log\log x}{x}\right)$$

and obtain

$$\sum_{\substack{n \\ n<\log^5 m \\ n^2 \equiv 0 \,(\mathrm{mod}\, k_1)}} \frac{1}{\phi(n^2)} = O\left\{ \sum_{\substack{n \\ n<\log^5 m \\ n^2 \equiv 0 \,(\mathrm{mod}\, k_1)}} \frac{\log\log n}{n^2} \right\}$$

$$= O(\log\log\log m) \sum_{\substack{n \\ n^2 \equiv 0 \,(\mathrm{mod}\, k_1)}} \frac{1}{n^2}.$$

We know from the last paragraph that k_1 or $k_1/4$ or $k_1/8$ is odd and quadratfrei, and it follows that we may write $n = rk_1/(2, k_1)$ in the last summation. Hence

$$\sum_{\substack{n \\ n<\log^5 m \\ n^2 \equiv 0 \,(\mathrm{mod}\, k_1)}} \frac{1}{\phi(n^2)} = O(\log\log\log m) \sum_r \frac{1}{r^2 {k_1}^2}$$

$$= O\left(\frac{\log\log\log m}{{k_1}^2}\right),$$

* Landau, *Primzahlen*, 1, 217.

and thus we may write (42) in the form

$$(43) \qquad Q_3 = O(me^{-A\sqrt{\log m}}) + O\left(\frac{m \log\log\log m}{\log m} \frac{e^{-A \log m/(\sqrt{k_1}\log^2 k_1)}}{k_1^2}\right).$$

In estimating the last error term we distinguish two cases. If $k_1 \geqslant \log^2 m$, it is clear that

$$\frac{e^{-A \log m/(\sqrt{k_1}\log^2 k_1)}}{k_1^2} = O\left(\frac{1}{\log^4 m}\right);$$

if $k_1 < \log^2 m$, we have, on using the result $e^{-x} < 4!/x^4$ $(x > 0)$,

$$\frac{e^{-A \log m/(\sqrt{k_1}\log^2 k_1)}}{k_1^2} = O\left(\frac{\log^8 k_1}{\log^4 m}\right) = O\left(\frac{(\log\log m)^8}{\log^4 m}\right).$$

It follows from (43) that

$$Q_3 = O(me^{-A\sqrt{\log m}}) + O\left(\frac{m}{\log^5 m} (\log\log m)^8 \log\log\log m\right),$$

and hence, on using (41), we see that in all cases

$$Q_3 = O\left(\frac{m}{\log^5 m} (\log\log m)^8 \log\log\log m\right).$$

Thus, from (39) and (40), we have

$$Q_1 - Q_2 = O\left(\frac{m}{\log^5 m} (\log\log m)^8 \log\log\log m\right),$$

and hence, from (37),

$$(44) \qquad Q(m) = Q_2 + O\left(\frac{m}{\log^5 m} (\log\log m)^8 \log\log\log m\right).$$

We now show that the finite series occurring in the formula (38), which defines Q_2, may be replaced by the infinite series without affecting the error term in (44). For let

$$(45) \qquad Q_5 = \left\{\int_2^m \frac{du}{\log u}\right\} \sum_{\substack{n \\ (n,\, m)=1}} \frac{\mu(n)}{\phi(n^2)}.$$

Then

$$|Q_2 - Q_5| = O\left(\frac{m}{\log m}\right) \sum_{n \geqslant \sqrt{m}} \frac{1}{\phi(n^2)}$$

$$= O\left(\frac{m}{\log m}\right) \sum_{n \geqslant \sqrt{m}} n^{-\frac{3}{2}} = O\left(\frac{m^{\frac{3}{4}}}{\log m}\right),$$

and hence, by (44),

$$Q(m) = Q_5 + O\left(\frac{m}{\log^5 m}(\log\log m)^8 \log\log\log m\right). \tag{46}$$

On noting that

$$\sum_{\substack{n \\ (n,\, m)=1}} \frac{\mu(n)}{\phi(n^2)} = \prod_{p \nmid m} [1-(p^2-p)^{-1}] = c \prod_{p \mid m} \left(\frac{p^2-p}{p^2-p-1}\right),$$

we have, from (45),

$$Q_5 = c\left\{\int_2^m \frac{du}{\log u}\right\} \prod_{p \mid m} \left(\frac{p^2-p}{p^2-p-1}\right),$$

which, with (46), proves Theorem III.

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A TABLE OF PARTITIONS*

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1. In connection with the Hardy-Ramanujan asymptotic formula† for $p(n)$, the number of partitions of n, valid for large values of n, a table of values of $p(n)$ for values of n up to 200 was computed by MacMahon. This table is published at the end of the Hardy-Ramanujan paper; the method of computation used by MacMahon consisted of direct applications of recurrence formulae derived from Euler's expansion

$$\sum_{n=0}^{\infty} p(n)x^n = \{(1-x)(1-x^2)(1-x^3)\ldots\}^{-1}$$
$$= \{1-x-x^2+x^5+x^7-\ldots\}^{-1},$$

in which the indices in the last expression are of the form $\frac{1}{2}m(3m\pm1)$.

I have constructed independently a table of values of $p(n)$ for values of n up to 300. My results up to 200 agree with those of MacMahon, and I have checked my value of $p(300)$ by means of the Hardy-Ramanujan formula. In §§ 2–4 I give a brief statement of the methods by which I was able to abbreviate considerably the actual work of calculation.

2. Let (n, m) denote the number of those partitions of n in which the smallest element that occurs is m. If such a partition is written down and one element equal to m is deleted, we are left with a partition of $n-m$ in which the smallest element that occurs is either m or some larger integer.

* I wish to express my grateful thanks to Prof. G. N. Watson for several useful suggestions in the preparation of this paper.

† G. H. Hardy and S. Ramanujan, "Asymptotic formulae in combinatory analysis", *Proc. London Math. Soc.* (2), 17 (1918), 75–115.

By considering all the (n, m) partitions in turn, we immediately obtain the formula

$$(n, m) = (n-m, m)+(n-m, m+1)+(n-m, m+2)+\ldots+(n-m, n-m),$$

whence the recurrence formula

$$(n, m) = (n-m, m)+(n+1, m+1) \tag{1}$$

follows at once.

This is the formula which I have used in the construction of my table. It leads to a table of double entry which is very much simpler than Euler's table of double entry described by Chrystal*.

The total number of partitions of n with this notation is $(n+1, 1)$. The expression

$$\frac{x^m}{(1-x^m)(1-x^{m+1})\ldots(1-x^{n-m})} = x^m \prod_{j=m}^{n-m} (1-x^j)^{-1}$$

may be regarded as a kind of generating function of (n, m), since

$$\sum_{n=m}^{\infty} (n, m)\, x^n = x^m \prod_{j=m}^{\infty} (1-x^j)^{-1}.$$

It may also be noticed that (n, m) is equal to the number of solutions of the equation

$$mx_1+(m+1)\,x_2+\ldots+nx_{n-m+1} = n-m$$

in positive integers (zero included).

3. Let $[x]$ denote the integral part of x. Then it is evident that

$$(n, n) = 1, \quad (n, m) = 0, \quad (n > m > [\tfrac{1}{2}n]).$$

These results may be combined in the formula

$$(n, > [\tfrac{1}{2}n]) = 1, \tag{2}$$

in which the expression on the left denotes the total number of partitions of n for which the smallest element is greater than $[\frac{1}{2}n]$.

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* G. Chrystal, *Algebra*, 2 (1900), 557–561.

Next we have

$$(n, m) = 1 \quad ([\tfrac{1}{2}n] \geqslant m > [\tfrac{1}{3}n]),$$

and hence

$$(n, >[\tfrac{1}{3}n]) = [\tfrac{1}{2}n] - [\tfrac{1}{3}n] + 1. \tag{3}$$

I then observed that the values of

$$(n, [\tfrac{1}{3}n]), \quad (n, [\tfrac{1}{3}n-1]), \quad (n, [\tfrac{1}{3}n-2]), \quad \ldots, \quad (n, [\tfrac{1}{4}n]+1)$$

form one or other of the three sets

(i) 3, 4, 6, 7, 9, 10, 12, 13, ...,

(ii) 2, 3, 5, 6, 8, 9, 11, 12, ...,

(iii) 2, 4, 5, 7, 8, 10, 11, 13, ...,

according as

$$n \equiv -1, \ 0, \ 1 \pmod 3,$$

the number of terms in the sets being $[\tfrac{1}{3}n] - [\tfrac{1}{4}n]$.

There are thus twenty-four forms obtainable for $(n, >[\tfrac{1}{4}n])$. If

$$n = 24m + 4k + l \quad (0 \leqslant k \leqslant 5, \ 0 \leqslant l \leqslant 3),$$

these are all included in the formula

$$(24m+4k+l, > 6m+k) = 3m^2 + (6+k+l)\,m + (k+l) + \left[\frac{(3+l)\,k+q}{7}\right], \tag{4}$$

where

$$q = 7, \ 1, \ 3, \ 0$$

according as

$$l = 0, \ 1, \ 2, \ 3.$$

4. The results of § 3 make it sufficient to consider (n, m) for values of m up to $[\tfrac{1}{4}n]$ only. My table of double entry consequently contains only seventy-five columns. A specimen of the calculations is as follows:

n	$(n, 1)$	$(n, 2)$	$(n, 3)$	$(n, 4)$	$(n, 5)$	$(n, 6)$	$(n, >[\tfrac{1}{4}n])$	
24	1255	210	60	24	10	6	+10	
25	1575	253	73	27	13	6	+11	
26		320	88	34	15	8	+13	
27			110	39	18	9	+15	
28				50	21	11	$(n, 7)$ 6	+12
29					26	12	7	+13
30						16	8	+16
31							10	+17

As each new row gets completed, a new diagonal is calculated, and this enables us to complete the next row. Thus the diagonal-elements in the specimen table are

$$10;\ 10+6=16;\ 16+10=26;\ 26+24=50;\ 50+60=110;\ \ldots;$$

the number in the last column with which we start being of course obtained from (4).

The following congruences, due to Ramanujan, were of considerable use in checking the accuracy of the results as the work of calculation proceeded.

$$(5m,\ 1)\equiv 0 \pmod 5,$$

$$(7m-1,\ 1)\equiv 0 \pmod 7,$$

$$(11m-4,\ 1)\equiv 0 \pmod{11}.$$

5. It may not be out of place here to give a direct method of calculating a table of partitions; the method is based on Euler's expansion already quoted.

In the column headed A are inserted alternately the values of $6p^2-7p+2$ and $6p^2-5p+1$ $(p=1,\ 2,\ 3,\ \ldots)$, and in column F are inserted the corresponding values of $6p^2-p$ and $6p^2+p$.

A	B	C	D	E	F
1	31	6842	3010	27	5
2	30	5604	1958	25	7
12	20	627	42	10	22
15	17	297	11	6	26
35					51
40					57
70					92
77					100
		13370	− 5021 = 8349.		

To find the number of partitions of any number, say 32, insert in columns B and E the numbers $(\geqslant 0)$ obtained by subtracting from 32 the numbers in columns A and F, and then insert in columns C and D the numbers of partitions of the numbers in columns B and E, $p(0)$ being taken to be 1. The difference of the sum of the entries in column C and the sum of the entries in column D is the required result.

6. It is easy to express the Hardy-Ramanujan asymptotic formula in the form

$$p(n) = \sum_{q=1}^{\nu} A_q \phi_q + O(n^{-\frac{1}{4}}),$$

where

$$\phi_q = B_n q^{\frac{1}{2}} \left(\frac{1}{q} - \frac{1}{\mu_n}\right) e^{\mu_n/q},$$

$$\mu_n = \frac{\pi\sqrt{(24n-1)}}{6}, \quad B_n = \frac{2\sqrt{3}}{24n-1},$$

and the A_q are given in Table II of the Hardy-Ramanujan paper. Taking $n = 300$, I find that

$$\mu_{300} = 44{\cdot}42574\ 39390\ 72800\ 69767,$$

$$1/\mu_{300} = 0{\cdot}02250\ 94711\ 15924\ 11447,$$

$$B_{300} = 0{\cdot}00048\ 11920\ 56554\ 76519.$$

Also

$$A_1 = 1, \quad A_2 = 1, \quad A_3 = 2\cos\frac{\pi}{18}, \quad A_4 = 2\cos\frac{\pi}{8},$$

$$A_5 = 4\cos^2\frac{\pi}{10}, \quad A_6 = 2\cos\frac{5\pi}{18},$$

$$A_7 = -2\left(2\cos\frac{5\pi}{14} + \cos\frac{3\pi}{14}\right), \quad A_8 = -2^{\frac{3}{2}}\cos\frac{3\pi}{16}.$$

The first eight terms of the formula give

$$\begin{aligned}
&+9\ 25308\ 29352\ 80987{\cdot}1131\\
&+14\ 41205{\cdot}5722\\
&+1377{\cdot}4002\\
&+26{\cdot}9413\\
&+4{\cdot}9914\\
&+0{\cdot}3589\\
&-0{\cdot}2884\\
&-0{\cdot}0847\\
\hline
&9\ 25308\ 29367\ 23602{\cdot}0040
\end{aligned}$$

which agrees with the result given in my table

$$p(300) = 9\,25308\,29367\,23602.$$

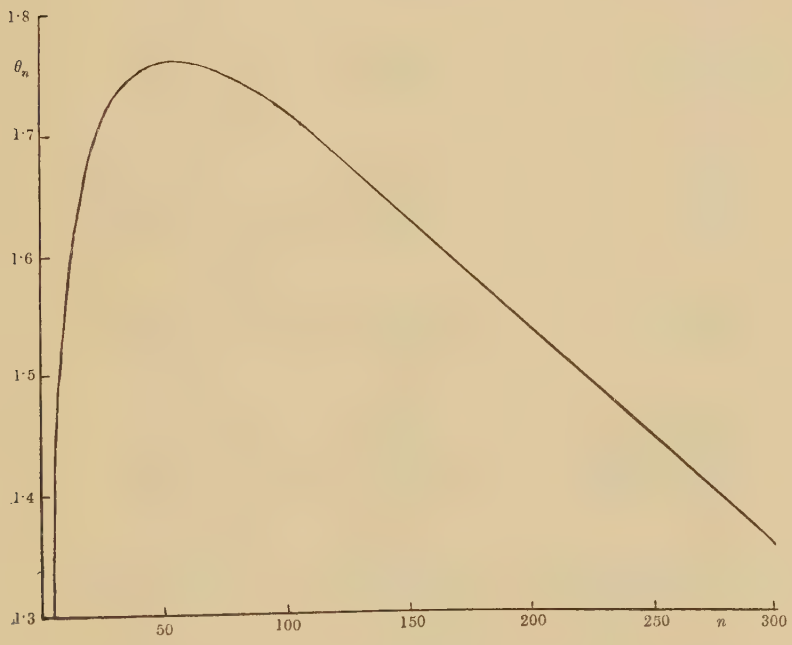
It may be observed that, within the limits of the table,

$$\sqrt{n} - \log_{10} p(n) = \theta_n,$$

where $\theta_n \leqslant 1{\cdot}761756$ and θ_n is a maximum when $n = 55$.

I give a graph of θ_n up to $n = 300$, from which it appears that, when n lies between 150 and 300, θ_n is very closely approximated by

$$1{\cdot}919 - 0{\cdot}00188n.$$



Graph of $\theta_n = \sqrt{n} - \log_{10} p(n)$.

n	$p(n)$	n	$p(n)$	n	$p(n)$
1	1	51	2 39943	101	2144 81126
2	2	52	2 81589	102	2412 65379
3	3	53	3 29931	103	2712 48950
4	5	54	3 86155	104	3048 01365
5	7	55	4 51276	105	3423 25709
6	11	56	5 26823	106	3842 76336
7	15	57	6 14154	107	4311 49389
8	22	58	7 15220	108	4835 02844
9	30	59	8 31820	109	5419 46240
10	42	60	9 66467	110	6071 63746
11	56	61	11 21505	111	6799 03203
12	77	62	13 00156	112	7610 02156
13	101	63	15 05499	113	8513 76628
14	135	64	17 41630	114	9520 50665
15	176	65	20 12558	115	10641 44451
16	231	66	23 23520	116	11889 08248
17	297	67	26 79689	117	13277 10076
18	385	68	30 87735	118	14820 74143
19	490	69	35 54345	119	16536 68665
20	627	70	40 87968	120	18443 49560
21	792	71	46 97205	121	20561 48051
22	1002	72	53 92783	122	22913 20912
23	1255	73	61 85689	123	25523 38241
24	1575	74	70 89500	124	28419 40500
25	1958	75	81 18264	125	31631 27352
26	2436	76	92 89091	126	35192 22692
27	3010	77	106 19863	127	39138 64295
28	3718	78	121 32164	128	43510 78600
29	4565	79	138 48650	129	48352 71870
30	5604	80	157 96476	130	53713 15400
31	6842	81	180 04327	131	59645 39504
32	8349	82	205 06255	132	66208 30889
33	10143	83	233 38469	133	73466 29512
34	12310	84	265 43660	134	81490 40695
35	14883	85	301 67357	135	90358 36076
36	17977	86	342 62962	136	1 00155 81680
37	21637	87	388 87673	137	1 10976 45016
38	26015	88	441 08109	138	1 22923 41831
39	31185	89	499 95925	139	1 36109 49895
40	37338	90	566 34173	140	1 50658 78135
41	44583	91	641 12359	141	1 66706 89208
42	53174	92	725 33807	142	1 84402 93320
43	63261	93	820 10177	143	2 03909 82757
44	75175	94	926 69720	144	2 25406 54445
45	89134	95	1046 51419	145	2 49088 58009
46	1 05558	96	1181 14304	146	2 75170 52599
47	1 24754	97	1332 30930	147	3 03886 71978
48	1 47273	98	1501 98136	148	3 35494 19497
49	1 73525	99	1692 29875	149	3 70273 55200
50	2 04226	100	1905 69292	150	4 08532 35313

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n	$p(n)$	n	$p(n)$	n	$p(n)$
151	4 50606 24582	201	432 83636 58647	251	24929 14511 68359
152	4 96862 88421	202	471 45668 86083	252	26923 27012 52579
153	5 47703 36324	203	513 42052 87973	253	29072 69579 16112
154	6 03566 73280	204	559 00883 17495	254	31389 19913 06665
155	6 64931 82097	205	608 52538 59260	255	33885 42642 48680
156	7 32322 43759	206	662 29877 08040	256	36574 95668 70782
157	8 06309 64769	207	720 68417 06490	257	39472 36766 55357
158	8 87517 78802	208	784 06562 26137	258	42593 30844 09356
159	9 76627 28555	209	852 85813 02375	259	45954 57504 48675
160	10 74381 59466	210	927 51025 75355	260	49574 19347 60846
161	11 81590 68427	211	1008 50658 85767	261	53471 50629 08609
162	12 99139 04637	212	1096 37072 05259	262	57667 26749 47168
163	14 27989 95930	213	1191 66812 36278	263	62183 74165 09615
164	15 69194 75295	214	1295 00959 25895	264	67044 81230 60170
165	17 23898 00255	215	1407 05456 99287	265	72276 09536 90372
166	18 93348 22579	216	1528 51512 48481	266	77905 06295 62167
167	20 78904 20102	217	1660 15981 07914	267	83961 17303 66814
168	22 82047 32751	218	1802 81825 16671	268	90476 01083 16360
169	25 04389 25115	219	1957 38561 61145	269	97483 43699 44625
170	27 47686 17130	220	2124 82790 09367	270	1 05019 74899 31117
171	30 13848 02048	221	2306 18711 73849	271	1 13123 85039 38606
172	33 04954 99613	222	2502 58737 60111	272	1 21837 43498 44333
173	36 23268 59895	223	2715 24089 25615	273	1 31205 18008 16215
174	39 71250 74750	224	2945 45499 41750	274	1 41274 95651 73450
175	43 51576 97830	225	3194 63906 96157	275	1 52098 04928 51175
176	47 67158 57290	226	3464 31263 22519	276	1 63729 39693 37171
177	52 21158 31195	227	3756 11335 82570	277	1 76227 84330 57269
178	57 17016 05655	228	4071 80636 27362	278	1 89656 41035 91584
179	62 58467 53120	229	4413 29348 84255	279	2 04082 58525 75075
180	68 49573 90936	230	4782 62397 45920	280	2 19578 63116 82516
181	74 94744 11781	231	5182 00518 38712	281	2 36221 91453 37711
182	81 98769 08323	232	5613 81486 70947	282	2 54095 25900 45698
183	89 66848 17527	233	6080 61354 38329	283	2 73287 31835 47535
184	98 04628 80430	234	6585 15859 70275	284	2 93892 97939 29555
185	107 18237 74337	235	7130 41855 14919	285	3 16013 78671 48997
186	117 14326 92373	236	7719 58926 63512	286	3 39758 40119 86773
187	128 00110 42268	237	8356 11039 25871	287	3 65243 08360 71053
188	139 83417 45571	238	9043 68396 68817	288	3 92592 21614 89422
189	152 72735 99625	239	9786 29337 03585	289	4 21938 85285 87095
190	166 77274 04093	240	10588 22467 22733	290	4 53425 31269 00886
191	182 07011 00652	241	11454 08845 53038	291	4 87203 80564 72084
192	198 72768 56363	242	12388 84430 77259	292	5 23437 10697 53672
193	216 86271 05469	243	13397 82593 44888	293	5 62299 26919 50605
194	236 60227 41845	244	14486 76924 96445	294	6 03976 38820 95515
195	258 08402 12973	245	15661 84125 27946	295	6 48667 41270 79088
196	281 45709 87591	246	16929 67223 91554	296	6 96585 01441 95831
197	306 88298 78530	247	18297 38898 54026	297	7 47956 50785 10584
198	334 53659 83698	248	19772 65166 81672	298	8 03024 83849 43040
199	364 60724 32125	249	21363 69198 20625	299	8 62049 62754 65025
200	397 29990 29388	250	23079 35543 64681	300	9 25308 29367 23602

GROUP CHARACTERS AND THE STRUCTURE OF GROUPS

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1. Simple methods for the calculation of the tables of characters of the symmetric groups are given in a paper by A. R. Richardson and myself, submitted to the Royal Society*. In the following paper a method is given for the calculation of the table of characters of a sub-group when the characters of the group are known. Hence, since any group is a sub-group of some symmetric group, a comparatively simple method is obtained for the calculation of the characters of any group.

Further, it is shown how many of the important properties of a group may be deduced from its table of characters, and hence is revealed a new method for the investigation of the structure of groups. At the end of the paper, together with the tables of characters of the symmetric groups of orders less than or equal to 9 !, is given a list of tables of characters of the more important sub-groups.

Some of the processes are tentative in nature, but they have the merit of practical utility. The tables of characters of the sub-groups, given at the end of this paper, were calculated by the methods given here. Most methods known for the determination of sub-groups are more or less tentative, and the method described here is no exception. However, many sub-groups may be found by this method, almost by inspection, and the easiest sub-groups to find are those with the largest orders, and these present most difficulty by other methods. The group of order 1344, which is discussed later, is an instance.

The possibilities of the method do not stop there. There is a possibility, almost a probability, of further theoretical developments that would render the methods much more definite, and make the exhaustive study of the sub-groups of large orders quite possible.

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* Littlewood and Richardson (13).

For example, there is good reason to suppose, though rigorous proof is lacking, that the coefficients of the simple characters in the compound character corresponding to a maximal sub-group are all zero or unity. This would follow, if the following theorem could be proved:

If G is a sub-group of H, and G has two linear invariants in an irreducible representation of H, then there is a sub-group G_1 of H, containing G, which has only one linear invariant.

In this form the theorem is plausible, and it is further found to be true in all known cases. If it could be proved, the exhaustive study of the sub-groups would be brought much nearer, for it is only necessary in the first instance to find the maximal sub-groups, the other sub-groups being found in turn from these.

Again, there is room for other discoveries concerning the possible compound characters, corresponding to sub-groups, that might greatly simplify the work. For example, in the case of the symmetric group, if the compound character includes the character $[1^n]$, then the sub-group is contained in the alternating group, and the compound character is self-associated. The compound character must obey many similar laws, which, if known, would greatly simplify the work.

2. *The theory of group characters.*

The theory of group characters* depends on the representation of an abstract group by means of isomorphic, but not necessarily simply isomorphic, groups of linear substitutions. The set of spurs of the corresponding matrices is the group *character*. There is an infinity of ways in which an abstract group may be represented by a group of linear substitutions, but the theory shows that, in general, such representations are reducible, that is, by transformation, each substitution is expressible as the direct sum† of the corresponding substitutions of a certain finite number of irreducible representations. The character of such a representation is, therefore, a linear function of the *simple* characters which belong to the component irreducible representations.

The number of these irreducible representations is the same as the number of classes of conjugate elements in a group, and a table of characters may therefore be written in the form of a square.

* Frobenius (5–11).

† Dickson (4), 33.

Every group has a simple character which takes the value unity for every substitution, corresponding to the representation of each element of the group by the unit matrix of degree 1.

The character elements are connected by the following very important orthogonal relations*.

Denote an element of the group by S, a class by ρ, and let h represent the order of the group, h_ρ the order of the class ρ. Let (λ) denote an irreducible representation with character $\chi^{(\lambda)}$, and represent the corresponding characteristic† of the element S by $\chi^{(\lambda)}(S)$, and of the class ρ by $\chi_\rho^{(\lambda)}$.

Then the following relations hold for real characters:

$$\sum_\lambda \chi_\rho^{(\lambda)} \chi_{\rho'}^{(\lambda)} = 0 \quad (\rho \neq \rho'),$$

$$h_\rho \sum_\lambda \chi_\rho^{(\lambda)^2} = h,$$

$$\sum_S \chi^{(\lambda)}(S)\chi^{(\mu)}(S) = 0 \quad [(\lambda) \neq (\mu)],$$

$$\sum_S \chi^{(\lambda)^2}(S) = h,$$

$$\sum_\rho h_\rho \chi_\rho^{(\lambda)} \chi_\rho^{(\mu)} = 0 \quad [(\lambda) \neq (\mu)],$$

$$\sum_\rho h_\rho \chi_\rho^{(\lambda)^2} = h.$$

The third and fourth equations are equivalent to the fifth and sixth, since the characteristics of the h_ρ elements of the class ρ are all equal. For complex characters, the second character in the product is replaced by the conjugate complex value.

3. *Determination of sub-groups.*

Let G be a sub-group of order g, of the group H of order h. Corresponding to this sub-group there is a representation of the group H as a permutation group‡.

Represent by A_1 the sum of the operations of the sub-group G. Let S_2 be any operation of H not included in A_1, and let $A_2 = S_2 A_1$. Similarly, S_3 is any operation not of A_1 or A_2, and $A_3 = S_3 A_1$.

* Frobenius (5).

† *I.e.* the value of the character for the element S.

‡ Burnside (1), 232.

Proceeding thus we obtain $\nu = h/g$ sets of operations

$$A_1, A_2, A_3, \ldots, A_\nu,$$

which include all the operations of H.

Pre-multiplication of any A_r by an operation of H will convert it into an A_s. If we multiply the sets

$$A_1, A_2, A_3, \ldots, A_\nu$$

on the left by an operation of H, these sets will be permuted amongst themselves. We thus obtain a permutation group which is isomorphic with the group H, and is therefore a representation of the group H as a permutation group.

Now any permutation group is simply isomorphic with a set of permutation matrices*. Hence we obtain a (reducible) representation of the group as a set of matrices. The spur of the matrix, which is the mark† of the corresponding cyclic sub-group of the permutation group, is therefore a compound character of H.

Thus, corresponding to any sub-group there is a representation as a permutation group, a matrix representation, and a compound character ϕ.

Also ϕ_0, the spur of the unit matrix, is the number of rows or columns of the matrix, namely $\nu = h/g$.

We next find $\phi(T)$, the value of the compound character for the operation T.

Multiply $$A_1, A_2, \ldots, A_\nu$$

on the left by T. We obtain

$$TA_1, TA_2, \ldots, TA_\nu.$$

Then $\phi(T)$ is the number of sets A_r that are left unchanged. Let A_r be any such set, and let S_r be any operation of A_r.

Then
$$TA_r = A_r,$$
$$A_r = S_r A_1,$$
$$TS_r A_1 = S_r A_1,$$
$$S_r^{-1} T S_r A_1 = A_1.$$

Hence $S_r^{-1} T S_r$ belongs to G.

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* *I.e.*, matrices of transformations which correspond to permutations.

† Burnside, *loc. cit.*, 236.

Corresponding to any set A_r which is unchanged when multiplied on the left by T, there will be g operations S_r such that $S_r^{-1} T S_r$ belongs to G.

Conversely, if $S_r^{-1} T S_r$ belongs to G and S_r belongs to A_r,

$$TA_r = A_r.$$

In all there will be $g\phi(T)$ operations S_r such that $S_r^{-1} T S_r$ belongs to G.

Let T belong to a class ρ containing h_ρ operations of H, of which g_ρ belong to G. If S_r runs through the h operations of H, $S_r^{-1} T S_r$ will run through the h_ρ operations of ρ, taking the value of each operation of ρ exactly h/h_ρ times. Consequently, hg_ρ/h_ρ of the operations $S_r^{-1} T S_r$ belong to G.

Thus
$$hg_\rho/h_\rho = g\phi(T),$$
$$\phi(T) = hg_\rho/gh_\rho.$$

It is clear, then, that from a knowledge of the sub-group and the number of operations in the classes, the compound character $\phi(T)$ may be obtained immediately.

Conversely, a knowledge of the character $\phi(T)$ defines the number of elements of each class that appear in the group G. This does not, of course, define the group G uniquely, unless it is self-conjugate, for any transform of G will correspond to the same character ϕ. We shall show later how to deduce from the table of characters of H, and the compound character ϕ, the table of characters of the sub-group G, and since it will further become apparent that all, or at least most, of the properties of the group are deducible from the table of characters, it may be inferred that, corresponding to one compound character ϕ, there can be but one conjugate set of sub-groups.

The problem of finding the sub-groups of a group, then, becomes the problem of finding suitable compound characters ϕ. Before seeking methods for solving this problem, we pass on to the characters of the sub-groups.

4. *Deduction of the characters of a sub-group from those of the group.*

We make use of Frobenius' relations between the characters of a group and those of a sub-group. Let the group H of order h have a sub-group G of order g, and let the class ρ of H contain h_ρ operations, of which g_ρ belong to G, and in G these operations split up into classes ρ', ρ'', etc., containing $g_{\rho'}$, $g_{\rho''}$, ..., operations respectively, so that

$$g_\rho = g_{\rho'} + g_{\rho''} + \ldots.$$

Denote by $\chi_\rho^{(i)}$ a character of H and by $\phi_\rho^{(j)}$ a character of G.

Then Frobenius proves that*

$$\chi_\rho^{(i)} = \sum_j g_{ij} \phi_{\rho'}^{(j)}, \tag{1}$$

$$\sum_{\rho'} \frac{h g_{\rho'}}{g h_\rho} \phi_{\rho'}^{(j)} = \sum_i g_{ij} \chi_\rho^{(i)}, \tag{2}$$

the summation on the left of (2) being taken over the classes ρ', ρ'', etc., into which ρ separates, and the coefficients g_{ij}, which are positive integers not greater than $\phi_0^{(j)}$ or $\chi_0^{(i)}$, being the same in both sets of equations.

Equation (2) may be written

$$\frac{h g_\rho}{g h_\rho} \phi_\rho^{(j)} = \sum_i g_{ij} \chi_\rho^{(i)}, \tag{3}$$

where $\phi_\rho^{(j)}$ is the mean character of the classes ρ', ρ'', etc.

Now suppose that we know the table of characters of a group H, and the value of $h g_\rho / (g h_\rho)$ for each class of H.

Equation (1) shows that the simple characters of H are compound characters of G, and we obtain as many compound characters of G as there are classes of H. If, however, a class of H contains no operation of G, these compound characters will be linearly dependent. We obtain as many linearly independent compound characters of G as there are classes of H containing operations of G. If each class ρ of H corresponds to but one class of G, we have sufficient compound characters to determine all the characters of G.

In general, however, some of the classes ρ of H will separate into two or more classes ρ', ρ'', ... in G. For each extra class introduced thus, we lack one equation to determine the characters of G, for the characters of H will only give us those compound characters of G which take the same values for the classes ρ', ρ'', etc. To provide the extra equations we use Frobenius' second equation.

First simplify the known compound characters as far as possible. The multiplicity of any compound character may be deduced from the orthogonal relations. The number of simple characters common to two characters may also be found. If one compound character entirely includes another, the latter may be subtracted, and a simpler character obtained. This will be illustrated in the example.

* Frobenius (8).

Now consider two compound characters of G, ψ and ψ', which have just one simple character in common. From (3),

$$\frac{hg_\rho}{gh_\rho}\phi_\rho^{(j)} = \Sigma_i g_{ij}\chi_\rho^{(i)}, \tag{4}$$

$\phi^{(j)}$ being the simple common character, in the case of classes of H which separate into more than one class of G, $\phi_\rho^{(j)}$ representing the mean character. We need first to determine the coefficients g_{ij}.

Since ψ and ψ' are compound characters,

$$\begin{cases} \dfrac{hg_\rho}{gh_\rho}\psi_\rho = \Sigma p_i\chi_\rho^{(i)}, \\ \dfrac{hg_\rho}{gh_\rho}\psi_\rho' = \Sigma q_i\chi_\rho^{(i)}. \end{cases} \tag{5}$$

p_i and q_i can be found as follows. Multiply the first equation of (5) by $h_\rho\chi_\rho^{(k)}$ and sum for all the classes.

$$\Sigma_\rho h_\rho \frac{hg_\rho}{gh_\rho}\psi_\rho\chi_\rho^{(k)} = \Sigma p_i h_\rho \chi_\rho^{(i)}\chi_\rho^{(k)} = hp_k.$$

Hence

$$p_k = \frac{1}{g}\Sigma g_\rho\psi_\rho\chi_\rho^{(k)}, \tag{6}$$

and, similarly,

$$q_k = \frac{1}{g}\Sigma g_\rho\psi_\rho'\chi_\rho^{(k)}.$$

Since ψ and ψ' both contain $\phi^{(j)}$,

$$p_i \geqslant g_{ij},$$

$$q_i \geqslant g_{ij}.$$

Hence $\Sigma g_{ij}\chi^{(i)}$ is contained in the common part of $\Sigma p_i\chi^{(i)}$ and $\Sigma q_i\chi^{(i)}$. In many cases $\Sigma g_{ij}\chi^{(i)}$ will be equal to the common part of $\Sigma p_i\chi^{(i)}$ and $\Sigma q_i\chi^{(i)}$, but it may happen that two simple characters of G, included respectively in ψ and ψ', correspond to two compound characters of H that have a simple character in common. If this is so, this simple character also will be included in the common part of $\Sigma p_i\chi^{(i)}$ and $\Sigma q_i\chi^{(i)}$. However, even in this case the correct compound character $\Sigma g_{ij}\chi^{(i)}$ is not difficult to find, since, first, $\Sigma g_{ij}\chi_0^{(i)}$ must be divisible by h/g from (3), and, secondly, $\Sigma g_{ij}\chi^{(i)}$ must vanish for those classes of H which contain no operations of G. These considerations usually determine g_{ij} without ambiguity.

We obtain thus the simple character $\phi^{(j)}$, though only the mean value for those classes ρ', ρ'', etc., which correspond to a single class of H. The equation

$$\Sigma g_{\rho'} \phi_{\rho'}^{(j)2} = g$$

usually enables us to determine the actual values for these classes.

Example. The table of characters of the symmetric group of order 8! is given on page 179. The compound character

$$\chi^{(8)}+\chi^{(4^2)}+\chi^{(2^4)}+\chi^{(1^8)}$$

corresponds to a permutation representation. Necessary criteria that a compound character should correspond to a permutation representation will be discussed later. For the present we assume it of this compound character. With this assumption, there must exist a sub-group of order

$$8!/\Sigma\chi_0 = 8!/30 = 1344.$$

The compound character is zero for the negative classes, and for the positive classes takes the values

$$30,\quad 0,\quad 6,\quad 0,\quad 2,\quad 6,\quad 0,\quad 2,\quad 6,\quad 2\quad 0,\quad 14,$$

respectively, for their order in the table referred to. Hence the numbers of operations of these classes in the sub-group are

$$1,\quad 0,\quad 42,\quad 0,\quad 168,\quad 224,\quad 0,\quad 384,\quad 252,\quad 224,\quad 0,\quad 49.$$

The simple characters of the symmetric group are compound characters of the sub-group, hence we obtain the following characters,

(α) $\qquad 1,\quad 1,\quad 1,\quad 1,\quad 1,\quad 1,\quad 1,\quad 1,$

for the eight existent classes of G. This is clearly simple.

(β) $\qquad 7,\quad 3,\quad 1,\quad 1,\quad 0,\quad -1,\quad -1,\quad -1.$

A rough approximation to $\Sigma g_\rho \psi_\rho^2$ for this character shows this to be about 1300. Since it must be a multiple of the order of the group 1344, it must be exactly 1344, and the character is simple.

(γ) $\qquad 20,\quad 4,\quad 0,\quad -1,\quad -1,\quad 0,\quad 1,\quad 4.$

Finding approximations to $\Sigma g_\rho \psi_\rho^{(\gamma)2}$ and $\Sigma g_\rho \psi_\rho^{(\beta)} \psi_\rho^{(\gamma)}$, we see that the character (γ) is the sum of two simple characters, neither of which is the character (β).

(δ) $\qquad 21,\quad 1,\quad -1,\quad 0,\quad 0,\quad 1,\quad 0,\quad -3.$

This proves to be a simple character not contained in (γ).

(ϵ) $\quad 28, \quad 4, \quad 0, \quad 1, \quad 0, \quad 0, \quad -1, \quad -4.$

This is the sum of the two characters (β) and (δ).

(ζ) $\quad 64, \quad 0, \quad 0, \quad -2, \quad 1, \quad 0, \quad 0, \quad 0.$

This is the sum of four simple characters, and includes the character (δ) once, since

$$\Sigma g_\rho \psi_\rho^{(\delta)} \psi_\rho^{(\xi)} = 1344.$$

Hence, subtracting this character, we obtain

(η) $\quad 43, \quad -1, \quad 1, \quad -2, \quad 1, \quad -1, \quad 0, \quad 3,$

the sum of three simple characters of which one is included in (γ), since

$$\Sigma g_\rho \psi_\rho^{(\gamma)} \psi_\rho^{(\eta)} = 1344.$$

The next step is to separate the common character of (γ) and (η). From (5),

$$\frac{hg_\rho}{g\bar{h}_\rho} \psi_\rho^{(\gamma)} = \Sigma p_i \chi_\rho^{(i)},$$

and, from (6),

$$p_i = \frac{1}{g} \Sigma g_\rho \psi_\rho^{(\gamma)} \chi_\rho^{(i)}.$$

$g_\rho \psi_\rho^{(\gamma)}$ takes the values

$$20, \quad 168, \quad 0, \quad -224, \quad -384, \quad 0, \quad 224, \quad 196$$

for the eight classes, and multiplying these numbers by the values of each of the characters of H in turn, and summing, an approximation again being sufficient, we see that $\Sigma p_i \chi^{(i)}$ includes the characters

$$2[62]+[521]+[4^2]+[431]+2[42^2]$$

together with the characters associated with each of these.

Similarly,

$$\frac{hg_\rho}{g\bar{h}_\rho} \psi^{(\eta)} = \Sigma q_i \chi^{(i)}$$

includes the characters

$$[62]+3[521]+[51^3]+2[431]+3[42^2],$$

together with the characters associated with these, and the self-associated character $2[421^2]$.

The common part of these is

$$[62]+[521]+[431]+2[42^2]+\text{associated characters.}$$

The corresponding χ_0 is

$$(20+64+70+2\times 56)\times 2.$$

This is not divisible by $h/g = 30$. By inspection it is apparent that the character $[42^2]$ cannot be repeated in $\Sigma g_{ij}\chi^{(i)}$ if $\Sigma g_{ij}\chi_0^{(i)}$ is to be divisible by 30.

The compound character

$$[62]+[521]+[431]+[42^2]+\text{associated characters}$$

must be the one required, and, as a check, this vanishes for the classes of H containing no operations of G. We are thus led to the simple character of G with mean values, for the classes of H,

$$(\zeta) \qquad 14, \quad 2, \quad 0, \quad -1, \quad 0, \quad -2/3, \quad 1, \quad 10/7.$$

It is clear that the sixth and last classes must split into separate classes. h_ρ/g_ρ must divide h/g. Hence the sixth class must divide into two classes of orders 84 and 168, with characters 0 and -1, or else -2 and 0. Other possibilities are excluded by the equation $\Sigma g_\rho \psi_\rho^{(\zeta)^2} = 1344$, the values 2 and -2, for example, making the left-hand side exceed the right.

Similarly the last class must split into two. The orders of these classes may be 42 and 7, with characters 2, -2, or 1, 4; or again 35 and 14, with characters 2, 0, or 0, 5; or lastly 28 and 21 with characters 1, 2, or 4, -2.

The only solution which satisfies

$$\Sigma g_\rho \psi_\rho^{(\zeta)^2} = 1344$$

is that the sixth class should split into two classes of orders

$$84 \quad \text{and} \quad 168$$

with characters $\qquad -2 \quad \text{and} \quad 0,$

and the last class into two classes of orders

$$42 \quad \text{and} \quad 7$$

with characters $\qquad 2 \quad \text{and} \quad -2.$

We obtain thus five simple characters of G. Since we have introduced two extra classes we need one more character not linearly dependent on

the characters of H. We obtain this by squaring the character (ζ). The square of a character is always a character, simple or compound. This, together with the remaining characters of H, yields, without difficulty, the following characters of G:

Order of Class	1	42	168	224	384	168	84	224	42	7
Character (a)	1	1	1	1	1	1	1	1	1	1
(b)	6	2	0	0	−1	0	2	0	2	6
(c)	7	3	1	1	0	−1	−1	−1	−1	−1
(d)	14	2	0	−1	0	0	−2	1	2	−2
(e)	21	1	−1	0	0	1	1	0	−3	−3
(f)	7	−1	−1	1	0	1	−1	−1	3	−1
(g)	21	−3	1	0	0	−1	1	0	1	−3
(h)	7	−1	−1	1	0	−1	−1	1	−1	7
(j)	8	0	0	−1	1	0	0	−1	0	8
$(k)+(l)$	6	−2	2	0	−1	2	−2	0	−2	6

All these characters are simple save the last, which is the sum of two simple characters.

For each class, if the characters are real,

$$\Sigma \chi_\rho^{(\lambda)^2} = g/g_\rho.$$

For complex characters this equation is replaced by

$$\Sigma \chi_\rho^{(\lambda)} \overline{\chi_\rho^{(\lambda)}} = g/g_\rho.$$

It is clear that the class of order 384 must split into two classes to provide the extra class, since 384 does not divide 1344. We obtain then, for the characters (k) and (l),

$$(k)\quad 3,\quad -1,\quad 1,\quad 0,\quad \frac{-1-\sqrt{7}i}{2},\quad \frac{-1+\sqrt{7}i}{2},\quad 1,\quad 1,\quad 0,\quad -1,\quad 3.$$

$$(l)\quad 3,\quad -1,\quad 1,\quad 0,\quad \frac{-1+\sqrt{7}i}{2},\quad \frac{-1-\sqrt{7}i}{2},\quad 1,\quad 1,\quad 0,\quad -1,\quad 3.$$

The characters take the complex values for the two classes of order 192, into which the class of order 384 divides. This is the only way of satisfying the above orthogonal relation.

Alternatively, the characters (k) and (l) could be obtained separately by using the characters of the alternating group instead of those of the symmetric group.

We thus complete the table of characters (see page 189).

5. *Determination of sub-groups*: (2) *Necessary criteria that a compound character should correspond to a permutation representation of the group.*

From the table of characters of the group we may obtain any compound character. If we can ascertain that any given compound character corresponds to a permutation representation of the group, we can immediately deduce the existence of a conjugate set of sub-groups, and the number of operations from each class of the group that belong to one of the sub-groups. Further, as shown in the last section, we can deduce the table of characters of the sub-group.

All that we require, then, to enable us to investigate the structure of a group from its table of characters is a set of criteria that will enable us to decide whether any given compound character corresponds to a permutation representation. We set forth the following criteria in order of simplicity, since the use of the criteria in this order reduces the labour of investigation to a minimum. Let ϕ be a compound character. The following criteria must be satisfied if it corresponds to a permutation representation.

CRITERION I. *ϕ_ρ is a positive or zero integer for every class.*

CRITERION II. *$\phi(S^r) \geqslant \phi(S)$ for all indices r.*

Those symbols left unchanged by a permutation corresponding to the operation S will be left unchanged by any power of S.

It follows from this, for example, that the value of the character for the class (6) of the symmetric group of order 6! cannot exceed the value for either of the classes (3^2) or (2^3). Again, for the symmetric group of order 7!, the character $[7]+[421]$ cannot correspond to a permutation representation, for its value for the class (61) is 2, and for the class $(3^2\,1)$ is zero. To the table of characters may be added extra columns for the differences $\chi(S^2)-\chi(S)$, $\chi(S^3)-\chi(S)$, etc., for those classes for which these may take negative values. In seeking a compound character that is positive or zero for every class, these columns may be taken into account.

CRITERION III. *ϕ_ρ divides the order of the group.*

CRITERION IV. *If $\phi = \Sigma g_i \chi^{(i)}$, then $g_i \leqslant \chi_0^{(i)}$.*

The group matrix corresponds to the group of permutations of the h operations of the group. It corresponds to the smallest possible sub-group, namely the identical element itself, and the compound character

$$\phi = \Sigma \chi_0^{(i)} \chi^{(i)}.$$

This permutation group is, in general, imprimitive, the elements being divided into sets corresponding to any sub-group of H. It follows that the matrices corresponding to a permutation group corresponding to any such sub-group may be obtained by a reduction of the group matrix. Hence

$$g_i \leqslant \chi_0^{(i)}.$$

CRITERION V. *ϕ contains the character that is unity for every operation exactly once.*

Since ϕ is essentially positive,

$$\Sigma g_\rho \phi_\rho > 0.$$

Hence ϕ must contain this character at least once. It cannot contain it more than once by Criterion IV.

CRITERION VI. *$\phi_\rho h_\rho / \phi_0$ is integral for every class, and if the group is symmetric, and the class ρ contains a cycle of prime order p, the integer is divisible by $p-1$.*

$\phi_\rho h_\rho / \phi_0$ is the number of operations of the class in the sub-group, and hence must be integral. Further, if the group is symmetric, and the class contains a cycle of prime order p, the number of operations from this class in the sub-group must be divisible by $p-1$, for the powers of any operation S of the sub-group must also belong to the sub-group. The latter part does not hold if the group is not symmetric, since the powers of S containing the same cycles as S may belong to a different class.

CRITERION VII. *The table of characters of the sub-group corresponding to the compound character ϕ may be found after the method of the previous section.*

This last criterion is the only one that may be taken as final and sufficient. The existence of a consistent table of characters may be taken as sufficient evidence for the existence of the sub-group. The calculation of the characters of the sub-group is a comparatively long process, and should not be attempted unless the compound character is found to satisfy the other six criteria.

Example. We proceed to find those sub-groups of the symmetric group of order 8! of which the orders are greater than 1000.

We have to find suitable compound characters ϕ such that

$$\phi_0 < 8!/1000 < 41.$$

Referring to the table of characters of the symmetric group of order 8!, which is given on page 179, we see that the inequality $\phi_0 < 41$ together with Criteria I and II give a series of diophantine inequalities with a very small number of solutions. A short examination only is sufficient to show that the only solutions which satisfy Criteria III, V, and VI are

$$\phi^{(1)} = \chi^{(8)} + \chi^{(1^8)},$$

$$\phi^{(2)} = \chi^{(8)} + \chi^{(71)},$$

$$\phi^{(3)} = \chi^{(8)} + \chi^{(71)} + \chi^{(62)},$$

$$\phi^{(4)} = \chi^{(8)} + \chi^{(62)} + \chi^{(4^2)},$$

$$\phi^{(5)} = \chi^{(8)} + \chi^{(4^2)} + \chi^{(2^4)} + \chi^{(1^8)},$$

$$\phi^{(6)} = \chi^{(8)} + \chi^{(71)} + \chi^{(21^6)} + \chi^{(1^8)}.$$

$\phi^{(1)}$ corresponds to the alternating group.

Remembering that*

$$\{7\}\{1\} = \{8\} + \{71\},$$

we see that $\phi^{(2)}$ corresponds to an intransitive sub-group, the symmetric group on 7 symbols.

Again $$\{6\}\{2\} = \{8\} + \{71\} + \{62\}.$$

Hence $\phi^{(3)}$ corresponds to an intransitive sub-group which is the direct product of the symmetric groups on 6 and 2 symbols respectively.

$\phi^{(6)}$ corresponds to the alternating group on 7 symbols, for the group is clearly the positive sub-group of the group corresponding to $\phi^{(2)}$.

$\phi^{(5)}$ would correspond to a sub-group of order 1344. It clearly satisfies the first six criteria. Since we have, in the last section, obtained the characters of the corresponding sub-group we may take this as sufficient evidence for the existence of the sub-group.

This group is one of the simplest to discover by the method which is given here. By the usual methods of group theory, however, it is by no means so easily found. It was omitted in the first lists of the sub-groups of the

* Littlewood and Richardson (13), "The multiplication of S-functions".

symmetric groups, as published by Cayley* in 1891, and by Cole† in 1893. Its existence was, however, known to Jordan‡.

Lastly, $\phi^{(4)}$ corresponds to a sub-group of order $8!/35 = 1152$. The table of characters may be found as shown in the last section (see page 190), and, as before, the successful completion of the table of characters may be taken as sufficient evidence for the existence of the group.

Referring again to the group of order 1344, we see by inspection that

$$\phi = \chi^{(a)} + \chi^{(b)},$$

$$\phi = \chi^{(a)} + \chi^{(c)},$$

$$\phi = \chi^{(a)} + \chi^{(f)},$$

$$\phi = \chi^{(a)} + \chi^{(h)}$$

satisfy the first six criteria.

There is no difficulty in showing that corresponding to these there is a sub-group of order $1344/7 = 192$, and three distinct sets of sub-groups of order $1344/8 = 168$, two of which are simply isomorphic, for we obtain the same character table, though the classes in the two cases are derived from different classes of the original group, the third being a distinct group.

By our method, not only may the sub-groups be found, but, as we shall see, many of the properties of the sub-groups may be read directly from their tables of characters.

6. *The properties of groups and character tables. Self-conjugate sub-groups.*

A necessary and sufficient condition that a group should possess a self-conjugate sub-group is that it should possess a character χ, not unity for every class, and a class ρ, other than the class containing identity, such that

$$\chi_\rho = \chi_0.$$

It is further necessary that characters $\chi^{(1)}, \chi^{(2)}, \ldots, \chi^{(p)}$ exist such that, if ϕ represents the compound character

$$\phi = \chi_0^{(1)}\chi^{(1)} + \chi_0^{(2)}\chi^{(2)} + \ldots + \chi_0^{(p)}\chi^{(p)},$$

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* Cayley (2).
† Cole (3).
‡ Jordan (12).

then $$\phi_\rho = \phi_0,$$

or $$\phi_\rho = 0,$$

for every class. The character ϕ corresponds to the self-conjugate sub-group which consists of the operations of those classes for which

$$\phi_\rho = \phi_0.$$

Suppose that the group H has a self-conjugate sub-group G, corresponding to the compound character ϕ. From the class ρ of H there are $\phi_\rho h_\rho/\phi_0$ operations in G. If this is neither zero nor h_ρ, then an operation of the class ρ belonging to G may be transformed into an operation not belonging to G, by an operation of H, and G cannot be self-conjugate.

Hence $$\phi_\rho h_\rho/\phi_0 = h_\rho \text{ or } 0,$$

i.e., $$\phi_\rho = \phi_0,$$

or $$\phi_\rho = 0.$$

If $$\phi = \Sigma g_i \chi^{(i)},$$

then $$\chi_\rho^{(i)} = \chi_0^{(i)}$$

for all classes ρ of H whose operations belong to G.

Hence the first condition is necessary. We now show that it is sufficient.

Let χ be a character of H, not equal to unity for every class, and let ρ be a class, not containing identity, such that

$$\chi_\rho = \chi_0.$$

Let S be an operation of the class ρ, and let P be the corresponding matrix in a matrix representation of the group corresponding to the character χ. Then P is a matrix of degree χ_0 and with spur χ_0.

If the order of the operation S is p, then

$$S^p = E,$$

$$P^p = E.$$

Hence the χ_0 roots of the characteristic equation of P are all p-th roots of unity, and since the sum of the χ_0 roots is the spur of $P = \chi_0$, the χ_0 roots must all be unity, and

$$P = E.$$

The set of operations S of H, which correspond to the unit matrix E in this representation, must form a self-conjugate sub-group G of H. The first condition is sufficient.

Lastly, suppose that the self-conjugate sub-group G of order g, of the group H of order h, exists, and corresponds to the compound character

$$\phi = \Sigma g_i \chi^{(i)}.$$

Denote by $\mathcal{G}$ the sum of the operations of G. Then

$$\mathcal{G}^2 = g\mathcal{G}.$$

But $$\Sigma \phi(S) \,.\, S = h\mathcal{G}/g,$$

the summation being taken over the group H, and

$$\Sigma \phi(S) \,.\, S = \Sigma g_i \chi^{(i)}(S) \,.\, S.$$

Now $(\chi_0^{(i)}/h)\Sigma\chi^{(i)}(S) \,.\, S$ is the modulus of an invariant sub-algebra of the group algebra, and is consequently idempotent. Further, the product of two such expressions corresponding to different characters is zero*.

Hence $$\{\Sigma \chi^{(i)}(S) \,.\, S\}^2 = \frac{h}{\chi_0^{(i)}} \Sigma \chi^{(i)}(S) \,.\, S$$

and $$\{\Sigma \phi(S) \,.\, S\}^2 = \Sigma \frac{hg_i^2}{\chi_0^{(i)}} \chi^{(i)}(S) \,.\, S.$$

But $$\{\Sigma \phi(S) \,.\, S\}^2 = \{h\mathcal{G}/g\}^2$$

$$= h^2 \mathcal{G}/g$$

$$= h\Sigma\phi(S) \,.\, S.$$

Hence $$\Sigma \frac{hg_i^2}{\chi_0^{(i)}} \chi^{(i)}(S) \,.\, S = \Sigma h g_i \chi^{(i)}(S) \,.\, S,$$

and $$g_i = \chi_0^{(i)}.$$

Thus $$\phi = \chi_0^{(i)} \chi^{(i)},$$

completing the proof that the second condition is necessary.

* Burnside (1), 316.

Examples. The symmetric group of order 4! has the following table of characters*:

Class	1^4	$1^2 2$	13	4	2^2
Order	1	6	8	6	3
[4]	1	1	1	1	1
[31]	3	1	0	−1	−1
$[2^2]$	2	0	−1	0	2
$[21^2]$	3	−1	0	1	−1
$[1^4]$	1	−1	1	−1	1

The self-conjugate sub-groups correspond to

$$\phi = \chi^{(4)} + \chi^{(1^4)},$$

which is the alternating group comprising the first, third, and fifth classes, and to

$$\phi = \chi^{(4)} + 2\chi^{(2^2)} + \chi^{(1^4)}.$$

This group is of order $24/(1+4+1) = 4$, and includes the identical operation and the three operations of the fifth class.

Consider once again the group of order 1344, the table of characters of which is given on page 189. The characters $\chi^{(a)}$, $\chi^{(b)}$, $\chi^{(h)}$, $\chi^{(k)}$, and $\chi^{(l)}$ satisfy

$$\chi_\rho = \chi_0$$

for the fourth class of order 7.

Corresponding to

$$\phi = \chi^{(a)} + 6\chi^{(b)} + 7\chi^{(h)} + 3\chi^{(k)} + 3\chi^{(l)},$$

we have a self-conjugate sub-group of order $1344/(1+36+49+9+9) = 8$, which comprises the identical operation and the seven operations of the fourth class.

The direct product of two groups.

Let G_1 and G_2 be two groups of orders g_1 and g_2 respectively, and let H be the direct product of the two groups.

Suppose that G_1 has p classes ρ_1 and G_2 q classes ρ_2. Then, if T belongs to H and

$$T = S_1 S_2,$$

* For cycles, classes, and all partitions 13, and $1^2 2$ denote the partitions 1, 3, and 1, 1, 2 of 4 respectively, etc.

where S_1 belongs to G_1 and S_2 to G_2, the class of T depends on the classes of S_1 and S_2. Hence there will be pq classes of H. If S_1 belongs to the class ρ_1 and S_2 to ρ_2, T will belong to a class ρ, and we may write symbolically

$$\rho = \rho_1 \rho_2.$$

If h_ρ, g_{ρ_1}, and g_{ρ_2} are the orders of these classes, then

$$h_\rho = g_{\rho_1} g_{\rho_2}.$$

Now the direct product of two matrix representations of G_1 and G_2 respectively must give a matrix representation of H. The spur of the direct product of two matrices is the product of the spurs.

Hence, if $\chi^{(i)}_{\rho_1}$ is a character of G_1, and $\chi^{(j)}_{\rho_2}$ is a character of G_2,

$$\chi^{(ij)}_{\rho_1 \rho_2} = \chi^{(i)}_{\rho_1} \chi^{(j)}_{\rho_2} \tag{1}$$

is a character of H.

Further,

$$\Sigma h_\rho \chi^{(ij)2}_\rho = \Sigma g_{\rho_1} \chi^{(i)2}_{\rho_1} \Sigma g_{\rho_2} \chi^{(j)2}_{\rho_2}$$

$$= g_1 g_2 = h.$$

Hence the character is simple, and we obtain thus the pq characters of H.

The equation (1), then, gives the complete table of characters of the direct product of two groups.

It is possible to infer almost by inspection of the table of characters of a group, whether it can be expressed as the direct product of two groups.

To begin with, there must be two perfectly distinct self-conjugate sub-groups, which may be distinguished as shown in the last paragraph. If this condition is satisfied, we must see if the whole table may be expressed in the form

$$\chi^{(ij)}_{\rho_1 \rho_2} = \chi^{(i)}_{\rho_1} \chi^{(j)}_{\rho_2},$$

the classes ρ_1 belonging to the first sub-group, and the classes ρ_2 to the second; the characters $\chi^{(i)}$ being those that satisfy $\chi^{(i)}_\rho = \chi^{(i)}_0$ for the classes of the second sub-group, and the characters $\chi^{(j)}$ those which satisfy $\chi^{(j)}_\rho = \chi^{(j)}_0$ for the classes of the first sub-group.

If this condition is satisfied, then the group is the direct product of two groups with character tables $[\chi^{(i)}_\rho]$ and $[\chi^{(j)}_\rho]$ respectively.

Example. From the table of characters of the symmetric group of order 7!, we obtain a sub-group of order 144, corresponding to the compound character

$$\chi^{(7)} + \chi^{(61)} + \chi^{(52)} + \chi^{(43)}.$$

The table of characters of this group, which may be found as shown in § 4, is as follows:

Class	A	B	C	D	E	F	G	H	I	J	K	L	M	N	P
Order	1	6	3	8	2	6	3	18	24	12	16	9	18	6	12
a	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
b	2	2	0	2	−1	2	2	0	0	−1	−1	0	0	−1	−1
c	1	1	−1	1	1	1	1	−1	−1	1	1	−1	−1	1	1
d	3	1	3	0	3	−1	−1	1	0	1	0	−1	−1	−1	−1
e	6	2	0	0	−3	−2	−2	0	0	−1	0	0	0	1	1
f	3	1	−3	0	3	−1	−1	−1	0	1	0	1	1	−1	−1
g	2	0	2	−1	2	0	2	0	−1	0	−1	2	0	2	0
h	4	0	0	−2	−2	0	4	0	0	0	1	0	0	−2	0
i	2	0	−2	−1	2	0	2	0	1	0	−1	−2	0	2	0
j	3	−1	3	0	3	1	−1	−1	0	−1	0	−1	1	−1	−1
k	6	−2	0	0	−3	2	−2	0	0	1	0	0	0	1	−1
l	3	−1	−3	0	3	1	−1	1	0	−1	0	1	−1	−1	1
m	1	−1	1	1	1	−1	1	−1	1	−1	1	1	−1	−1	1
n	2	−2	0	2	−1	−2	2	0	0	1	−1	0	0	1	−1
p	1	−1	−1	1	1	−1	1	1	−1	−1	1	−1	1	1	−1

There is a self-conjugate sub-group which comprises the classes A, B, D, F, and G, and another comprising the classes A, C, and E. The corresponding characters are a, d, g, j, and m, and a, b, and c.

Hence, if the group is the direct product of these two sub-groups, then the characters of these sub-groups must be:

Order of Class	1	6	8	6	3
	1	1	1	1	1
	3	1	0	−1	−1
	2	0	−1	0	2
	3	−1	0	1	−1
	1	−1	1	−1	1

Order of Class	1	3	2
	1	1	1
	2	0	−1
	1	−1	1

It is easily verified that the table of characters of the group may be obtained from these two tables of characters of the sub-group by means of the equation

$$\chi^{(ij)}_{\rho_1\rho_2} = \chi^{(i)}_{\rho_1}\chi^{(j)}_{\rho_2}.$$

Also the orders of the classes of the group are the products of the orders of classes of the two sub-groups.

The tables of characters of the sub-groups are easily recognised as those of the symmetric groups of orders 4! and 3! respectively. Hence the group is the direct product of two symmetric groups of orders 4! and 3! respectively.

This might, of course, have been deduced at the beginning from the compound character

$$\phi = \chi^{(7)}+\chi^{(61)}+\chi^{(52)}+\chi^{(43)}.$$

Since

$$\{4\}\{3\} = \{7\}+\{61\}+\{52\}+\{43\},$$

a sub-group of the symmetric group of order 7!, corresponding to the compound character ϕ, must be the direct product of two symmetric groups of orders 4! and 3! respectively. The object of the example, however, was to illustrate the deduction of the property from the character table.

Characters of intransitive sub-groups.

The intransitive sub-groups of the symmetric group of degree $n = p+q+r+\ldots$, that leave sets of p, q, r, ... symbols invariant, are sub-groups of the direct product of the symmetric groups of substitutions on p, q, r, ... symbols respectively. Hence, to find these sub-groups and their characters, we may start with the table of characters of this group, instead of the symmetric group on n symbols.

More generally, for particular intransitive sub-groups, any of these symmetric groups in the direct product may be replaced by one of its sub-groups.

The table of characters is given by

$$\chi^{(ijk\ldots)}_{\rho\rho'\rho''\ldots} = \chi^{(i)}_{\rho}\chi^{(j)}_{\rho'}\chi^{(k)}_{\rho''}\ldots,$$

$\chi^{(i)}$ being a character of the symmetric group on p symbols, or a sub-group of this, $\chi^{(j)}$ on q symbols, $\chi^{(k)}$ on r symbols, etc.

From the table of characters we may seek sub-groups corresponding to compound characters

$$\phi = \Sigma g_{ijk\ldots}\chi^{(ijk\ldots)}.$$

If such a compound character can be expressed in the form

$$\phi = \Sigma g_i \chi^{(i)} \Sigma g_{jk\ldots}\chi^{(jk\ldots)},$$

then the sub-group is the direct product of a group of substitutions on p symbols and an intransitive sub-group on $n-p$ symbols leaving sets of q, r, ... symbols invariant.

It is sometimes more convenient to start with the direct product of two such groups and seek compound characters which cannot be expressed in the form

$$\phi = \Sigma g_i \chi^{(i)} \Sigma g_j \chi^{(j)}.$$

Example. We seek sub-groups of the symmetric group of order 6!, which leave two sets of three symbols invariant.

The table of characters of the direct product of two symmetric groups, each of order 3!, is as follows:

Class	$\rho_1\sigma_1$	$\rho_1\sigma_2$	$\rho_1\sigma_3$	$\rho_2\sigma_1$	$\rho_2\sigma_2$	$\rho_2\sigma_3$	$\rho_3\sigma_1$	$\rho_3\sigma_2$	$\rho_3\sigma_3$
Order	1	3	2	3	9	6	2	6	4
(a) $[3.3]$	1	1	1	1	1	1	1	1	1
(b) $[3.21]$	2	0	−1	2	0	−1	2	0	−1
(c) $[3.1^3]$	1	−1	1	1	−1	1	1	−1	1
(d) $[21.3]$	2	2	2	0	0	0	−1	−1	−1
(e) $[21.21]$	4	0	−2	0	0	0	−2	0	1
(f) $[21.1^3]$	2	−2	2	0	0	0	−1	1	−1
(g) $[1^3.3]$	1	1	1	−1	−1	−1	1	1	1
(h) $[1^3.21]$	2	0	−1	−2	0	1	2	0	−1
(j) $[1^3.1^3]$	1	−1	1	−1	1	−1	1	−1	1

There are three sub-groups of order 18, corresponding to the characters $(a)+(c)$, $(a)+(g)$, $(a)+(j)$, of which the first two may be expressed as direct products of two groups of degree 3.

There are two sub-groups of order 12, corresponding to $(a)+(b)$ and $(a)+(d)$, which may be expressed as direct products.

There are three sub-groups of order 6, corresponding to $(a)+(e)+(j)$, $(a)+(b)+(h)+(j)$, and $(a)+(d)+(f)+(j)$. The last two groups are really of degree 5, each being the positive sub-group of the direct product of symmetric groups of orders 3! and 2!, the other symbol being left invariant.

All other sub-groups are of degree less than 6.

Characters of quotient groups.

Suppose that a group H of order h has an invariant sub-group G of order g. We proceed to show how to deduce the characters of the quotient group $\Gamma = H/G$ from the character table of H.

If we also use G and H to represent the sums of the operations of the respective groups, we may find $\nu = h/g$ operations of H, $S_1 = E$, S_2, ..., S_ν,

such that $$S_1 G + S_2 G + \ldots + S_\nu G = H,$$

where S_r may be replaced by any operation of the set $S_r G$ without altering the set, and

$$S_r G = G S_r.$$

Now $$G^2 = gG,$$

and $$S_i G S_j G = S_i S_j G^2$$
$$= g S_k G,$$

where $S_i S_j$ belongs to the set $S_k G$.

The quantities

$$\frac{1}{g} S_1 G, \quad \frac{1}{g} S_2 G, \quad \ldots, \quad \frac{1}{g} S_\nu G$$

form a group which is the quotient group Γ.

In any matrix representation of Γ, the quantity $(1/g) S_r G$ is represented by M_r. We obtain thus a matrix representation of the whole group H by representing every operation of the set $S_r G$ by M_r. Every operation of G will be represented by the unit matrix.

Hence every character $\chi^{(\lambda)}$ of Γ is a character of H in which all classes ρ containing operations of G satisfy

$$\chi_\rho^{(\lambda)} = \chi_0^{(\lambda)}.$$

Again, $\Sigma \chi_0^{(\lambda)} \chi^{(\lambda)}$, summed for all the characters of H whose values are the same for all operations of G, is the compound character of H corresponding to the sub-group G.

Hence $\Sigma \chi_0^{(\lambda)^2} = h/g = \nu$, and these characters must give the complete table of characters of the quotient group Γ.

Classes of H which take the same values for all these characters $\chi^{(\lambda)}$ correspond to the same class of Γ. The number of operations in a class of Γ is equal to the total number of operations in the classes of H that go to form this class, divided by g.

Example. The table of characters of the symmetric group of order 4! is given on page 177.

Referring to this, we observe that there is an invariant sub-group of order 4 corresponding to $[4]+2[2^2]+[1^4]$, and another of order 12 corresponding to $[4]+[1^4]$.

The orders of the classes of the first quotient group are

$$(1+3)/4 = 1, \quad (6+6)/4 = 3, \quad 8/4 = 2.$$

We obtain the table of characters

Order of Class	1	3	2
[4]	1	1	1
[2^2]	2	0	−1
[1^4]	1	−1	1

The orders of the classes of the second quotient group are

$$(1+8+3)/12 = 1, \quad (6+6)/12 = 1.$$

We obtain the character table

Order of Class	1	1
[4]	1	1
[1^4]	1	−1

Hence the two quotient groups are the symmetric groups of orders 3! and 2! respectively.

Commutator sub-groups and perfect groups.

If S and T are operations of a group H, the operation $S^{-1}T^{-1}ST$ is called a commutator. The commutators of a group generate an invariant sub-group called the commutator sub-group.

The quotient group of a group H corresponding to its commutator sub-group is Abelian. Conversely, every invariant sub-group which is complementary to an Abelian quotient group includes the commutator sub-group*.

An Abelian group of order n has n characters for each of which $\chi_0 = 1$. Conversely, if a group of order n has n characters for which $\chi_0 = 1$, the group is Abelian. For an Abelian group must have as many classes as operations, and hence as many characters, and, conversely, such a group must be Abelian.

Hence, following the last paragraph, it follows that from the table of characters of a group we may immediately deduce the commutator sub-group. For the corresponding compound character is

$$\phi = \Sigma \chi^{(\lambda)},$$

* Miller, Blichfeldt, and Dickson (14).

summed for all characters $\chi^{(\lambda)}$ satisfying

$$\chi_0^{(\lambda)} = 1.$$

The quotient group complementary to this sub-group must be Abelian, for all its characters satisfy $\chi_0 = 1$. Hence the group contains the commutator sub-group. Also, no smaller invariant sub-group can be found of which the quotient group is Abelian. Hence this must be the commutator sub-group.

A group is perfect if it is identical with its commutator sub-group. Hence the condition that a group is perfect is that it possesses no character satisfying $\chi_0 = 1$ save that character which is unity for every operation.

Solvable groups.

A group is said to be solvable if and only if it contains a series of invariant sub-groups, such that the last of the series is identity, and the index of each of these invariant sub-groups under the next larger invariant sub-group is a prime number*. Every Abelian group is solvable.

The solvability of a group may be deduced from its table of characters as follows.

Let H be a group of which the characters are known, and let G_1 be its commutator sub-group. Deduce the table of characters of G_1 as has been shown.

From the table of characters of G_1 deduce the commutator sub-group G_2 and its table of characters.

Proceeding thus, we arrive either at a perfect group G_r or at an Abelian group G_r with commutator sub-group the identity. In the first case the group H is insolvable, in the second case solvable.

The quotient group at each stage is Abelian, and hence solvable. A perfect group is insolvable, and any group containing an insolvable sub-group is insolvable.

In this process, if the simple characters of H lead to compound characters of a sub-group, it is not necessary to separate these compound characters, as at each stage the commutator sub-group is invariant in H, and takes complete classes and complete characters from H. It must be remembered, however, that the character is compound, in deducing the next commutator sub-group.

Example. The symmetric group of order 6! has a sub-group of order 72 corresponding to the compound character $\phi = \chi^{(6)} + \chi^{(42)}$. Its table of

* Miller, Blichfeldt, and Dickson (**14**), 174.

characters is as follows:

Class	A	B	C	D	E	F	G	H	J
Order	1	6	4	9	12	12	4	6	18
(a)	1	1	1	1	1	1	1	1	1
(b)	1	−1	1	1	−1	−1	1	−1	1
(c)	1	1	1	1	1	−1	1	−1	−1
(d)	1	−1	1	1	−1	1	1	1	−1
(e)	4	2	1	0	−1	0	−2	0	0
(f)	4	−2	1	0	1	0	−2	0	0
(g)	4	0	−2	0	0	−1	1	2	0
(h)	4	0	−2	0	0	1	1	−2	0
(j)	2	0	2	−2	0	0	2	0	0

The commutator sub-group corresponds to the compound character $(a)+(b)+(c)+(d)$, and consists of the classes A, C, D, and G. It has the following characters. (a) is simple. (j) is a simple character taken twice over. (e) and (g) are each the sum of two characters.

The next commutator sub-group corresponds to the compound character $(a)+\frac{1}{2}(j)$, and consists of the classes A, C, and G. At this stage we obtain the characters

Class	A	C	G
Order	1	4	4
(a)	1	1	1
(e)	4	1	−2
(g)	4	−2	1

(e) and (g) are each the sum of four simple characters. Hence the group is Abelian, and the original group of order 72 is solvable.

Contragredient isomorphisms.

No case is known of a contragredient isomorphism which changes each class into itself. On the other hand, it is still an open question whether or no such isomorphisms exist*.

If a group contains no invariant sub-group, its group of cogredient isomorphisms is simply isomorphic with the group itself. The existence

* Burnside (1), 89.

of contragredient isomorphisms that interchange some of the classes may be deduced from the table of characters. Hence, apart from the questionable existence of contragredient isomorphisms that leave all classes invariant, a complete knowledge of the isomorphisms may be obtained from the character table.

A contragredient isomorphism, not leaving all classes invariant, must correspond to an interchange of the classes. By reason of the isomorphism, however, the table of characters must be left unchanged, and, since all the characters cannot take the same values for two different classes, there must also be an interchange of characters. Further, the classes that are interchanged must have the same number of operations, and these operations must have the same order.

Hence, if a group admits a contragredient isomorphism, not leaving all classes invariant, then there is a possible rearrangement of the classes, which may be expressed as a series of cyclic interchanges of classes containing the same number of operations of the same order, and a rearrangement of the characters, which, taken together, leaves the table of characters unchanged.

Conversely, if such a rearrangement is possible, there exists a contragredient isomorphism.

The quotient group of the group of isomorphisms by the group of cogredient isomorphisms (or the set which leaves the classes invariant, if this is different) is the group of rearrangements of the classes under the above conditions, for if two contragredient isomorphisms T and U give rise to the same interchange of classes, then $T^{-1}U$ leaves all the classes unchanged.

Examples. In the table of characters of the symmetric group of order 6! (see page 178), interchange the classes $(1^3\,3)$ and (3^2), $(1^4\,2)$ and (2^3), (123) and (6), and simultaneously the characters [51] and $[2^3]$, $[41^2]$ and $[31^3]$, $[3^2]$ and $[21^4]$. The table of characters remains unchanged, and there is a contragredient isomorphism. This is the only rearrangement satisfying the above conditions, and the quotient group of the isomorphisms by the cogredient set is a group of order 2.

For the group of order 1344 with table of characters on page 189, a rearrangement of the classes and characters is possible satisfying all the conditions save that the orders of the operations are different. This condition failing, no contragredient isomorphism exists.

Consider the group of order 18,

$$S^9 = E, \quad T^2 = E, \quad ST = TS^8.$$

If $\alpha = 2\cos 40^\circ$, $\beta = 2\cos 80^\circ$, $\gamma = 2\cos 160^\circ$, the table of characters is as

follows:

Class	a	b	c	d	e	f
Order	1	2	2	2	2	9
A	1	1	1	1	1	1
B	1	1	1	1	1	−1
C	2	α	β	γ	−1	0
D	2	γ	α	β	−1	0
E	2	β	γ	α	−1	0
F	2	−1	−1	−1	2	0

The table is left unchanged if we interchange cyclically the classes (b), (c), and (d), and simultaneously the characters $[C]$, $[D]$, and $[E]$. Hence the quotient group of the group of isomorphisms by the cogredient set is a cyclic group of order 3.

7. *Tables of characters of the symmetric groups.*

Degree 2.

Class	1^2	2
Order	1	1
[2]	1	1
$[1^2]$	1	−1

Degree 3.

Class	1^3	12	3
Order	1	3	2
[3]	1	1	1
*[21]	2	0	−1
$[1^3]$	1	−1	1

Degree 4.

Class	1^4	$1^2 2$	13	4	2^2
Order	1	6	8	6	3
[4]	1	1	1	1	1
[31]	3	1	0	−1	−1
$*[2^2]$	2	0	−1	0	2
$[21^2]$	3	−1	0	1	−1
$[1^4]$	1	−1	1	−1	1

Degree 5.

Class	1^5	$1^3 2$	$1^2 3$	14	12^2	23	5
Order	1	10	20	30	15	20	24
[5]	1	1	1	1	1	1	1
[41]	4	2	1	0	0	−1	−1
[32]	5	1	−1	−1	1	1	0
$*[31^2]$	6	0	0	0	−2	0	1
$[2^2 1]$	5	−1	−1	1	1	−1	0
$[21^3]$	4	−2	1	0	0	1	−1
$[1^5]$	1	−1	1	−1	1	−1	1

* Denotes a self-associated partition and character.

Degree 6.

Class	1^6	$1^4 2$	$1^3 3$	$1^2 4$	$1^2 2^2$	123	15	6	24	2^3	3
Order	1	15	40	90	45	120	144	120	90	15	40
[6]	1	1	1	1	1	1	1	1	1	1	1
[51]	5	3	2	1	1	0	0	−1	−1	−1	−1
[42]	9	3	0	−1	1	0	−1	0	1	3	0
$[41^2]$	10	2	1	0	−2	−1	0	1	0	−2	1
$[3^2]$	5	1	−1	−1	1	1	0	0	−1	−3	2
*[321]	16	0	−2	0	0	0	1	0	0	0	−2
$[2^3]$	5	−1	−1	1	1	−1	0	0	−1	3	2
$[31^3]$	10	−2	1	0	−2	1	0	−1	0	2	1
$[2^2 1^2]$	9	−3	0	1	1	0	−1	0	1	−3	0
$[21^4]$	5	−3	2	−1	1	0	0	1	−1	1	−1
$[1^6]$	1	−1	1	−1	1	−1	1	−1	1	−1	1

Degree 7.

Class	1^7	$1^5 2$	$1^4 3$	$1^3 4$	$1^3 2^2$	$1^2 23$	$1^2 5$	16	124	12^3	13^2	25	$2^2 3$	34	7
Order	1	21	70	210	105	420	504	840	630	105	280	504	210	420	720
[7]	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
[61]	6	4	3	2	2	1	1	0	0	0	0	−1	−1	−1	−1
[52]	14	6	2	0	2	0	−1	−1	0	2	−1	1	2	0	0
$[51^2]$	15	5	3	1	−1	−1	0	0	−1	−3	0	0	−1	1	1
[43]	14	4	−1	−2	2	1	−1	0	0	0	2	−1	−1	1	0
[421]	35	5	−1	−1	−1	−1	0	1	1	1	−1	0	−1	−1	0
$[3^2 1]$	21	1	−3	−1	1	1	1	0	−1	−3	0	1	1	−1	0
*$[41^3]$	20	0	2	0	−4	0	0	0	0	0	2	0	2	0	−1
$[32^2]$	21	−1	−3	1	1	−1	1	0	−1	3	0	−1	1	1	0
$[321^2]$	35	−5	−1	1	−1	1	0	−1	1	−1	−1	0	−1	1	0
$[2^3 1]$	14	−4	−1	2	2	−1	−1	0	0	0	2	1	−1	−1	0
$[31^4]$	15	−5	3	−1	−1	1	0	0	−1	3	0	0	−1	−1	1
$[2^2 1^3]$	14	−6	2	0	2	0	−1	1	0	−2	−1	−1	2	0	0
$[21^5]$	6	−4	3	−2	2	−1	1	0	0	0	0	1	−1	1	−1
$[1^7]$	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1

* Denotes a self-associated partition and character.

Degree 8.

Class	1^8	$1^6 2$	$1^5 3$	$1^4 4$	$1^4 2^2$	$1^3 23$	$1^3 5$	$1^2 6$	$1^2 24$	$1^2 2^3$	$1^2 3^2$	125	$12^2 3$	134	17	8	4^2	$2^2 4$	26	23^2	35	2^4
Order	1	28	112	420	210	1120	1344	3360	2520	420	1120	4032	1680	3360	5760	5040	1260	1260	3360	1120	2688	105
[8]	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
[71]	7	5	4	3	3	2	2	1	1	1	1	0	0	0	0	−1	−1	−1	−1	−1	−1	−1
[62]	20	10	5	2	4	1	0	−1	0	2	−1	0	1	−1	−1	0	0	2	1	1	0	4
$[61^2]$	21	9	6	3	1	0	1	0	−1	−3	0	−1	−2	0	0	1	1	−1	0	0	1	−3
[53]	28	10	1	−2	4	1	−2	−1	0	2	1	0	1	1	0	0	0	−2	−1	1	1	−4
[521]	64	16	4	0	0	−2	−1	0	0	0	−2	1	0	0	1	0	0	0	0	−2	−1	0
$[51^3]$	35	5	5	1	−5	−1	0	0	−1	−3	2	0	1	1	0	−1	−1	1	0	2	0	3
$[4^2]$	14	4	−1	−2	2	1	−1	0	0	0	2	−1	−1	1	0	0	2	2	0	−2	−1	6
[431]	70	10	−5	−4	2	1	0	1	0	−2	1	0	−1	−1	0	0	−2	0	1	1	0	−2
$[42^2]$	56	4	−4	0	0	−2	1	1	0	4	−1	−1	0	0	0	0	0	0	−1	1	1	8
*$[421^2]$	90	0	0	0	−6	0	0	0	2	0	0	0	0	0	−1	0	2	0	0	0	0	−6
*$[3^2 2]$	42	0	−6	0	2	0	2	0	−2	0	0	0	2	0	0	0	2	0	0	0	−1	−6
$[3^3 1^2]$	56	−4	−4	0	0	2	1	−1	0	−4	−1	1	0	0	0	0	0	0	−1	−1	1	8
$[32^2 1]$	70	−10	−5	4	2	−1	0	−1	0	2	1	0	−1	1	0	0	−2	0	1	−1	0	−2
$[2^4]$	14	−4	−1	2	2	−1	−1	0	0	0	2	1	−1	−1	0	0	2	−2	0	2	−1	6
$[41^4]$	35	−5	5	−1	−5	1	0	0	−1	3	2	0	1	−1	0	1	−1	−1	0	−2	0	3
$[321^3]$	64	−16	4	0	0	2	−1	0	0	0	−2	−1	0	0	1	0	0	0	0	2	−1	0
$[2^3 1^2]$	28	−10	1	2	4	−1	−2	1	0	−2	1	0	1	−1	0	0	0	2	−1	−1	1	−4
$[31^5]$	21	−9	6	−3	1	0	1	0	−1	3	0	1	−2	0	0	−1	1	1	0	0	1	−3
$[2^2 1^4]$	20	−10	5	−2	4	−1	0	1	0	−2	−1	0	1	1	−1	0	0	−2	1	−1	0	4
$[21^6]$	7	−5	4	−3	3	−2	2	−1	1	−1	1	0	0	0	0	1	−1	1	−1	1	−1	−1
$[1^8]$	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	1

* Denotes a self-associated partition and character.

Degree 9.

Class	1^9	$1^7 2$	$1^6 3$	$1^5 4$	$1^5 2^2$	$1^4 23$	$1^4 5$	$1^3 6$	$1^3 24$	$1^3 2^3$	$1^3 3^2$	$1^2 25$	$1^2 2^2 3$	$1^2 34$
Order	1	36	168	756	378	2520	3024	10080	7560	1260	3360	18144	7560	15120
[9]	1	1	1	1	1	1	1	1	1	1	1	1	1	1
[81]	8	6	5	4	4	3	3	2	2	2	2	1	1	1
[72]	27	15	9	5	7	3	2	0	1	3	0	0	1	−1
$[71^2]$	28	14	10	6	4	2	3	1	0	−2	1	−1	−2	0
[63]	48	20	6	0	8	2	−2	−2	0	4	0	0	2	0
[621]	105	35	15	5	5	−1	0	−1	−1	−1	−3	0	−1	−1
$[61^3]$	56	14	11	4	−4	−1	1	0	−2	−6	2	−1	−1	1
[54]	42	14	0	−4	6	2	−3	−1	0	2	3	−1	0	2
[531]	162	36	0	−6	6	0	−3	0	0	0	0	1	0	0
$[52^2]$	120	20	0	0	0	−4	0	1	0	4	−3	0	0	0
$[521^2]$	189	21	9	1	−11	−3	−1	0	1	−3	0	1	1	1
$[4^2 1]$	84	14	−6	−6	4	2	−1	1	0	−2	3	−1	−2	0
[432]	168	14	−15	−4	4	−1	3	2	−2	2	0	−1	1	−1
$[431^2]$	216	6	−9	−4	−4	3	1	0	2	−6	0	1	−1	−1
*$[51^4]$	70	0	10	0	−10	0	0	0	−2	0	4	0	2	0
*$[3^3]$	42	0	−6	0	2	0	2	0	−2	0	0	0	2	0
$[42^2 1]$	216	−6	−9	4	−4	−3	1	0	2	6	0	−1	−1	1
$[3^2 21]$	168	−14	−15	4	4	1	3	−2	−2	−2	0	1	1	1
$[32^3]$	84	−14	−6	6	4	−2	−1	−1	0	2	3	1	−2	0
$[421^3]$	189	−21	9	−1	−11	3	−1	0	1	3	0	−1	1	−1
$[3^2 1^3]$	120	−20	0	0	0	4	0	−1	0	−4	−3	0	0	0
$[32^2 1^2]$	162	−36	0	6	6	0	−3	0	0	0	0	−1	0	0
$[2^4 1]$	42	−14	0	4	6	−2	−3	1	0	−2	3	1	0	−2
$[41^5]$	56	−14	11	−4	−4	1	1	0	−2	6	2	1	−1	−1
$[321^4]$	105	−35	15	−5	5	1	0	1	−1	1	−3	0	−1	1
$[2^3 1^3]$	48	−20	6	0	8	−2	−2	2	0	−4	0	0	2	0
$[31^6]$	28	−14	10	−6	4	−2	3	−1	0	2	1	1	−2	0
$[2^2 1^5]$	27	−15	9	−5	7	−3	2	0	1	−3	0	0	1	1
$[21^7]$	8	−6	5	−4	4	−3	3	−2	2	−2	2	−1	1	−1
$[1^9]$	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1

* Denotes a self-associated partition and character.

$1^2\,7$	18	14^2	$12^2\,4$	126	123^2	135	12^4	45	3^3	36	234	27	$2^2\,5$	$2^3\,3$	9
25920	45360	11340	11340	30240	10080	24192	945	18144	2240	20160	15120	25920	9072	2520	40320
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1
−1	−1	−1	1	0	0	−1	3	0	0	0	1	1	2	3	0
0	0	0	−2	−1	−1	0	−4	1	1	1	0	0	−1	−2	1
−1	0	0	0	0	2	1	0	0	3	1	0	−1	−2	−2	0
0	1	1	1	1	−1	0	1	0	−3	−1	−1	0	0	−1	0
0	0	0	0	0	2	1	0	−1	2	0	1	0	1	3	−1
0	0	2	0	−1	−1	0	2	1	−3	−1	0	0	1	2	0
1	0	−2	−2	0	0	0	−6	−1	0	0	0	1	1	0	0
1	0	0	0	−1	−1	0	8	0	3	1	0	−1	0	4	0
0	−1	1	1	0	0	−1	−3	1	0	0	1	0	−1	−3	0
0	0	0	2	1	−1	−1	4	−1	3	1	0	0	−1	−2	0
0	0	0	0	0	2	0	0	1	−3	−1	1	0	−1	−1	0
−1	0	0	0	0	0	1	0	1	0	0	−1	−1	1	3	0
0	0	−2	0	0	0	0	6	0	−2	0	−2	0	0	0	1
0	0	2	0	0	0	−1	−6	0	6	0	−2	0	2	0	0
−1	0	0	0	0	0	1	0	−1	0	0	−1	1	1	−3	0
0	0	0	0	0	−2	0	0	−1	−3	1	1	0	−1	1	0
0	0	0	−2	1	1	−1	4	1	3	−1	0	0	−1	2	0
0	1	1	−1	0	0	−1	−3	−1	0	0	1	0	−1	3	0
1	0	0	0	−1	1	0	8	0	3	−1	0	1	0	−4	0
1	0	−2	2	0	0	0	−6	1	0	0	0	−1	1	0	0
0	0	2	0	−1	1	0	2	−1	−3	1	0	0	1	−2	0
0	0	0	0	0	−2	1	0	1	2	0	1	0	1	−3	−1
0	−1	1	−1	1	1	0	1	0	−3	1	−1	0	0	1	0
−1	0	0	0	0	−2	1	0	0	3	−1	0	1	−2	2	0
0	0	0	2	−1	1	0	−4	−1	1	−1	0	0	−1	2	1
−1	1	−1	−1	0	0	−1	3	0	0	0	1	−1	2	−3	0
1	0	0	0	0	0	0	0	1	−1	1	−1	1	−1	1	−1
1	−1	1	−1	1	−1	1	1	−1	1	−1	1	−1	1	−1	1

Degree 10.

Class	1^{10}	$1^8 2$	$1^7 3$	$1^6 4$	$1^6 2^2$	$1^5 23$	$1^5 5$	$1^4 6$	$1^4 24$	$1^4 2^3$	$1^4 3^2$	$1^3 25$	$1^3 2^2 3$	$1^3 34$	$1^3 7$	$1^2 8$	$1^2 4^2$	$1^2 2^2 4$	$1^2 26$
Order	1	45	240	1260	630	5040	6048	25200	18900	3150	8400	60480	25200	50400	86400	226800	56700	56700	151200
[10]	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
[91]	9	7	6	5	5	4	4	3	3	3	3	2	2	2	2	1	1	1	1
[82]	35	21	14	9	11	6	5	2	3	5	2	1	2	0	0	−1	−1	1	0
$[81^2]$	36	20	15	10	8	5	6	3	2	0	3	0	−1	1	1	0	0	−2	−1
[73]	75	35	15	5	15	5	0	−2	1	7	0	0	3	−1	−2	−1	−1	1	0
[721]	160	64	34	16	16	4	5	0	0	0	−2	−1	−2	−2	−1	0	0	0	0
$[71^3]$	84	28	21	10	0	1	4	1	−2	−8	3	−2	−3	1	0	0	0	−2	−1
[64]	90	34	6	−4	14	4	−5	−3	0	6	3	−1	2	2	−1	0	2	0	−1
[631]	315	91	21	−1	19	1	−5	−3	−1	3	−3	1	1	−1	0	1	−1	−1	1
$[62^2]$	225	55	15	5	5	−5	0	0	−1	3	6	0	−1	−1	1	1	1	1	0
$[621^2]$	350	70	35	10	−10	−5	0	−1	−2	−10	−1	0	−1	1	0	0	2	2	1
$[61^4]$	126	14	21	4	−14	−1	1	0	−4	−6	6	−1	1	1	0	0	−2	0	0
$[5^2]$	42	14	0	−4	6	2	−3	−1	0	2	3	−1	0	2	0	0	2	0	−1
[541]	288	64	−6	−16	16	4	−7	0	0	0	6	−1	−2	2	1	0	0	0	0
[532]	450	70	−15	−10	10	−5	0	3	−2	6	−3	0	1	−1	2	0	−2	−2	−1
$[531^2]$	567	63	0	−9	−9	0	−3	0	3	−9	0	3	0	0	0	−1	−1	−1	0
$[52^21]$	525	35	0	5	−15	−10	0	1	3	7	−3	0	0	2	0	−1	1	1	−1
$[4^32]$	252	28	−21	−10	8	1	2	3	−2	0	3	−2	−1	−1	0	0	0	2	1
$[4^21^2]$	300	20	−15	−10	0	5	0	1	2	−8	3	0	−3	1	−1	0	0	2	1
$[43^2]$	210	14	−21	−4	6	−1	5	2	−4	2	0	−1	3	−1	0	0	2	0	0
*[4321]	768	0	−48	0	0	0	8	0	0	0	0	0	0	0	−2	0	0	0	0
*$[521^3]$	448	0	28	0	−32	0	−2	0	0	0	4	0	4	0	0	0	0	0	0
$[3^31]$	210	−14	−21	4	6	1	5	−2	−4	−2	0	1	3	1	0	0	2	0	0
$[42^3]$	300	−20	−15	10	0	−5	0	−1	2	8	3	0	−3	1	−1	0	0	−2	1
$[3^22^2]$	252	−28	−21	10	8	−1	2	−3	−2	0	3	2	−1	1	0	0	0	−2	1
$[431^3]$	525	−35	0	−5	−15	10	0	−1	3	−7	−3	0	0	−2	0	1	1	−1	−1
$[42^21^2]$	567	−63	0	9	−9	0	−3	0	3	9	0	−3	0	0	0	1	−1	1	0
$[3^21^4]$	450	−70	−15	10	10	5	0	−3	−2	−6	−3	0	1	1	2	0	−2	2	−1
$[32^31]$	288	−64	−6	16	16	−4	−7	0	0	0	6	1	−2	−2	1	0	0	0	0
$[2^5]$	42	−14	0	4	6	−2	−3	1	0	−2	3	1	0	−2	0	0	2	0	−1
$[51^5]$	126	−14	21	−4	−14	1	1	0	−4	6	6	1	1	−1	0	0	−2	0	0
$[421^4]$	350	−70	35	−10	−10	5	0	1	−2	10	−1	0	−1	−1	0	0	2	−2	1
$[3^21^4]$	225	−55	15	−5	5	5	0	0	−1	−3	−6	0	−1	1	1	−1	1	−1	0
$[32^31^3]$	315	−91	21	1	19	−1	−5	3	−1	−3	−3	−1	1	1	0	−1	−1	1	1
$[2^41^2]$	90	−34	6	4	14	−4	−5	3	0	−6	3	1	2	−2	−1	0	2	0	−1
$[41^6]$	84	−28	21	−10	0	−1	4	−1	−2	8	3	2	−3	−1	0	0	0	2	−1
$[321^5]$	160	−64	34	−16	16	−4	5	0	0	0	−2	1	−2	2	−1	0	0	0	0
$[2^31^4]$	75	−35	15	−5	15	−5	0	2	1	−7	0	0	3	1	−2	1	−1	−1	0
$[31^7]$	36	−20	15	−10	8	−5	6	−3	2	0	3	0	−1	−1	1	0	0	2	−1
$[2^21^6]$	35	−21	14	−9	11	−6	5	−2	3	−5	2	−1	2	0	0	1	−1	−1	0
$[21^8]$	9	−7	6	−5	5	−4	4	−3	3	−3	3	−2	2	−2	2	−1	1	−1	1
$[1^{10}]$	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1

* Denotes a self-associated partition and character.

$2^2 3^2$	$1^2 35$	$1^2 2^4$	145	13^3	136	1234	127	$12^2 5$	$12^3 3$	19	10	82	62^2	42^3	2^5	73	532	64	$4^2 2$	5^2	43^2	$3^2 2^2$
50400	120960	4725	181440	22400	201600	151200	259200	90720	25200	403200	362880	226800	75600	18900	945	72800	20960	51200	56700	72576	50400	25200
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	0	0	0	0	0	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
0	−1	3	−1	−1	−1	0	0	1	2	−1	0	1	2	3	5	0	1	0	1	0	0	2
−1	0	−4	0	0	0	−1	−1	−2	−3	0	1	0	−1	−2	−4	1	0	1	0	1	1	−1
2	0	3	0	3	1	1	0	0	1	0	0	−1	−2	−3	−5	1	0	0	−1	0	2	0
−2	−1	0	1	−2	0	0	1	1	0	1	0	0	0	0	0	−1	−1	0	0	0	−2	−2
1	1	−4	0	3	1	1	0	0	1	0	−1	0	1	2	4	0	1	−1	0	−1	1	3
1	1	2	1	0	0	0	−1	−1	0	0	0	0	1	4	10	−1	−1	1	2	0	−1	−1
1	1	−5	−1	0	0	−1	0	−1	−3	0	0	1	1	−1	−5	0	1	−1	−1	0	−1	1
−2	0	9	0	0	0	−1	−1	0	3	0	0	−1	0	3	15	1	0	0	−1	0	2	2
1	0	−2	0	−1	−1	1	0	0	−1	−1	0	0	−1	−2	−10	0	0	1	2	0	1	−1
2	1	6	−1	0	0	−1	0	1	3	0	1	0	0	0	6	0	−1	0	−2	1	−2	−2
−1	0	2	1	−3	−1	0	0	1	2	0	0	0	−1	−4	−10	0	2	−1	−2	2	−1	3
−2	−1	0	−1	0	0	0	1	1	0	0	0	0	0	0	0	1	−1	0	0	−2	2	−2
1	0	2	0	0	0	1	0	0	3	0	0	0	−1	−2	−10	−1	0	1	2	0	−1	1
0	0	−9	1	0	0	0	0	1	0	0	0	−1	0	3	15	0	0	0	−1	2	0	0
−1	0	5	0	3	1	0	0	0	−2	0	0	1	1	−1	−5	0	0	−1	−1	0	−1	−3
1	−1	4	0	0	0	1	0	−2	−3	0	0	0	−1	2	20	0	1	−1	0	2	−1	−1
−1	0	4	0	3	1	−1	−1	0	1	0	0	0	1	−2	−20	−1	0	1	0	0	−1	3
2	−1	−6	1	3	−1	−1	0	1	−1	0	0	0	2	0	−10	0	−1	0	−2	0	2	0
0	2	0	0	−6	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	−2	0	0
0	−2	0	0	−2	0	0	0	−2	0	1	0	0	0	0	0	0	0	0	0	−2	0	4
−2	−1	−6	−1	3	1	−1	0	1	1	0	0	0	−2	0	10	0	1	0	2	0	−2	0
1	0	4	0	3	−1	−1	1	0	−1	0	0	0	−1	−2	20	−1	0	1	0	0	1	3
−1	−1	4	0	0	0	1	0	−2	3	0	0	0	1	2	−20	0	−1	−1	0	2	1	−1
1	0	5	0	3	−1	0	0	0	2	0	0	1	−1	−1	5	0	0	−1	1	0	1	−3
0	0	−9	−1	0	0	0	0	1	0	0	0	−1	0	3	−15	0	0	0	1	2	0	0
−1	0	2	0	0	0	1	0	0	−3	0	0	0	1	−2	10	−1	0	1	−2	0	1	1
2	−1	0	1	0	0	0	−1	1	0	0	0	0	0	0	0	1	1	0	0	−2	−2	−2
1	0	2	−1	−3	1	0	0	1	−2	0	0	0	1	−4	10	0	−2	−1	2	2	1	3
−2	1	6	1	0	0	−1	0	1	−3	0	−1	0	0	0	−6	0	1	0	2	1	2	−2
−1	0	−2	0	−1	1	1	0	0	1	−1	0	0	1	−2	10	0	0	1	−2	0	−1	−1
2	0	9	0	0	0	−1	1	0	−3	0	0	−1	0	3	−15	1	0	0	1	0	−2	2
−1	1	−5	1	0	0	−1	0	−1	3	0	0	1	−1	−1	5	0	−1	−1	1	0	1	1
−1	1	2	−1	0	0	0	1	−1	0	0	0	0	−1	4	−10	−1	1	1	−2	0	1	−1
−1	1	−4	0	3	−1	1	0	0	−1	0	1	0	−1	2	−4	0	−1	−1	0	−1	−1	3
2	−1	0	−1	−2	0	0	−1	1	0	1	0	0	0	0	0	−1	1	0	0	0	2	−2
−2	0	3	0	3	−1	1	0	0	−1	0	0	−1	2	−3	5	1	0	0	1	0	−2	0
1	0	−4	0	0	0	−1	1	−2	3	0	−1	0	1	−2	4	1	0	1	0	1	−1	−1
0	−1	3	1	−1	1	0	0	1	−2	−1	0	1	−2	3	−5	0	−1	0	−1	0	0	2
−1	1	1	0	0	0	0	0	0	0	0	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1
−1	1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1

8. *Tables of characters of transitive sub-groups. Alternating groups.*

The simple characters of the symmetric groups which are not self-associated are also simple characters of the alternating groups.

Every self-associated character of the symmetric group (marked with a * in the tables) is the sum of two simple characters of the alternating group. These characters of the alternating group take exactly half the values of the character of the symmetric group, save for that class for which the character of the symmetric group is ± 1. This class splits into two for the alternating group, and it is for these classes alone that the two characters of the alternating group differ, the characteristics of the two classes being interchanged for the second character.

We give the corresponding characteristics of these classes for the alternating groups of degrees up to 9, since this is all that is necessary to complete the table of characters, when the characters of the symmetric groups are known.

Degree 3.

Character $[21]$. Class (3). Values $\frac{1}{2}(-1\pm i\sqrt{3})$.

Degree 4.

Character $[2^2]$. Class (13). Values $\frac{1}{2}(-1\pm i\sqrt{3})$.

Degree 5.

Character $[31^2]$. Class (5). Values $\frac{1}{2}(1\pm\sqrt{5})$.

Degree 6.

Character $[321]$. Class (15). Values $\frac{1}{2}(1\pm\sqrt{5})$.

Degree 7.

Character $[41^3]$. Class (7) Values $\frac{1}{2}(-1\pm i\sqrt{7})$.

Degree 8.

Character $[421^2]$. Class (17). Values $\frac{1}{2}(-1\pm i\sqrt{7})$.

Character $[3^2 2]$. Class (35). Values $\frac{1}{2}(-1\pm i\sqrt{15})$.

Degree 9.

Character $[51^4]$. Class (9). Values $\frac{1}{2}(1\pm 3)=2$ and -1.

Character $[3^3]$. Class (135). Values $\frac{1}{2}(-1\pm i\sqrt{15})$.

If $n=\lambda_1+\lambda_2+\lambda_3+\ldots$ is a self-associated partition of n, then

$$n=p_1+p_2+p_3+\ldots,$$

where

$$p_1 = 2\lambda_1 - 1,$$

$$p_2 = 2\lambda_2 - 3,$$

$$p_3 = 2\lambda_3 - 5,$$

$$\ldots \quad \ldots \quad \ldots$$

is a partition of n into odd numbers. The characteristic of the class $(p_1 p_2 p_3 \ldots)$ of the symmetric group, corresponding to the partition

$$n = \lambda_1 + \lambda_2 + \ldots$$

of n, is given by

$$\chi_p^{(\lambda)} = (-1)^{\frac{1}{2}(p_1 p_2 p_3 \ldots - 1)}.$$

The corresponding characteristics of the two classes of the alternating group into which this class separates are

$$\frac{(-1)^{\frac{1}{2}(p_1 p_2 \ldots - 1)} \pm \sqrt{(p_1 p_2 p_3 \ldots)}\,(i)^{\frac{1}{2}(p_1 p_2 \ldots - 1)}}{2}.$$

Frobenius* proves these properties, which may also be deduced from the orthogonal properties.

General cyclic group of order n.

Let S be any operation of the group whose order is n, and let ω be a primitive n-th root of unity.

There is a character χ taking the value ω^r for the operation S^r. The n distinct powers of this character give the n characters of the group.

Other transitive sub-groups.

Degree 4.

Sub-group of order 8. Compound character $[4]+[2^2]$.

Cycles	1^4	$1^2 2$	4	2^2	2^2
Order	1	2	2	2	1
	1	1	1	1	1
	1	−1	−1	1	1
	1	1	−1	−1	1
	1	−1	1	−1	1
	2	0	0	0	−2

* Frobenius (10).

Self-conjugate sub-group of order 4. Compound character $[4]+[2^2]+[1^4]$.

Cycles	1^4	2^2	2^2	2^2
Order	1	1	1	1
	1	1	1	1
	1	1	−1	−1
	1	−1	1	−1
	1	−1	−1	1

Degree 5.

Sub-group of order 20. Compound character $[5]+[2^2 1]$.

Cycles	1^5	14	14	12^2	5
Order	1	5	5	5	4
	1	1	1	1	1
	1	−1	−1	1	1
	1	$-i$	i	−1	1
	1	i	$-i$	−1	1
	4	0	0	0	−1

Sub-group of order 10 of above group.

Cycles	1^5	12^2	5	5
Order	1	5	2	2
	1	1	1	1
	1	−1	1	1
	2	0	$-\frac{1}{2}(1+\sqrt{5})$	$-\frac{1}{2}(1-\sqrt{5})$
	2	0	$-\frac{1}{2}(1-\sqrt{5})$	$-\frac{1}{2}(1+\sqrt{5})$

Degree 6.

Sub-group of order 120. Compound character $[6]+[2^3]$.

Cycles	1^6	$1^2 4$	$1^2 2^2$	15	6	2^3	3^2
Order	1	30	15	24	20	10	20
	1	1	1	1	1	1	1
	4	0	0	−1	−1	2	1
	5	−1	1	0	1	1	−1
	6	0	−2	1	0	0	0
	5	1	1	0	−1	−1	−1
	4	0	0	−1	1	−2	1
	1	−1	1	1	−1	−1	1

This group is clearly simply isomorphic with the symmetric group of order 5!. The characters of sub-groups of this group, then, may be obtained from the characters of groups of degree 5.

Sub-group of order 72. Compound character $[6]+[42]$.

Cycles	1^6	$1^4 2$	$1^3 3$	$1^2 2^2$	123	6	24	2^3	3^2
Order	1	6	4	9	12	12	18	6	4
	1	1	1	1	1	1	1	1	1
	1	−1	1	1	−1	−1	1	−1	1
	1	1	1	1	1	−1	−1	−1	1
	1	−1	1	1	−1	1	−1	1	1
	4	2	1	0	−1	0	0	0	−2
	4	−2	1	0	1	0	0	0	−2
	4	0	−2	0	0	−1	0	2	1
	4	0	−2	0	0	1	0	−2	1
	2	0	2	−2	0	0	0	0	2

Sub-group of order 48. Compound character $[6]+[42]+[2^3]$.

Cycles	1^6	$1^4 2$	$1^2 4$	$1^2 2^2$	$1^2 2^2$	6	24	2^3	2^3	3^2
Order	1	3	6	3	6	8	6	6	1	8
	1	1	1	1	1	1	1	1	1	1
	1	−1	−1	1	1	−1	1	−1	−1	1
	1	1	−1	1	−1	1	−1	−1	1	1
	1	−1	1	1	−1	−1	−1	1	−1	1
	2	2	0	2	0	−1	0	0	2	−1
	2	−2	0	2	0	1	0	0	−2	−1
	3	1	1	−1	1	0	−1	−1	−3	0
	3	−1	−1	−1	1	0	−1	1	3	0
	3	1	−1	−1	−1	0	1	1	−3	0
	3	−1	1	−1	−1	0	1	−1	3	0

Degree 7.

Sub-group of order 168. Compound character $[7]+[43]+[2^3 1]+[1^7]$.

Cycles	1^7	$1^3 2^2$	124	13^2	7	7
Order	1	21	42	56	24	24
	1	1	1	1	1	1
	6	2	0	0	−1	−1
	7	−1	−1	1	0	0
	8	0	0	−1	1	1
	3	−1	1	0	$\frac{1}{2}(-1+i\sqrt{7})$	$\frac{1}{2}(-1-i\sqrt{7})$
	3	−1	1	0	$\frac{1}{2}(-1-i\sqrt{7})$	$\frac{1}{2}(-1+i\sqrt{7})$

Sub-group of order 42. Compound character

$[7]+[43]+[421]+[41^3]+[32^2]+[2^3 1]+[31^4]$.

Cycles	1^7	16	16	12^3	13^2	13^2	7
Order	1	7	7	7	7	7	6
	1	1	1	1	1	1	1
	1	ω	ω^2	1	ω	ω^2	1
	1	ω^2	ω	1	ω^2	ω	1
	1	$-\omega$	$-\omega^2$	−1	ω	ω^2	1
	1	$-\omega^2$	$-\omega$	−1	ω^2	ω	1
	1	−1	−1	−1	1	1	1
	6	0	0	0	0	0	−1

Sub-group of order 21 of above groups.

Cycles	1^7	13^2	13^2	7	7
Order	1	7	7	3	3
	1	1	1	1	1
	1	ω	ω^2	1	1
	1	ω^2	ω	1	1
	3	0	0	$\frac{1}{2}(-1+i\sqrt{7})$	$\frac{1}{2}(-1-i\sqrt{7})$
	3	0	0	$\frac{1}{2}(-1-i\sqrt{7})$	$\frac{1}{2}(-1+i\sqrt{7})$

Degree 8.

Sub-group of order 1344. Compound character $[8]+[4^2]+[2^4]+[1^8]$.

Cycles	1^8	$1^4 2^2$	2^4	2^4	$1^2 3^2$	$1^2 24$	26	17	17	4^2	4^2
Order	1	42	42	7	224	168	224	192	192	168	84
(*a*)	1	1	1	1	1	1	1	1	1	1	1
(*b*)	6	2	2	6	0	0	0	−1	−1	0	2
(*c*)	7	3	−1	−1	1	1	−1	0	0	−1	−1
(*d*)	14	2	2	−2	−1	0	1	0	0	0	−2
(*e*)	21	1	−3	−3	0	−1	0	0	0	1	1
(*f*)	7	−1	3	−1	1	−1	−1	0	0	1	−1
(*g*)	21	−3	1	−3	0	1	0	0	0	−1	1
(*h*)	7	−1	−1	7	1	−1	1	0	0	−1	−1
(*j*)	8	0	0	8	−1	0	−1	1	1	0	0
(*k*)	3	−1	−1	3	0	1	0	$\frac{1}{2}(-1-i\sqrt{7})$	$\frac{1}{2}(-1+i\sqrt{7})$	1	−1
(*l*)	3	−1	−1	3	0	1	0	$\frac{1}{2}(-1+i\sqrt{7})$	$\frac{1}{2}(-1-i\sqrt{7})$	1	−1

Sub-group of order 1152. Compound character $[8]+[62]+[4^2]$.

Cycles	1^8	1^62	1^53	1^44	1^22^3	1^42^2	1^323	1^224	1^42^2	1^33^2	12^23	134	8	4^2	4^2	2^24	2^24	26	2^1	2^1
Order	1	12	16	12	36	6	96	72	36	64	48	96	144	72	36	144	36	192	9	24
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	−1	1	−1	−1	1	−1	1	1	1	1	−1	−1	1	1	−1	−1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	1	−1	1	−1	1	−1
	1	−1	1	−1	−1	1	−1	1	1	1	1	−1	1	−1	1	1	−1	−1	1	−1
	6	4	3	2	0	2	1	0	2	0	−1	−1	0	0	−2	0	−2	0	−2	0
	6	−4	3	−2	0	2	−1	0	2	0	−1	1	0	0	−2	0	2	0	−2	0
	2	0	2	0	0	2	0	−2	−2	2	2	0	0	0	−2	0	0	0	2	0
	18	0	0	0	0	−6	0	2	−2	0	0	0	0	0	−2	0	0	0	2	0
	12	2	−3	−2	2	4	−1	0	0	0	1	1	0	0	0	0	−2	0	−4	0
	12	−2	−3	2	−2	4	1	0	0	0	1	−1	0	0	0	0	2	0	−4	0
	9	3	0	−3	1	−3	0	−1	1	0	0	−1	1	1	1	−1	1	0	1	−3
	9	−3	0	3	−1	−3	0	−1	1	0	0	1	−1	1	1	1	−1	0	1	−3
	9	3	0	−3	1	−3	0	−1	1	0	0	−1	−1	−1	1	1	1	0	1	3
	9	−3	0	3	−1	−3	0	−1	1	0	0	1	1	−1	1	−1	−1	0	1	3
	4	2	1	2	2	4	−1	0	0	−2	1	0	0	0	0	0	2	0	4	0
	4	−2	1	−2	−2	4	1	0	0	−2	1	0	0	0	0	0	−2	0	4	0
	4	0	−2	0	0	4	0	0	0	1	−2	0	0	2	0	0	0	−1	4	2
	4	0	−2	0	0	4	0	0	0	1	−2	0	0	−2	0	0	0	1	4	−2
	6	2	3	4	−2	2	−1	0	−2	0	−1	0	0	0	2	0	0	0	−2	0
	6	−2	3	−4	2	2	1	0	−2	0	−1	0	0	0	2	0	0	0	−2	0

Sub-group* of order 192 of above group of order 1344. Compound character $[8]+[62]+2[4^2]+[42^2]+[3^21^2]+2[2^4]+[2^21^4]+[1^8]$.

Cycles	1^8	1^42^2	1^42^2	2^4	2^4	2^4	2^4	1^23^2	1^224	26	4^2	4^2	4^2
Order	1	12	6	12	6	1	6	32	24	32	24	24	12
	1	1	1	1	1	1	1	1	1	1	1	1	1
	3	1	−1	1	−1	3	3	0	−1	0	−1	1	−1
	2	0	2	0	2	2	2	−1	0	−1	0	0	2
	3	−1	−1	−1	−1	3	3	0	1	0	1	−1	−1
	1	−1	1	−1	1	1	1	1	−1	1	−1	−1	1
	3	−1	3	−1	−1	3	−1	0	−1	0	1	−1	−1
	3	1	3	1	−1	3	−1	0	1	0	−1	−1	−1
	3	1	−1	1	3	3	−1	0	−1	0	1	−1	−1
	3	−1	−1	−1	3	3	−1	0	1	0	−1	1	−1
	6	0	−2	0	−2	6	−2	0	0	0	0	0	2
	4	2	0	−2	0	−4	0	1	0	−1	0	0	0
	4	−2	0	2	0	−4	0	1	0	−1	0	0	0
	8	0	0	0	0	8	0	−1	0	1	0	0	0

* This group is the positive group of the group of order 384 given below. The characters of the latter group may be found by the method given in the text, but an easier method is to find the self-associated characters of the symmetric group corresponding to

Sub-group of order 384. Compound character

$$[8]+[62]+[4^2]+[42^2]+[2^4].$$

Cycles	1^8	$1^6 2$	$1^4 4$	$1^4 2^2$	$1^4 2^2$	16	$1^2 24$	$1^2 2^3$	$1^2 2^3$	$1^2 3^2$	8	4^2	4^2	$2^2 4$	$2^2 4$	26	23^2	2^4	2^4	2^4
Order	1	4	12	12	6	32	24	24	4	32	48	48	12	24	12	32	32	12	12	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	−1	−1	1	1	−1	1	−1	−1	1	−1	1	1	−1	−1	1	−1	1	1	1
	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	1	1	−1	1	1
	1	−1	1	−1	1	−1	−1	1	−1	1	1	−1	1	−1	1	1	−1	−1	1	1
	2	2	0	0	2	−1	0	0	2	−1	0	0	2	2	0	−1	−1	0	2	2
	2	−2	0	0	2	1	0	0	−2	−1	0	0	2	−2	0	−1	1	0	2	2
	6	0	0	2	−2	0	−2	0	0	0	0	0	−2	0	0	0	0	2	2	6
	6	0	0	−2	−2	0	2	0	0	0	0	0	−2	0	0	0	0	−2	2	6
	3	3	1	1	3	0	1	1	3	0	−1	−1	−1	−1	1	0	0	1	−1	3
	3	−3	−1	1	3	0	1	−1	−3	0	1	−1	−1	1	−1	0	0	1	−1	3
	3	3	−1	−1	3	0	−1	−1	3	0	1	1	−1	−1	−1	0	0	−1	−1	3
	3	−3	1	−1	3	0	−1	1	−3	0	−1	1	−1	1	1	0	0	−1	−1	3
	4	2	2	2	0	1	0	0	−2	1	0	0	0	0	−2	−1	−1	−2	0	−4
	4	−2	−2	2	0	−1	0	0	2	1	0	0	0	0	2	−1	1	−2	0	−4
	4	2	−2	−2	0	1	0	0	−2	1	0	0	0	0	2	−1	−1	2	0	−4
	4	−2	2	−2	0	−1	0	0	2	1	0	0	0	0	−2	−1	1	2	0	−4
	6	0	2	0	−2	0	0	−2	0	0	0	0	2	0	2	0	0	0	−2	6
	6	0	−2	0	−2	0	0	2	0	0	0	0	2	0	−2	0	0	0	−2	6
	8	4	0	0	0	−1	0	0	−4	−1	0	0	0	0	0	1	1	0	0	−8
	8	−4	0	0	0	1	0	0	4	−1	0	0	0	0	0	1	−1	0	0	−8

Sub-group of order 168 of above group of order 1344. Compound character $[8]+[51^4]+[4^2]+[431]+[41^3]+[32^2 1]+[2^4]+[1^8]$.

Cycles	1^8	2^4	$1^2 3^2$	$1^2 3^2$	26	26	17	17
Order	1	7	28	28	28	28	24	24
	1	1	1	1	1	1	1	1
	3	3	0	0	0	0	$\frac{1}{2}(-1+i\sqrt{7})$	$\frac{1}{2}(-1-i\sqrt{7})$
	3	3	0	0	0	0	$\frac{1}{2}(-1-i\sqrt{7})$	$\frac{1}{2}(-1+i\sqrt{7})$
	7	−1	1	1	−1	−1	0	0
	7	−1	ω	ω^2	$-\omega$	$-\omega^2$	0	0
	7	−1	ω^2	ω	$-\omega^2$	$-\omega$	0	0
	1	1	ω	ω^2	ω	ω^2	1	1
	1	1	ω^2	ω	ω^2	ω	1	1

(ω is a complex cube root of unity.)

the simple characters of the above group. These may be divided into two associated compound characters which correspond to simple characters of the group of order 384. The latter group is a maximal sub-group of the symmetric group.

Sub-group* of order 168 of above group of order 1344. Compound character $[8]+[51^3]+2[4^2]+[42^2]+[41^4]+[3^2 1^2]+2[2^4]+[1^8]$.

Cycles	1^8	2^4	$1^2 3^2$	4^2	17	17
Order	1	21	56	42	24	24
	1	1	1	1	1	1
	6	2	0	0	-1	-1
	7	-1	1	-1	0	0
	8	0	-1	0	1	1
	3	-1	0	1	$\frac{1}{2}(-1+i\sqrt{7})$	$\frac{1}{2}(-1-i\sqrt{7})$
	3	-1	0	1	$\frac{1}{2}(-1-i\sqrt{7})$	$\frac{1}{2}(-1+i\sqrt{7})$

Sub-group of order 336. Compound character

$$[8]+[4^2]+[42^2]+[41^4]+[2^4].$$

Cycles	1^8	2^4	$1^2 3^2$	4^2	17	$1^2 6$	$1^2 2^3$	8	8
Order	1	21	56	42	48	56	28	42	42
	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	-1	-1	-1	-1
	6	2	0	0	-1	0	0	$\sqrt{2}$	$-\sqrt{2}$
	6	2	0	0	-1	0	0	$-\sqrt{2}$	$\sqrt{2}$
	7	-1	1	-1	0	1	1	-1	-1
	7	-1	1	-1	0	-1	-1	1	1
	8	0	-1	0	1	-1	2	0	0
	8	0	-1	0	1	1	-2	0	0
	6	-2	0	2	-1	0	0	0	0

* This group is the positive group of the group of order 336 given below, of which the characters may be found by the same method as for the group of order 384.

Degree 9.

Sub-group of order 1512. Compound character

$$[9]+[4^2\,1]+[51^4]+[32^3]+[1^9].$$

Cycles	1^9	$1^3 3^2$	$1^3 3^2$	$1^2 7$	126	126	12^4	3^3	9	9	9
Order	1	84	84	216	252	252	63	56	168	168	168
	1	1	1	1	1	1	1	1	1	1	1
	1	ω	ω^2	1	ω	ω^2	1	1	1	ω	ω^2
	1	ω^2	ω	1	ω^2	ω	1	1	1	ω^2	ω
	7	1	1	0	-1	-1	-1	-2	1	1	1
	7	ω	ω^2	0	$-\omega$	$-\omega^2$	-1	-2	1	ω	ω^2
	7	ω^2	ω	0	$-\omega^2$	$-\omega$	-1	-2	1	ω^2	ω
	8	2	2	1	0	0	0	-1	-1	-1	-1
	8	2ω	$2\omega^2$	1	0	0	0	-1	-1	$-\omega$	$-\omega^2$
	8	$2\omega^2$	2ω	1	0	0	0	-1	-1	$-\omega^2$	$-\omega$
	21	0	0	0	0	0	-3	3	0	0	0
	27	0	0	-1	0	0	3	0	0	0	0

Sub-group of order 504 of above group.

Cycles	1^9	$1^2 7$	$1^2 7$	$1^2 7$	12^4	3^3	9	9	9
Order	1	72	72	72	63	56	56	56	56
	1	1	1	1	1	1	1	1	1
	7	0	0	0	-1	-2	1	1	1
	7	0	0	0	-1	1	-2	1	1
	7	0	0	0	-1	1	1	-2	1
	7	0	0	0	-1	1	1	1	-2
	9	0	-2	1	1	0	0	0	0
	9	-2	1	0	1	0	0	0	0
	9	1	0	-2	1	0	0	0	0
	8	1	1	1	0	-1	-1	-1	-1

Sub-group of order 1296. Compound character
$[9]+[72]+[63]+[52^2]+[4^2 1]$.

Cycles	1^9	$1^7 2$	$1^6 3$	$1^5 2^2$	$1^4 23$	$1^3 6$	$1^3 24$	$1^3 2^3$	$1^3 2^3$	$1^3 3^2$	$1^2 2^2 3$	$12^2 4$	126	123^2	12^4	3^3	3^3	36	36	234	$2^3 3$	9
Order	1	9	6	27	36	36	54	18	27	12	54	162	108	36	54	8	72	72	216	108	36	144
A	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
B	1	−1	1	1	−1	−1	1	−1	−1	1	1	−1	1	−1	1	1	1	−1	−1	1	−1	1
C	1	1	1	1	1	−1	−1	−1	1	1	1	−1	−1	1	−1	1	1	−1	1	−1	−1	1
D	1	−1	1	1	−1	1	−1	1	−1	1	1	1	−1	−1	−1	1	1	1	−1	−1	1	1
E	2	2	2	2	2	0	0	0	2	2	2	0	0	2	0	2	−1	0	−1	0	0	−1
F	2	−2	2	2	−2	0	0	0	−2	2	2	0	0	−2	0	2	−1	0	1	0	0	−1
G	6	4	3	2	1	2	2	2	0	0	−1	0	0	−2	0	−3	0	−1	0	−1	−1	0
H	6	−4	3	2	−1	−2	2	−2	0	0	−1	0	0	2	0	−3	0	1	0	−1	1	0
J	6	4	3	2	1	−2	−2	−2	0	0	−1	0	0	−2	0	−3	0	1	0	1	1	0
K	6	−4	3	2	−1	2	−2	2	0	0	−1	0	0	2	0	−3	0	−1	0	1	−1	0
L	12	4	0	0	−2	1	0	−2	0	−3	0	0	−1	1	2	3	0	−1	0	0	2	0
M	12	−4	0	0	2	−1	0	2	0	−3	0	0	−1	−1	2	3	0	1	0	0	−2	0
N	12	4	0	0	−2	−1	0	2	0	−3	0	0	1	1	−2	3	0	1	0	0	−2	0
P	12	−4	0	0	2	1	0	−2	0	−3	0	0	1	−1	−2	3	0	−1	0	0	2	0
Q	3	−1	3	−1	−1	−1	1	−1	3	3	−1	1	−1	−1	−1	3	0	−1	0	1	−1	0
R	3	1	3	−1	1	1	1	1	−3	3	−1	−1	−1	1	−1	3	0	1	0	1	1	0
S	3	−1	3	−1	−1	1	−1	1	3	3	−1	−1	1	−1	1	3	0	1	0	−1	1	0
T	3	1	3	−1	1	−1	−1	−1	−3	3	−1	1	1	1	1	3	0	−1	0	−1	−1	0
U	8	0	−4	0	0	2	0	−4	0	2	0	0	0	0	0	−1	2	1	0	0	−2	−1
V	8	0	−4	0	0	−2	0	4	0	2	0	0	0	0	0	−1	2	−1	0	0	2	−1
W	12	0	6	−4	0	0	0	0	0	0	2	0	0	0	0	−6	0	0	0	0	0	0
X	16	0	−8	0	0	0	0	0	0	4	0	0	0	0	0	−2	−2	0	0	0	0	1

Sub-group of order 648 of above group of order 1296.
Compound character $A+B$.

Cycles	1^9	$1^6 3$	$1^5 2^2$	$1^3 24$	$1^3 3^2$	$1^3 2^2 3$	126	12^4	3^3	3^3	234	234	9	9
Order	1	6	27	54	12	54	108	54	8	72	54	54	72	72
	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	−1	1	1	−1	−1	1	1	−1	−1	1	1
	2	2	2	0	2	2	0	0	2	−1	0	0	−1	−1
	6	3	2	2	0	−1	0	0	−3	0	−1	−1	0	0
	6	3	2	−2	0	−1	0	0	−3	0	1	1	0	0
	12	0	0	0	−3	0	−1	2	3	0	0	0	0	0
	12	0	0	0	−3	0	1	−2	3	0	0	0	0	0
	3	3	−1	1	3	−1	−1	−1	3	0	1	1	0	0
	3	3	−1	−1	3	−1	1	1	3	0	−1	−1	0	0
	8	−4	0	0	2	0	0	0	−1	2	0	0	−1	−1
	6	3	−2	0	0	1	0	0	−3	0	$\omega^2-\omega$	$\omega-\omega^2$	0	0
	6	3	−2	0	0	1	0	0	−3	0	$\omega-\omega^2$	$\omega^2-\omega$	0	0
	8	−4	0	0	2	0	0	0	−1	−1	−1	−1	2	−1
	8	−4	0	0	2	0	0	0	−1	−1	−1	−1	−1	2

Sub-group of order 648 of above group of order 1296.

Compound character $A+C$.

Cycles	1^9	$1^7 2$	$1^6 3$	$1^5 2^2$	$1^4 23$	$1^3 2^3$	$1^3 3^2$	$1^2 2^2 3$	123^2	123^2	3^3	3^3	3^3	36	36	9	9
Order	1	9	6	27	36	27	12	54	18	18	8	36	36	108	108	72	72
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	−1	1	1	−1	−1	1	1	−1	−1	1	1	1	−1	−1	1	1
	1	1	1	1	1	1	1	1	1	1	1	ω	ω^2	ω	ω^2	ω	ω^2
	1	1	1	1	1	1	1	1	1	1	1	ω^2	ω	ω^2	ω	ω^2	ω
	1	−1	1	1	−1	−1	1	1	−1	−1	1	ω	ω^2	$-\omega$	$-\omega^2$	ω	ω^2
	1	−1	1	1	−1	−1	1	1	−1	−1	1	ω^2	ω	$-\omega^2$	$-\omega$	ω^2	ω
	6	4	3	2	1	0	0	−1	−2	−2	−3	0	0	0	0	0	0
	6	−4	3	2	−1	0	0	−1	2	2	−3	0	0	0	0	0	0
	12	4	0	0	−2	0	−3	0	1	1	3	0	0	0	0	0	0
	12	−4	0	0	2	0	−3	0	−1	−1	3	0	0	0	0	0	0
	3	1	3	−1	1	−3	3	−1	1	1	3	0	0	0	0	0	0
	3	−1	3	−1	−1	3	3	−1	−1	−1	3	0	0	0	0	0	0
	6	0	3	−2	0	0	0	1	−3	3	−3	0	0	0	0	0	0
	6	0	3	−2	0	0	0	1	3	−3	−3	0	0	0	0	0	0
	8	0	−4	0	0	0	2	0	0	0	−1	2	2	0	0	−1	−1
	8	0	−4	0	0	0	2	0	0	0	−1	2ω	$2\omega^2$	0	0	$-\omega$	$-\omega^2$
	8	0	−4	0	0	0	2	0	0	0	−1	$2\omega^2$	2ω	0	0	$-\omega^2$	$-\omega$

Sub-group of order 648 of above group of order 1296.

Compound character $A+D$.

Cycles	1^9	$1^6 3$	$1^5 2^2$	$1^3 6$	$1^3 2^3$	$1^3 3^2$	$1^2 2^2 3$	$12^2 4$	3^3	3^3	3^3	36	36	$2^3 3$	$2^3 3$	9	9
Order	1	6	27	36	18	12	54	162	4	4	72	36	36	18	18	72	72
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	−1	−1	1	1	−1	1	1	1	−1	−1	−1	−1	1	1
	2	2	2	0	0	2	2	0	2	2	−1	0	0	0	0	−1	−1
	6	3	2	2	2	0	−1	0	−3	−3	0	−1	−1	−1	−1	0	0
	6	3	2	−2	−2	0	−1	0	−3	−3	0	1	1	1	1	0	0
	12	0	0	1	−2	−3	0	0	3	3	0	−1	−1	2	2	0	0
	12	0	0	−1	2	−3	0	0	3	3	0	1	1	−2	−2	0	0
	3	3	−1	1	1	3	−1	−1	3	3	0	1	1	1	1	0	0
	3	3	−1	−1	−1	3	−1	1	3	3	0	−1	−1	−1	−1	0	0
	4	−2	0	1	−2	1	0	0	$1+3\omega$	$1+3\omega^2$	1	ω	ω^2	-2ω	$-2\omega^2$	ω	ω^2
	4	−2	0	1	−2	1	0	0	$1+3\omega^2$	$1+3\omega$	1	ω^2	ω	$-2\omega^2$	-2ω	ω^2	ω
	4	−2	0	−1	2	1	0	0	$1+3\omega$	$1+3\omega^2$	1	$-\omega$	$-\omega^2$	2ω	$2\omega^2$	ω	ω^2
	4	−2	0	−1	2	1	0	0	$1+3\omega^2$	$1+3\omega$	1	$-\omega^2$	$-\omega$	$2\omega^2$	2ω	ω^2	ω
	6	3	−2	0	0	0	1	0	−3	−3	0	$-\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	0	0
	6	3	−2	0	0	0	1	0	−3	−3	0	$\sqrt{3}$	$-\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	0	0
	8	−4	0	0	0	2	0	0	$2+6\omega$	$2+6\omega^2$	0	0	0	0	0	$-\omega$	$-\omega^2$
	8	−4	0	0	0	2	0	0	$2+6\omega^2$	$2+6\omega$	0	0	0	0	0	$-\omega^2$	$-\omega$

Sub-group of order 324 of above group of order 1296.
Compound character $A+B+C+D$.

Cycles	1^9	$1^6 3$	$1^5 2^2$	$1^3 3^2$	$1^2 2^2 3$	3^3	3^3	3^3	3^3	9	9	9	9
Order	1	6	27	12	54	4	4	36	36	36	36	36	36
	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	ω	ω^2	ω	ω	ω^2	ω^2
	1	1	1	1	1	1	1	ω^2	ω	ω^2	ω^2	ω	ω
	6	3	2	0	-1	-3	-3	0	0	0	0	0	0
	6	3	-2	0	1	-3	-3	0	0	0	0	0	0
	12	0	0	-3	0	3	3	0	0	0	0	0	0
	3	3	-1	3	-1	3	3	0	0	0	0	0	0
	4	-2	0	1	0	$1+3\omega$	$1+3\omega^2$	1	1	ω	ω^2	ω	ω^2
	4	-2	0	1	0	$1+3\omega^2$	$1+3\omega$	1	1	ω^2	ω	ω^2	ω
	4	-2	0	1	0	$1+3\omega$	$1+3\omega^2$	ω	ω^2	ω^2	1	1	ω
	4	-2	0	1	0	$1+3\omega^2$	$1+3\omega$	ω^2	ω	ω	1	1	ω^2
	4	-2	0	1	0	$1+3\omega$	$1+3\omega^2$	ω	ω^2	1	ω^2	ω	1
	4	-2	0	1	0	$1+3\omega^2$	$1+3\omega$	ω^2	ω	1	ω	ω^2	1

Sub-group of order 324 of above group of order 1296.
Compound character $A+S$.

Cycles	1^9	$1^6 3$	$1^3 6$	$1^3 2^3$	$1^3 2^3$	$1^3 3^2$	$1^3 3^2$	126	12^4	3^3	3^3	3^3	36	36	36	$2^3 3$	9
Order	1	6	18	9	27	6	6	54	27	6	2	18	18	18	54	18	36
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	-1	-1	-1	1	1	1	1	1	1	1	-1	-1	-1	-1	1
	1	1	-1	-1	1	1	1	-1	-1	1	1	1	-1	-1	1	-1	1
	1	1	1	1	-1	1	1	-1	-1	1	1	1	1	1	-1	1	1
	2	2	0	0	2	2	2	0	0	2	2	-1	0	0	-1	0	-1
	2	2	0	0	-2	2	2	0	0	2	2	-1	0	0	1	0	-1
	2	-1	-1	2	0	-1	2	0	0	-1	2	2	1	-2	0	1	-1
	2	-1	1	-2	0	-1	2	0	0	-1	2	2	-1	2	0	-1	-1
	6	3	2	2	0	0	0	0	0	-3	-3	0	-1	-1	0	-1	0
	6	3	-2	-2	0	0	0	0	0	-3	-3	0	1	1	0	1	0
	6	0	1	-2	0	-3	0	0	0	3	-3	0	-1	-1	0	2	0
	6	0	-1	2	0	-3	0	0	0	3	-3	0	1	1	0	-2	0
	6	-3	1	-2	0	3	0	0	0	0	-3	0	2	-1	0	-1	0
	6	-3	-1	2	0	3	0	0	0	0	-3	0	-2	1	0	1	0
	6	0	0	0	0	0	-3	-1	2	0	6	0	0	0	0	0	0
	6	0	0	0	0	0	-3	1	-2	0	6	0	0	0	0	0	0
	4	-2	0	0	0	-2	4	0	0	-2	4	-2	0	0	0	0	1

Sub-group of order 162 of above group of order 1296.

Compound character $A+B+S+T$.

Cycles	1^9	$1^6 3$	$1^3 3^2$	$1^3 3^2$	$1^3 3^2$	126	126	12^4	3^3	3^3	3^3	9	9
Order	1	6	6	3	3	27	27	27	6	2	18	18	18
	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	−1	−1	−1	1	1	1	1	1
	2	−1	−1	2	2	0	0	0	−1	2	−1	2	−1
	2	−1	−1	2	2	0	0	0	−1	2	−1	−1	2
	2	−1	−1	2	2	0	0	0	−1	2	2	−1	−1
	2	2	2	2	2	0	0	0	2	2	−1	−1	−1
	3	0	0	3ω	$3\omega^2$	ω	ω^2	1	0	3	0	0	0
	3	0	0	$3\omega^2$	3ω	ω^2	ω	1	0	3	0	0	0
	3	0	0	3ω	$3\omega^2$	$-\omega$	$-\omega^2$	−1	0	3	0	0	0
	3	0	0	$3\omega^2$	3ω	$-\omega^2$	$-\omega$	−1	0	3	0	0	0
	6	3	0	0	0	0	0	0	−3	−3	0	0	0
	3	0	−3	0	0	0	0	0	3	−3	0	0	0
	6	−3	3	0	0	0	0	0	0	−3	0	0	0

Sub-group of order 81 of above group of order 162.

Cycles	1^9	$1^6 3$	$1^6 3$	$1^3 3^2$	$1^3 3^2$	$1^3 3^2$	$1^3 3^2$	3^3	3^3	3^3	3^3	3^3	3^3	9	9	9	9
Order	1	3	3	3	3	3	3	3	3	1	1	9	9	9	9	9	9
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	ω	ω^2	ω	ω^2	1	1	ω	ω^2	1	1	ω	ω^2	1	1	ω	ω^2
	1	ω^2	ω	ω^2	ω	1	1	ω^2	ω	1	1	ω^2	ω	1	1	ω^2	ω
	1	ω^2	ω	ω^2	ω	1	1	ω^2	ω	1	1	ω	ω^2	ω	ω^2	1	1
	1	ω	ω^2	ω	ω^2	1	1	ω	ω^2	1	1	ω^2	ω	ω^2	ω	1	1
	1	1	1	1	1	1	1	1	1	1	1	ω	ω^2	ω^2	ω	ω^2	ω
	1	1	1	1	1	1	1	1	1	1	1	ω^2	ω	ω	ω^2	ω	ω^2
	1	ω	ω^2	ω	ω^2	1	1	ω	ω^2	1	1	1	1	ω	ω^2	ω^2	ω
	1	ω^2	ω	ω^2	ω	1	1	ω^2	ω	1	1	1	1	ω^2	ω	ω	ω^2
	3	0	0	0	0	3ω	$3\omega^2$	0	0	3	3	0	0	0	0	0	0
	3	0	0	0	0	$3\omega^2$	3ω	0	0	3	3	0	0	0	0	0	0
	3	$1-\omega$	$1-\omega^2$	$\omega-\omega^2$	$\omega^2-\omega$	0	0	ω^2-1	$\omega-1$	3ω	$3\omega^2$	0	0	0	0	0	0
	3	$1-\omega^2$	$1-\omega$	$\omega^2-\omega$	$\omega-\omega^2$	0	0	$\omega-1$	ω^2-1	$3\omega^2$	3ω	0	0	0	0	0	0
	3	$\omega-\omega^2$	$\omega^2-\omega$	ω^2-1	$\omega-1$	0	0	$1-\omega$	$1-\omega^2$	3ω	$3\omega^2$	0	0	0	0	0	0
	3	$\omega^2-\omega$	$\omega-\omega^2$	$\omega-1$	ω^2-1	0	0	$1-\omega^2$	$1-\omega$	$3\omega^2$	3ω	0	0	0	0	0	0
	3	ω^2-1	$\omega-1$	$1-\omega$	$1-\omega^2$	0	0	$\omega-\omega^2$	$\omega^2-\omega$	3ω	$3\omega^2$	0	0	0	0	0	0
	3	$\omega-1$	ω^2-1	$1-\omega^2$	$1-\omega$	0	0	$\omega^2-\omega$	$\omega-\omega^2$	$3\omega^2$	3ω	0	0	0	0	0	0

Metacyclic group of order 54, sub-group of above group of order 162.

Cycles	1^9	$1^3 3^2$	$1^3 3^2$	126	126	12^4	3^3	9	9	9
Order	1	3	3	9	9	9	2	6	6	6
	1	1	1	1	1	1	1	1	1	1
	1	1	1	-1	-1	-1	1	1	1	1
	1	ω	ω^2	ω	ω^2	1	1	1	ω	ω^2
	1	ω^2	ω	ω^2	ω	1	1	1	ω^2	ω
	1	ω	ω^2	$-\omega$	$-\omega^2$	-1	1	1	ω	ω^2
	1	ω^2	ω	$-\omega^2$	$-\omega$	-1	1	1	ω^2	ω
	2	2	2	0	0	0	2	-1	-1	-1
	2	2ω	$2\omega^2$	0	0	0	2	-1	$-\omega$	$-\omega^2$
	2	$2\omega^2$	2ω	0	0	0	2	-1	$-\omega^2$	$-\omega$
	6	0	0	0	0	0	-3	0	0	0

Sub-group of order 432. Compound character

$$[9]+[63]+[52^2]+[521^2]+[4^2 1]+[3^3]+[32^3]+[42^2 1]+[41^5].$$

Cycles	1^9	$1^3 2^3$	$1^3 3^2$	18	18	14^2	126	12^4	3^3	3^3	36
Order	1	36	24	54	54	54	72	9	8	48	72
	1	1	1	1	1	1	1	1	1	1	1
	1	-1	1	-1	-1	1	1	1	1	1	-1
	2	0	-1	0	0	2	-1	2	2	-1	0
	2	0	-1	$i\sqrt{2}$	$-i\sqrt{2}$	0	1	-2	2	-1	0
	2	0	-1	$-i\sqrt{2}$	$i\sqrt{2}$	0	1	-2	2	-1	0
	3	1	0	-1	-1	-1	0	3	3	0	1
	3	-1	0	1	1	-1	0	3	3	0	-1
	8	2	2	0	0	0	0	0	-1	-1	-1
	8	-2	2	0	0	0	0	0	-1	-1	1

Sub-group of order 144 of above group of order 432.

Cycles	1^9	$1^3 2^3$	18	18	14^2	14^2	12^4	3^3	36
Order	1	12	18	18	36	18	9	8	24
	1	1	1	1	1	1	1	1	1
	1	−1	−1	−1	1	1	1	1	−1
	1	−1	1	1	−1	1	1	1	−1
	1	1	−1	−1	−1	1	1	1	1
	2	0	0	0	0	−2	2	2	0
	2	0	$i\sqrt{2}$	$-i\sqrt{2}$	0	0	−2	2	0
	2	0	$-i\sqrt{2}$	$i\sqrt{2}$	0	0	−2	2	0
	8	2	0	0	0	0	0	−1	−1
	8	−2	0	0	0	0	0	−1	1

References.

1. W. Burnside, *The theory of groups* (second edition), (Cambridge, 1911).
2. A. Cayley, *Quart. J. of Math.*, 25 (1891), 77–88, 137–155; *Collected Works*, 13 (Cambridge, 1897), 117–149.
3. F. W. Cole, *Bull. New York Math. Soc.*, 2 (1893), 184–190.
4. L. E. Dickson, *Algebras and their arithmetics* (Chicago, 1923).

G. Frobenius, *Berliner Sitzungsberichte*:

5. ———— "Ueber Gruppencharaktere", (1896), 985–1021.
6, 7. ———— "Ueber die Darstellung der endliche Gruppen durch lineare Substitutionen", (1897), 994–1015; (1899), 482–500.
8. ———— "Relationen zwischen die Charaktern einer Gruppe und denen ihrer Untergruppen", (1898), 501–515.
9, 10. ———— "Ueber die Charaktere der symmetrischen und alternierenden Gruppen", (1900), 516–534; (1901), 303–315.
11. ———— "Ueber die charakteristischen Einheiten der symmetrischen Gruppe", (1903), 328–358.
12. M. E. C. Jordan, *Comptes rendus*, 73 (1872), 175–7.
13. D. E. Littlewood and A. R. Richardson, "Group characters and algebra", *Phil. Trans. Roy. Soc.* (A), 233 (1934), 99.
14. G. A. Miller, H. F. Blichfeldt, and L. E. Dickson, *Finite groups* (New York, 1916).

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THE CHARACTERS OF THE SYMMETRIC GROUP OF ORDER 11!

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This paper is a supplement to the paper by D. E. Littlewood (*supra*, 150–199), which it follows.

The method of calculation is essentially the same as described by Littlewood and Richardson in their paper on "Immanants of some special matrices" in *Quart. J. of Math.* (Oxford), 5 (1934), 269-282.

One half of the table only is given here. The characteristics corresponding to the partitions

$$[432^2],\ [3^3\,2],\ [4^2\,1^3],\ [52^2\,1^2],\ [4321^2],\ [3^3\,1^2],\ [42^3\,1],$$

$$[3^2\,2^2\,1],\ [32^4],\ [521^4],\ [431^4],\ [42^2\,1^3],\ [3^2\,21^3],$$

$$[32^3\,1^2],\ [2^5\,1],\ [51^6],\ [421^5],\ [3^2\,1^5],\ [32^2\,1^4],$$

$$[2^4\,1^3],\ [41^7],\ [321^6],\ [2^3\,1^5],\ [31^8],\ [2^2\,1^7],\ [21^9],$$

of the other half, are obtained by multiplying the corresponding characteristics for the conjugate partitions, namely,

$$[4^2\,21],\ [4^2\,3],\ [52^3],\ [531^3],\ [5321],$$

$$[53^2],\ [541^2],\ [542],\ [5^2\,1],\ [621^3],\ [62^2\,1],$$

$$[631^2],\ [632],\ [641],\ [65],\ [71^4],\ [721^2],\ [72^2],$$

$$[731],\ [74],\ [81^3],\ [821],\ [83],\ [91^2],\ [92],\ [10,\ 1],$$

respectively, by the characteristics of $[1^{11}]$ given in the last row of the table.

I am indebted to Prof. Richardson and Mr. Littlewood for their guidance.

Degree 11.

Class	1^{11}	$1^9 2$	$1^8 3$	$1^7 4$	$1^7 2^2$	$1^6 23$	$1^6 5$	$1^5 6$	$1^5 24$	$1^5 2^3$	$1^5 3^2$	$1^4 25$	$1^4 2^2 3$	$1^4 34$
Order	1	55	330	1980	990	9240	11088	55440	41580	6930	18480	1 66320	69300	1 38600
[11]	1	1	1	1	1	1	1	1	1	1	1	1	1	1
[10, 1]	10	8	7	6	6	5	5	4	4	4	4	3	3	3
[92]	44	28	20	14	16	10	9	5	6	8	5	3	4	2
[91^2]	45	27	21	15	13	9	10	6	5	3	6	2	1	3
[83]	110	56	29	14	26	11	5	0	4	12	2	1	5	−1
[821]	231	105	63	35	35	15	16	5	5	5	3	0	−1	−1
[81^3]	120	48	36	20	8	6	10	4	0	−8	6	−2	−4	2
[74]	165	69	21	1	29	9	−5	−5	1	13	3	−1	5	1
[731]	550	190	70	20	50	10	0	−5	0	10	−5	0	2	−4
[72^2]	385	119	49	21	21	−1	5	0	−1	3	−8	−1	−3	−3
[721^2]	594	162	90	36	6	0	9	0	−4	−18	0	−3	−6	0
[71^4]	210	42	42	14	−14	0	5	1	−6	−14	9	−3	−2	2
[65]	132	48	6	−8	20	6	−8	−4	0	8	6	−2	2	4
[641]	693	189	21	−21	49	9	−17	−6	−1	9	6	−1	1	3
[632]	990	216	21	−6	34	−9	−5	0	−4	12	−12	1	1	−3
[631^2]	1232	224	56	0	0	−4	−8	−4	0	−16	−4	4	0	0
[$62^2 1$]	1100	160	50	20	−20	−20	0	0	0	0	−10	0	−2	2
[621^3]	924	84	84	14	−56	−6	−1	−1	−6	−16	9	−1	4	2
[$5^2 1$]	330	78	−6	−20	22	6	−10	−1	0	2	9	−2	−2	4
[542]	990	162	−42	−36	34	0	−5	6	−4	6	6	−3	−2	0
[541^2]	1155	147	−21	−35	7	9	−10	1	5	−17	9	2	−5	1
[53^2]	660	84	−36	−14	16	−6	5	5	−6	8	−3	−1	4	−2
[5321]	2310	168	−63	−14	−14	−15	5	4	4	4	−6	3	1	1
[531^3]	1540	−28	28	14	−56	−10	−5	1	6	16	1	−3	4	2
[52^3]	825	15	−15	15	−15	−15	0	0	5	15	0	0	−3	3
[$4^2 3$]	462	42	−42	−14	14	0	7	5	−6	2	3	−3	2	−2
[$4^2 21$]	1320	48	−84	−20	8	6	10	4	0	−8	6	−2	−4	−2
*[61^5]	252	0	42	0	−28	0	2	0	−8	0	12	0	2	0
*[$43^2 1$]	1188	0	−90	0	12	0	18	0	−8	0	0	0	6	0
[1^{11}]	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1

* Denotes a self-associated partition and character.

Degree 11—*continued.*

Class	$1^4 7$	$1^3 8$	$1^3 4^2$	$1^3 2^2 4$	$1^3 26$	$1^3 23^2$	$1^3 35$	$1^3 2^1$	$1^2 45$	$1^2 3^3$	$1^2 36$	$1^2 234$	$1^2 27$	$1^2 2^2 5$
Order	2 37600	8 31600	2 07900	2 07900	5 54400	1 84800	4 43520	17325	9 97920	1 23200	11 08800	8 31600	14 25600	4 98960
[11]	1	1	1	1	1	1	1	1	1	1	1	1	1	1
[10, 1]	3	2	2	2	2	2	2	2	1	1	1	1	1	1
[92]	2	0	0	2	1	1	0	4	−1	−1	−1	0	0	1
$[91^2]$	3	1	1	−1	0	0	1	−3	0	0	0	−1	−1	−2
[83]	−2	−2	−2	2	0	2	−1	6	−1	2	0	1	0	1
[821]	0	−1	−1	−1	−1	−3	−2	−1	0	−3	−1	−1	0	0
$[81^3]$	1	0	0	−4	−2	0	1	−8	0	3	1	0	−1	−2
[74]	−3	−1	1	1	−1	3	1	5	1	3	1	1	−1	−1
[731]	−3	0	−2	0	1	1	0	−2	0	1	1	0	1	0
$[72^3]$	0	1	1	1	0	−4	−1	9	1	−2	0	−1	0	1
$[721^2]$	−1	0	2	0	0	0	0	−6	1	0	0	2	1	1
$[71^4]$	0	0	−2	−2	−1	3	2	2	−1	3	1	0	0	1
[65]	−1	0	4	0	−2	0	1	4	2	−3	−1	0	−1	0
[641]	0	1	1	−1	0	0	1	−3	−1	0	0	−1	0	−1
[632]	3	2	−2	−2	0	0	1	6	−1	0	0	−1	−1	−1
$[631^2]$	0	0	0	0	2	2	1	−16	0	−1	−1	0	0	0
$[62^2 1]$	1	0	4	4	0	−2	0	12	0	2	0	0	−1	0
$[621^3]$	0	0	0	2	1	3	−1	4	−1	−3	−1	0	0	−1
$[5^2 1]$	1	0	2	0	−1	−3	−1	2	0	−3	−1	0	1	2
[542]	3	0	−2	0	0	0	−2	6	−1	0	0	2	1	−1
$[541^2]$	0	−1	−1	1	1	−3	−1	−5	0	3	1	−1	0	2
$[53^2]$	2	0	0	−2	−1	3	−1	−4	1	3	−1	0	0	1
[5321]	0	−2	−2	−2	−2	0	2	−2	1	−3	1	1	0	1
$[531^3]$	0	0	0	2	−1	−1	−2	−4	−1	1	1	0	0	−1
$[52^3]$	−1	−1	1	−1	0	0	0	9	0	6	0	−1	1	0
$[4^2 3]$	0	0	2	2	1	3	−2	−2	1	3	−1	0	0	−1
$[4^2 21]$	−3	0	0	4	2	0	1	8	0	−3	1	0	−1	−2
*$[61^5]$	0	0	−4	0	0	0	2	12	0	0	0	−2	0	2
*$[43^2 1]$	−2	0	4	0	0	0	0	−12	0	0	0	−2	0	2
$[1^{11}]$	1	−1	1	−1	1	−1	1	1	−1	1	−1	1	−1	1

* Denotes a self-associated partition and character.

Degree 11—*continued.*

Class	$1^2 2^3 3$	$1^2 9$	1, 10	182	162^2	142^3	12^5	173	1532	164	$14^2 2$	15^2	143^2	$13^2 2^2$
Order	1 38600	22 17600	39 91680	24 94800	8 31600	2 07900	10395	19 00800	13 30560	16 63200	6 23700	7 98336	5 54400	2 77200
[11]	1	1	1	1	1	1	1	1	1	1	1	1	1	1
[10, 1]	1	1	0	0	0	0	0	0	0	0	0	0	0	0
[92]	2	−1	−1	0	1	2	4	−1	0	−1	0	−1	−1	1
$[91^2]$	−3	0	0	−1	−2	−3	−5	0	−1	0	−1	0	0	−2
[83]	3	−1	0	0	0	0	0	1	1	0	0	0	2	2
[821]	−1	0	1	1	1	1	1	0	0	1	1	1	−1	−1
$[81^3]$	−2	0	0	0	0	0	0	1	1	0	0	0	2	2
[74]	1	0	0	−1	−1	1	5	0	−1	1	1	0	1	−1
[731]	−2	1	0	0	−1	−4	−10	0	0	−1	−2	0	−1	−1
$[72^2]$	3	1	0	−1	0	3	15	0	−1	0	−1	0	0	0
$[721^2]$	0	0	−1	0	0	0	−6	−1	0	0	2	−1	0	0
$[71^4]$	4	0	0	0	1	2	10	0	0	−1	−2	0	−1	1
[65]	2	0	0	0	0	0	0	−1	1	0	0	2	−2	2
[641]	−3	0	0	1	2	3	5	0	−1	0	1	−2	0	−2
[632]	3	0	0	0	0	0	0	0	1	0	0	0	0	4
$[631^2]$	−4	−1	0	0	0	0	0	0	1	0	0	2	0	0
$[62^2 1]$	0	−1	0	0	0	0	0	1	0	0	0	0	2	−2
$[621^3]$	2	0	1	0	−1	−2	−4	0	−1	1	0	−1	−1	1
$[5^2 1]$	2	0	0	0	−1	−4	−10	1	1	−1	−2	0	1	1
[542]	0	0	0	0	−2	0	10	0	0	0	2	0	0	−2
$[541^2]$	1	0	0	−1	1	1	−5	0	−1	1	−1	0	1	1
$[53^2]$	2	0	0	0	1	−2	−20	−1	−1	1	0	0	1	1
[5321]	1	0	0	0	0	0	0	0	0	0	0	0	−2	−2
$[531^3]$	−2	1	0	0	1	2	−20	0	0	−1	0	0	−1	1
$[52^3]$	−3	0	0	1	0	−3	15	−1	0	0	−1	0	0	0
$[4^2 3]$	−4	0	0	0	1	2	10	0	0	−1	−2	2	1	−1
$[4^2 21]$	−2	0	0	0	0	0	0	0	1	0	0	0	−2	2
*$[61^5]$	0	0	0	0	0	0	0	0	0	0	0	2	0	−4
*$[43^2 1]$	0	0	0	0	0	0	0	1	0	0	0	−2	0	0
$[1^{11}]$	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1

* Denotes a self-associated partition and character.

Degree 11—*continued.*

Class	56	245	29	$2^4 3$	$2^3 5$	$2^2 7$	23^3	236	$2^2 34$	$3^2 5$	47	34^2	38	11
Order	13 30560	9 97920	22 17600	34650	1 66320	7 12800	1 23200	11 08800	4 15800	4 43520	14 25600	4 15800	16 63200	36 28800
[11]	1	1	1	1	1	1	1	1	1	1	1	1	1	1
[10, 1]	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
[92]	0	1	1	4	3	2	1	1	2	0	0	0	0	0
$[91^2]$	1	0	0	−3	−2	−1	0	0	−1	1	1	1	1	1
[83]	0	−1	−1	−3	−3	−2	2	0	−1	2	0	1	1	0
[821]	0	0	0	−1	0	0	−3	−1	−1	−2	0	−1	−1	0
$[81^3]$	−1	0	0	4	2	1	3	1	2	1	−1	0	0	−1
[74]	0	1	0	5	3	1	−3	−1	1	−2	1	1	−1	0
[731]	0	0	1	−2	0	1	1	1	0	0	−1	−2	0	0
$[72^2]$	0	−1	−1	9	3	0	2	0	1	2	0	1	1	0
$[721^2]$	0	1	0	−6	−3	−1	0	0	0	0	1	2	0	0
$[71^4]$	1	−1	0	2	1	0	−3	−1	−2	−1	0	−2	0	1
[65]	1	0	0	−2	−2	−1	3	1	0	1	−1	−2	0	0
[641]	−1	−1	0	−3	−1	0	0	0	−1	1	0	1	1	0
[632]	0	1	0	−3	−3	−1	0	0	1	−2	1	1	−1	0
$[631^2]$	1	0	−1	8	4	0	−1	−1	0	1	0	0	0	0
$[62^2 1]$	0	0	1	−6	0	1	−2	0	−2	0	−1	−2	0	0
$[621^3]$	−1	−1	0	4	−1	0	3	1	2	−1	0	0	0	0
$[5^2 1]$	−1	0	0	2	2	1	−3	−1	0	−1	1	2	0	0
[542]	1	1	0	6	1	−1	0	0	0	1	−1	−2	0	0
$[541^2]$	1	0	0	−5	−2	0	3	1	1	−1	0	−1	−1	0
$[53^2]$	0	−1	0	−4	3	2	3	−1	−2	2	0	0	0	0
[5321]	−1	−1	0	1	−1	0	−3	1	1	−1	0	1	1	0
$[531^3]$	1	1	−1	−4	1	0	−1	−1	2	1	0	0	0	0
$[52^3]$	0	0	0	9	0	−1	6	0	−1	0	1	1	−1	0
$[4^2 3]$	0	−1	0	−2	−3	0	−3	1	2	−2	0	2	0	0
$[4^2 21]$	−1	0	0	−4	2	1	3	−1	−2	1	1	0	0	0
$*[61^5]$	0	2	0	−6	0	0	0	0	0	2	0	2	0	−1
$*[43^2 1]$	0	2	0	6	0	−2	0	0	0	0	0	−2	0	0
$[1^{11}]$	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1

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* Denotes a self-associated partition and character.

NOTES ON TWO-DIMENSIONAL ELECTRIC FIELD PROBLEMS

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Note 1.

The ordinary condenser in two dimensions.

1. The problem of determining the electric field between the plates of an ordinary condenser in two dimensions is solved by the transformation

$$z = A\chi + Z(\chi), \tag{1}$$

where Z denotes the Jacobian Zeta-function, for two values of A, viz. $A = 0$ and $A = \pi/(2KK')$. The second of these values is used by Love, in view of later developments in his paper*. The first is used in this note, the purpose of which is to show that an approximate value of the capacity can be readily calculated for any ratio of the breadth of the plates to the distance between them.

2. The correspondence between the z and χ planes determined by the transformation

$$z = Z(\chi) = \int_0 \left(\mathrm{dn}^2\chi - \frac{E}{K}\right) d\chi \tag{2}$$

is indicated in Figs. 1, 2. In the z plane, OM represents half of one plate of the complete condenser, and BC is midway between the plates and extends to infinity.

Let $2b$ be the breadth of the plates, and $2h$ the distance between them. Then, at the point O, $\chi = 0$, $z = 0$. At A, $\chi = K$, $z = 0$. At B, $\chi = K + iK'$, $z = -ih = -\pi i/(2K)$; hence $h = \pi/(2K)$.

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* A. E. H. Love, *Proc. London Math. Soc.* (2), 22 (1924), 337. In this paper references are given to papers by Michell and Bromwich.

Also, as χ varies from 0 to K, $Z(\chi)$ varies from 0 to 0, passing through a maximum, Z_m say, at M where $Z_m = Z(\phi_0)$ and ϕ_0 is given by the equation

$$\mathrm{dn}^2 \phi_0 = \frac{E}{K}, \tag{3}$$

so that, at M, $\chi = \phi_0$, $z = b = Z_m$. Hence, and from the value of h above,

$$\frac{b}{h} = \frac{2K}{\pi} Z_m. \tag{4}$$

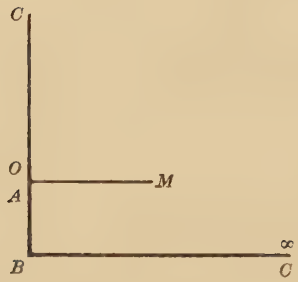

Fig. 1. z-plane.

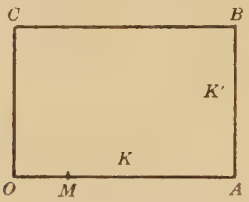

Fig. 2. χ-plane.

The capacity of the condenser is $K/(4\pi K')$ and so, by (4), the ratio r of the capacity to $b/(4\pi h)$, which is the well-known first approximation, is given by

$$r = \frac{\pi}{2K'} \frac{1}{Z_m}. \tag{5}$$

3. Given k, we can obtain b/h from (4) and r from (5), after Z_m has been found.

The value of Z_m for ordinary values of k can be read at once from the tables of Milne-Thomson*. But the ordinary values of k do not give the cases of practical interest in which b/h is large. In these cases, k is very near to unity, and we have approximately $\mathrm{dn}^2 \chi \doteqdot \mathrm{sech}^2 \chi$, $K' \doteqdot \frac{1}{2}\pi$, $E \doteqdot 1$, and hence

$$\mathrm{sech}^2 \phi_0 \doteqdot 1/K,$$

$$\cosh \phi_0 \doteqdot \sqrt{K}, \tag{7}$$

$$Z_m \doteqdot \tanh \phi_0 - \phi_0/K, \tag{8}$$

$$r \doteqdot 1/Z_m. \tag{9}$$

* Milne-Thomson, "The zeta-function of Jacobi", *Proc. Royal Soc. Edinburgh*, 52 (1932), 236.

These formulae are easy to use with a table of hyperbolic functions, and appear to be accurate to within one-tenth per cent. when $k = \cdot 99970$, $K = 5 \cdot 0988$, and, of course, still more accurate for values of k closer to unity, or larger values of K.

The table below shows how the ratio r varies for values of b/h from $\cdot 302$ to $61 \cdot 4$. The values of Z_m in the first two rows were taken from Milne-Thomson's tables; those in the remaining rows were calculated from the above approximate formulae.

We note that, when b/h is as large as 20, the ratio r exceeds unity, *i.e.* the capacity exceeds $b/(4\pi h)$, by as much as 10 per cent. nearly.

k^2	K	Z_m	b/h	r
·7	2·075	·229	·302	4·00
·9	2·578	·355	·582	2·74
·99	3·695	·510	1·20	1·96
	6·25	·666	2·65	1·50
	9	·747	4·28	1·34
	16	·839	8·55	1·19
	25	·888	14·1	1·13
	36	·917	21·0	1·09
	49	·936	29·2	1·07
	100	·965	61·4	1·04

4. From (7) and (8) we find further

$$\phi_0 \doteqdot \tfrac{1}{2} \log 4K, \tag{10}$$

$$Z_m \doteqdot 1 - \frac{1}{2K}(1 + \log 4K), \tag{11}$$

and note, in passing, that $\phi_0/K \to 0$ and $Z_m \to 1$, when $k \to 1$.

From (11), (4), and (5) we find the approximation*

$$r \doteqdot 1 + \frac{h}{\pi b}\left(1 + \log \frac{2\pi b}{h}\right), \tag{12}$$

giving r in terms of b/h. But (10), (11), (12) are less accurate than (7), (8), (9). Thus (12) gives a value of r about 4 per cent. in defect when $b/h = 2$, though, of course, the percentage error diminishes to zero as K increases.

* Love, *loc. cit.*

Note 2.

A two-dimensional condenser formed by a finite plate between two infinite parallel plates.

1. The correspondence between the z and χ planes determined by the transformation

$$Z = \chi Z(a) - \Pi(\chi, a), \tag{1}$$

or

$$z = \tfrac{1}{2} \log \frac{\Theta(\chi+a)}{\Theta(\chi-a)}, \tag{2}$$

where Π denotes the standard Jacobian elliptic integral of the third kind, is indicated in Figs. 1, 2.

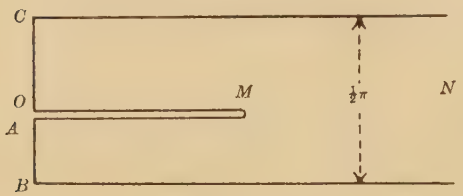

Fig. 1. z-plane.

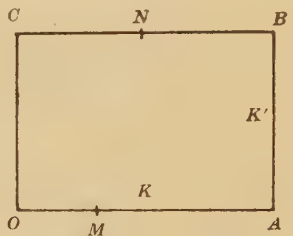

Fig. 2. χ-plane.

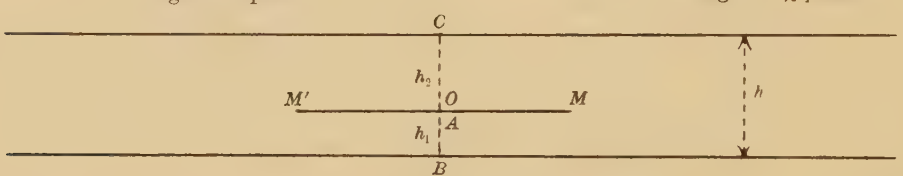

Fig. 3.

By reflexion in the line BC we obtain the figure of a condenser consisting of a finite plate between, and at unequal distances from, two infinite plates (Fig. 3). Our present object is to approximate to the capacity of such a condenser when the two infinite plates are at the same potential, and the breadth of the finite plate is large compared with the distance between the two infinite plates.

By repeated reflexion in the lines BN, CN we could obtain the figure of a condenser consisting of a pile of such plates, and the capacity found below would then be the capacity of such a multiple condenser per pair of plates.

2. To determine the dimensions of the figure in the z-plane, we have, at B, $\chi = K+iK'$, and

$$z = (K+iK')\,Z(a) - (K+iK')\,Z(a) - \frac{\pi i a}{2K} = -\frac{\pi i a}{2K},$$

and hence

$$AB = \frac{\pi}{2}\frac{a}{K}. \tag{3}$$

At C, $\chi = iK'$, and

$$z = iK'\,Z(a) + i\left\{\frac{\pi}{2} - K'\,Z(a) - \frac{\pi a}{2K}\right\} = \frac{\pi i}{2}\,\frac{K-a}{K},$$

and hence

$$OC = \frac{\pi}{2}\,\frac{K-a}{K}. \tag{4}$$

From (3) and (4),

$$BC = \frac{\pi}{2}, \quad \frac{AB}{OC} = \frac{a}{K-a}. \tag{5}$$

To find OM, we note that the value of χ at M is given by putting $dz/d\chi = 0$. Now, from (2),

$$2\frac{dz}{d\chi} = \frac{\Theta'(\chi+a)}{\Theta(\chi+a)} - \frac{\Theta'(\chi-a)}{\Theta(\chi-a)} = Z(\chi+a) - Z(\chi-a),$$

and consequently, if $\chi = \phi_0$ at M, the value of ϕ_0 is given by the equation

$$Z(\phi_0+a) = Z(\phi_0-a), \tag{6}$$

and then

$$OM = \tfrac{1}{2}\log\frac{\Theta(\phi_0+a)}{\Theta(\phi_0-a)}. \tag{7}$$

3. *The capacity.* Let b be the whole breadth of the middle plate of the condenser in Fig. 3, and h the distance between the two infinite plates. Then, by (5) and (7),

$$\frac{b}{h} = \frac{2OM}{BC} = \frac{2}{\pi}\log\frac{\Theta(\phi_0+a)}{\Theta(\phi_0-a)}. \tag{8}$$

If we now put $a = \theta K$ $(0 < \theta < 1)$, so that the finite plate divides the distance between the infinite plates in the ratio $\theta/(1-\theta)$, then (6) and (8) determine ϕ_0 and k in terms of θ and b/h, and when k has been found the capacity C is given by

$$C = \frac{2K}{4\pi K'} = \frac{K}{2\pi K'}. \tag{9}$$

4. We shall attempt to approximate to the solution of these equations only when K is large. The breadth of the middle plate is then large compared with the distance between the outer plates, as we shall see from equation (12) below.

The ratio in which the finite plate divides the distance between the infinite plates may be supposed known, and so for the sake of definiteness we shall assume that K is large compared with BC/AB, *i.e.* K is large compared with $1/\theta$; and hence that $a = \theta K$ is large compared with unity.

Now, when K is large it is plain from the behaviour of the zeta-function that $\phi_0 = a$ is a first approximation to the solution of equation(6). More accurately, this equation can then be written approximately

$$\tanh(\phi_0+a)-\frac{\phi_0+a}{K} \doteqdot \tanh(\phi_0-a)-\frac{\phi_0-a}{K},$$

which reduces to

$$\frac{\sinh 2a}{\cosh 2\phi_0+\cosh 2a} \doteqdot \frac{a}{K},$$

that is, since a and ϕ_0 are both large compared with unity,

$$\frac{e^{2a}}{e^{2\phi_0}+e^{2a}} \doteqdot \theta,$$

and hence

$$\phi_0 \doteqdot a+\tfrac{1}{2}\log\frac{1-\theta}{\theta}, \tag{10}$$

i.e.

$$\phi_0 \doteqdot \theta K+\tfrac{1}{2}\log\frac{1-\theta}{\theta}. \tag{11}$$

Further, when K is large,

$$\begin{aligned}\Theta(u) &= \Theta(0)\, e^{\int_0 Z(u)\,du} \\ &\doteqdot \Theta(0)\, e^{\int_0 (\tanh u - u/K)\,du} \\ &= \Theta(0)\, e^{-u^2/(2K)} \cosh u,\end{aligned}$$

and hence (8) becomes

$$\begin{aligned}\frac{b}{h} &\doteqdot \frac{2}{\pi}\log\frac{e^{-(\phi_0+a)^2/(2K)}\cosh(\phi_0+a)}{e^{-(\phi_0-a)^2/(2K)}\cosh(\phi_0-a)} \\ &= \frac{2}{\pi}\log e^{-2a\phi_0/K}\frac{\cosh(\phi_0+a)}{\cosh(\phi_0-a)};\end{aligned}$$

but

$$\cosh(\phi_0+a) \doteqdot \tfrac{1}{2}e^{\phi_0+a},$$

and, from (10),

$$\cosh(\phi_0-a) \doteqdot \frac{1}{2\theta^{\frac{1}{2}}(1-\theta)^{\frac{1}{2}}},$$

and hence

$$\begin{aligned}\frac{b}{h} &\doteqdot \frac{2}{\pi}\log\{e^{-2a\phi_0/K}e^{\phi_0+a}\theta^{\frac{1}{2}}(1-\theta)^{\frac{1}{2}}\} \\ &= \frac{2}{\pi}\{-2\theta\phi_0+\phi_0+\theta K+\tfrac{1}{2}\log\theta+\tfrac{1}{2}\log(1-\theta)\};\end{aligned}$$

this, by (11), reduces to

$$\frac{b}{h} \doteqdot \frac{2}{\pi}\left\{2\theta(1-\theta)K - \theta\log\frac{1}{\theta} - (1-\theta)\log\frac{1}{1-\theta}\right\}, \tag{12}$$

from which we have

$$K \doteqdot \frac{1}{2\theta(1-\theta)}\left\{\frac{\pi b}{2h} + \theta\log\frac{1}{\theta} + (1-\theta)\log\frac{1}{1-\theta}\right\}. \tag{13}$$

Further, since $K' \doteqdot \frac{1}{2}\pi$, by (9) the capacity is given by $C \doteqdot K/\pi^2$. Hence, putting $h_1 = \theta h$, $h_2 = (1-\theta)h$, we get

$$C \doteqdot \frac{h^2}{2\pi^2 h_1 h_2}\left(\frac{\pi b}{2h} + \frac{h_1}{h}\log\frac{h}{h_1} + \frac{h_2}{h}\log\frac{h}{h_2}\right),$$

which can be written

$$C \doteqdot \frac{b}{4\pi h_1} + \frac{b}{4\pi h_2} + \frac{1}{2\pi^2}\left(\frac{h}{h_1}\log\frac{h}{h_2} + \frac{h}{h_2}\log\frac{h}{h_1}\right). \tag{14}$$

In particular, when $h_1 = h_2$, so that the middle plate is midway between the outer plates,

$$C \doteqdot \frac{b}{\pi h} + \frac{\log 4}{\pi^2}, \tag{15}$$

as can be verified independently.

Note 3.

A square within a square.

1. The problem of the electric field in a prismatic condenser, the cross section of which is bounded by two concentric regular polygons, similarly situated, is well known*. The case of two squares was considered by Moulton†, but it has not, I believe, been pointed out that in this case the integration can be completed in terms of elliptic integrals, and the capacity expressed in terms of complete elliptic integrals and so evaluated at once by means of existing tables, for any ratio of the sides of the two squares.

* *E.g.* Greenhill, *Report* 19 *of the Advisory Committee for Aeronautics*, 1910.

† H. F. Moulton, *Proc. London Math. Soc.* (2), 3 (1905), 104.

2. Let the plane of a cross section of the condenser be the z-plane. By symmetry, we need only consider one-eighth of the cross section, $OABC$, and this is represented on a half-t-plane by the Schwarzian transformation

$$dz = \frac{M dt}{t^{\frac{3}{4}}(1-t)^{\frac{1}{2}}(1-k^2 t)^{\frac{1}{2}}}, \tag{1}$$

where M is a constant that fixes the scale.

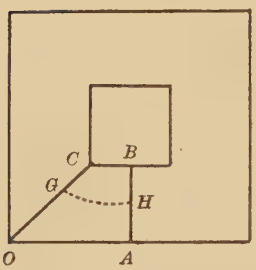

Fig. 1. z-plane.

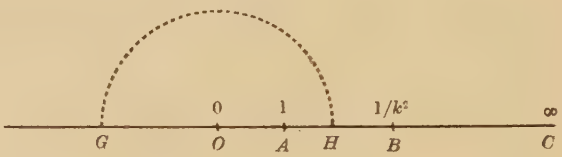

Fig. 2. t-plane.

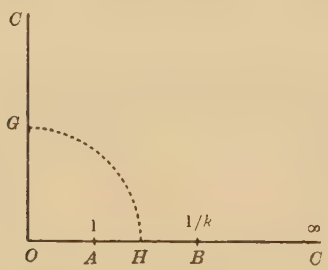

Fig. 3. u-plane.

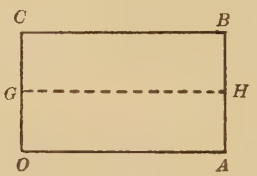

Fig. 4. χ-plane.

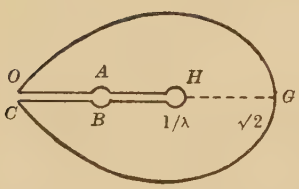

Fig. 5. ζ-plane.

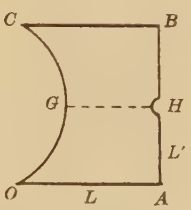

Fig. 6. w-plane.

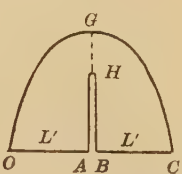

Fig. 7. w'-plane.

Putting $t = u^2$, $u = \operatorname{sn}\chi$, we have also

$$dz = \frac{2M\,du}{\sqrt{\{u(1-u^2)(1-k^2u^2)\}}}, \tag{2}$$

$$dz = \frac{2M\,d\chi}{\sqrt{(\operatorname{sn}\chi)}}. \tag{3}$$

To integrate (2) we put*

$$\lambda = \frac{1+\sqrt{k}}{\sqrt{2}\sqrt{(1+k)}}, \quad \lambda' = \frac{1-\sqrt{k}}{\sqrt{2}\sqrt{(1+k)}}, \quad k = \left(\frac{\lambda-\lambda'}{\lambda+\lambda'}\right)^2, \tag{4}$$

$$\lambda^2+\lambda'^2 = 1, \quad \lambda > 1/\sqrt{2}, \quad \lambda' < 1/\sqrt{2}, \tag{5}$$

$$\sqrt{u} = \frac{(\lambda+\lambda')\,\zeta}{\sqrt{(1-\lambda^2\zeta^2)}+\sqrt{(1-\lambda'^2\zeta^2)}}, \tag{6}$$

or

$$\zeta^2 = \frac{2(1+k)\,u}{(1+u)(1+ku)}. \tag{7}$$

We then find that

$$dz = \frac{M\sqrt{2}}{\sqrt{(1+k)}}\left\{\frac{d\zeta}{\sqrt{\{(1-\zeta^2)(1-\lambda^2\zeta^2)\}}}+\frac{d\zeta}{\sqrt{\{(1-\zeta^2)(1-\lambda'^2\zeta^2)\}}}\right\},$$

and by integration, putting $M = \sqrt{(1+k)}/\sqrt{2}$ for convenience, we have

$$z = w+w', \tag{8}$$

where

$$\zeta = \mathrm{sn}(w, \lambda) = \mathrm{sn}(w', \lambda'). \tag{9}$$

3. The boundary curves in the Argand diagrams of the several variables are shown in the figures on p. 212. The curve in the ζ-plane is one half of a lemniscate, the transformation of the circle $|\zeta'-1| = 1$ in the ζ'-plane when we put $\zeta' = \zeta^2$. The ζ'-plane is not shown.

Putting $a = OA$, $b = AB$, in the z-plane, we find from equation (8), on comparing the z, w, w' figures,

$$a = L+L', \quad b = 2L', \tag{10}$$

and hence

$$\frac{L}{L'} = \frac{2a}{b}-1, \tag{11}$$

where L, L' are complete elliptic integrals of the first kind to moduli λ, λ'; and it follows that the capacity C per unit length of the condenser is given by

$$C = 8\times\frac{K}{4\pi K'} = \frac{2K}{\pi K'},$$

where

$$k = \left(\frac{\lambda-\lambda'}{\lambda+\lambda'}\right)^2$$

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* Cayley, *Elliptic functions* (1895), 360; Legendre, *Traité des fonctions elliptiques*, 3 (1828), 335.

and λ is given in terms of the ratio a/b by

$$\frac{L}{L'} = \frac{2a}{b} - 1.$$

4. From (3) we have

$$\left|\frac{dz}{d\chi}\right| = \frac{2M}{|\mathrm{sn}\,\chi|^{\frac{1}{2}}},$$

from which it follows that the mean equipotential curve $\chi = \phi + \frac{1}{2}iK'$ is one of constant electric stress. The transformations of this curve in the several planes are indicated by dotted lines joining the points G, H. Along this curve we can put (see Figs. 6, 7)

$$w = X + iL', \quad w' = L' + iY,$$

where X, Y are real, and hence

$$z = x + iy = X + iY + (1+i)\,L', \tag{12}$$

so that (X, Y) are Cartesian coordinates of a point on this equipotential GH, referred to the point that bisects OC as origin (not with a corner of the square as origin, as stated by Greenhill, *loc. cit.*, p. 69). The equation of this curve in the z-plane is then given by (9), viz.

$$\mathrm{sn}(X + iL', \lambda) = \mathrm{sn}(L' + iY, \lambda'),$$

or
$$\lambda\,\mathrm{sn}\,(X, \lambda) = \mathrm{dn}(Y, \lambda), \tag{13}$$

or
$$\mathrm{dn}^2 X + \mathrm{dn}^2 Y = 1 \pmod{\lambda}. \tag{14}$$

Note 4.

1. The following theorem follows at once from a general existence theorem in conformal representation:

Let $OAO'A'$ (Fig. 1) be any curve having an axis of symmetry OO', the points A, A' being a symmetrical pair of points. Suppose this curve to be the cross section of a cylindrical conductor through which electricity is flowing between two electrodes the traces of which in the cross section are

OA, $O'A'$. Then *the resistance per unit length of the conductor is equal to the specific resistance**.

2. *Proof.* By an existence theorem in conformal representation, the area OAO' (Fig. 1) can be mapped on any triangle with the points O, A, O' corresponding to the vertices of the triangle; in particular, on

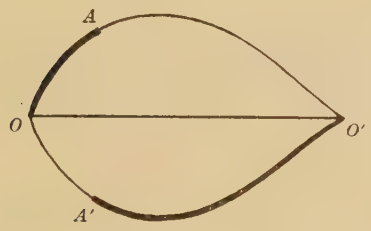

Fig. 1.

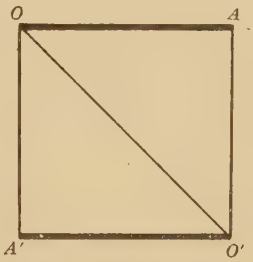

Fig. 2.

the right-angled isosceles triangle OAO' (Fig. 2). At the same time, as we know from Schwarz's principle of reflexion, the area $OA'O'$ is mapped on the triangle $OA'O'$ which is the reflexion of OAO' in the diagonal OO'. Further, in the application of conformal transformations to problems in the two-dimensional flow of electric current, it is known that equivalent resistance is conserved. But the resistance per unit length of a cylinder of square cross section, when the current is flowing between two faces whose traces in the cross section are opposite sides of the square, is the specific resistance. Hence the theorem follows.

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* The particular case in which $OAO'A'$ is a quadrilateral symmetrical about a diagonal was discussed by C. H. Lees, "Electrical resistance of a quadrilateral, etc.", *Proc. Manchester Literary and Philosophical Society*, 44 (1899), 1.

CONCERNING INTEGRALS OF MODULI OF REGULAR FUNCTIONS ALONG CONVEX CURVES

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Introduction. About eight years ago Prof. J. E. Littlewood suggested to me the probable truth of the following

THEOREM I. *If $F(Z)$ is regular inside and on a closed convex curve Γ and C is any convex curve inside Γ, then, if $\lambda \geqslant 0$,*

$$\int_C |F^\lambda(Z)||dZ| \leqslant A\int_\Gamma |F^\lambda(Z)||dZ|,$$

where A is an absolute constant.

He also suggested that A might take the value 2, this certainly being a best possible result if it is true.

Shortly after that time I proved* the special case of the theorem in which the outer curve Γ is a circle, by means of arguments involving ideas of inversion and reflection, inextricably associated with the special nature of Γ, and in this special case I obtained $A = 2$. I now give an argument which proves the general theorem as stated above; it does not, however, yield $A = 2$. I also obtain in the final section of this paper a result analogous to Theorem I in which the condition of the convexity of Γ is relaxed. The method consists of transforming the interior of Γ on to the

* See R. M. Gabriel, "Some results concerning the integrals of moduli of regular functions along curves of certain types", *Proc. London Math. Soc.* (2), 28 (1928), 121–127.

interior of a circle and of proving the transformed theorem. In this paper λ denotes a strictly positive number.

1. We begin by setting out certain results necessary to our proof.

THEOREM 1.1. *If $f(z)$ is regular in $|z|<R$ and*

$$I(r)=\int_{|z|=r}|f^{\lambda}(z)|\,|dz|$$

is continuous in $0\leqslant r\leqslant R$, then

$$\int_0^R \max_{|z|=r}|f^{\lambda}(z)|\,dr\leqslant\tfrac{1}{2}\int_{|z|=R}|f^{\lambda}(z)|\,|dz|.$$

By the continuity of $I(r)$ in $0\leqslant r\leqslant R$, it is sufficient to deal with the case in which $f(z)$ is regular in $|z|\leqslant R$, and we may then suppose that

$$f(z)=\sum_{n=0}^{\infty}a_n z^n,$$

where the series is absolutely and uniformly convergent in $|z|\leqslant R$. We have

$$\begin{aligned}\int_0^R \max_{0\leqslant\theta\leqslant 2\pi}|f^2(re^{\theta i})|\,dr &\leqslant\int_0^R\Big\{\sum_{n=0}^{\infty}|a_n|r^n\Big\}^2 dr\\ &=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{|a_n|\,|a_m|\,R^{n+m+1}}{(n+m+1)}\\ &\leqslant\pi R\sum_{n=0}^{\infty}|a_n|^2R^{2n}=\tfrac{1}{2}\int_{|z|=R}|f^2(z)|\,|dz|,\end{aligned}$$

by Hilbert's inequality*.

This proves our theorem in the special case where $\lambda=2$. The extension to the general case $\lambda>0$ may be carried out in the usual manner†.

We also require a result analogous to Theorem 1.1, in which we deal with $\max|f(re^{\theta i})|$ for given θ in place of for given r. A theorem of this type has been established by Hardy and Littlewood‡; it lies much deeper than

* See Riesz and Fejér, "Über einige funktionentheoretische Ungleichungen", *Math. Zeitschrift*, 11 (1921), 305–314. The proof given by these authors is closely connected with the ideas involved in the above theorem. *Cf.* R. M. Gabriel, *loc. cit.*

† See Riesz and Fejér, *loc. cit.*

‡ See Hardy and Littlewood, "A maximal theorem with function-theoretic applications", *Acta Math.*, 54 (1930), 81–115. In particular, see I, 1, p. 81, and V, 18, pp. 103–107.

our corresponding Theorem 1.1. We state the Hardy-Littlewood result in the following form:

THEOREM 1.2. *If $u(re^{\theta i})$ is subharmonic in $0 \leqslant r \leqslant R$, $0 \leqslant \theta \leqslant 2\pi$, and $\kappa > 1$, then*

$$\int_0^{2\pi} \max_{0 \leqslant r \leqslant R} |u^\kappa(re^{\theta i})|\, d\theta \leqslant \left(\frac{\kappa}{\kappa-1}\right)^\kappa \int_0^{2\pi} |u^\kappa(re^{\theta i})|\, d\theta.$$

Our requirements are in terms of regular functions and we apply Theorem 1.2 at once to the case in which $u(z) = |f^{\lambda/\kappa}(z)|$, where $f(z)$ is regular in $|z| < R$. In this case it is well known that $u(z)$ is subharmonic for all $\kappa > 0$ and we remark that, for all sufficiently large values of κ, $\{\kappa/(\kappa-1)\}^\kappa$ differs from e by as little as we please. We extend our boundary conditions concerning $f(z)$ and we easily obtain

THEOREM 1.3. *If $f(z)$ is a regular function of $z = re^{\theta i}$ in $|z| < R$, and if*

$$I(r) = \int_0^{2\pi} |f^\lambda(re^{\theta i})|\, d\theta$$

is continuous in $0 \leqslant r \leqslant R$, then

$$\int_0^{2\pi} \max_{0 \leqslant r \leqslant R} |f^\lambda(re^{\theta i})|\, d\theta \leqslant e \int_0^{2\pi} |f^\lambda(Re^{\theta i})|\, d\theta.$$

We have now established the requisite integral inequalities.

2. We now discuss some results associated with conformal representation. Given a closed finite simple rectifiable curve Γ in the Z-plane, we know that there is a function $Z(z)$, regular and schlicht in $|z| < 1$ and continuous in $|z| \leqslant 1$, which represents the interior of the circle γ, $|z| = 1$, conformally upon the interior of Γ and which gives a continuous (1, 1) correspondence between the points of the boundaries Γ and γ, the positional order of points on these curves being conserved. By $\Gamma(r)$ we denote the transform by $Z = Z(z)$ of the circle $\gamma(r)$, $|z| = r \leqslant 1$, so that $\Gamma = \Gamma(1)$ and $\gamma = \gamma(1)$. Any simple curve k in $|z| < 1$ corresponds to a simple curve K inside Γ and conversely; further, if there is a tangent to k at a point z, then there is a tangent to K at the corresponding point Z and conversely. If $\psi = \psi(z)$, $\Psi = \Psi(Z)$ are the inclinations of these tangents to the real axes of their respective z- and Z-planes, we have

$$\Psi = \psi + \arg Z'(z).$$

We denote by $\Lambda\{C\}$ the total variation along C of the inclination of the tangent to C. Clearly

$$\Lambda\{C\} = \int_C |d\psi|,$$

and it can also be shown that

$$\Lambda\{C\} = \text{upper bound } \Sigma\,|\arg(\zeta_{2n+2}-\zeta_{2n+1})-\arg(\zeta_{2n}-\zeta_{2n-1})|,$$

where the points $\{\zeta_\nu\}$ may be any finite set whatever of points in order along C.

We prove the following

THEOREM 2.1. *If γ, Γ, and $\Gamma(r)$ are defined as above, and if $D = D(d)$ is the transform in the Z-plane of a diameter d of γ, then*

$$\Lambda\{\Gamma(r)\} \leqslant \Lambda\{\Gamma\}, \tag{2.11}$$

$$\Lambda\{D\} \leqslant \tfrac{1}{2}\Lambda\{\Gamma\}. \tag{2.12}$$

Consider any finite set of points $\{\zeta_\nu\}$ $(\nu = 1, 2, \ldots, 2N)$, in order on the circle $|z| = r$; the points $\{Z(\zeta_\nu)\}$ $(\nu = 1, 2, \ldots, 2N)$ are then in order on $\Gamma(r)$ and, if $|z| = \rho$, the points $\{(z\zeta_\nu/r)\}$ $(\nu = 1, 2, \ldots, 2N)$ are in order on $\gamma(\rho)$ and the points $\{Z(z\zeta_\nu/r)\}$ are in order on $\Gamma(\rho)$. We take a set of η's such that each $\eta = \pm 1$ and we write

$$\Phi(z) = \sum_{n=1}^{N-1} \eta_n \{\arg[Z(z\zeta_{2n+2}/r)-Z(z\zeta_{2n+1}/r)]-\arg[Z(z\zeta_{2n}/r)-Z(z\zeta_{2n-1}/r)]\}.$$

Clearly, we may choose the set of η's so that

$$\Phi(r) = \sum_{n=1}^{N-1} |\arg[Z(\zeta_{2n+2})-Z(\zeta_{2n+1})]-\arg[Z(\zeta_{2n})-Z(\zeta_{2n-1})]|.$$

Now it is easily seen that

$$\Phi(z) = \mathfrak{I} \sum_{n=1}^{N-1} \log\left\{\frac{Z(z\zeta_{2n+2}/r)-Z(z\zeta_{2n+1}/r)}{Z(z\zeta_{2n}/r)-Z(z\zeta_{2n-1}/r)}\right\}^{\eta_n}, \tag{2.111}$$

which is harmonic in $|z| < 1$ and without singularities in that region, since $Z(z)$ is schlicht in $|z| < 1$, so that

$$Z(z\zeta_{2n}/r)-Z(z\zeta_{2n-1}/r) \neq 0 \quad \text{for} \quad |z| < 1,$$

except at $z = 0$, and at that point the expression in (2.111) whose logarithm

is considered may be replaced by its limit as $z \to 0$, namely

$$\frac{\zeta_{2n+2}-\zeta_{2n+1}}{\zeta_{2n}-\zeta_{2n-1}},$$

which is finite and non-zero. Thus $|\Phi(z)|$ is subharmonic in $|z|<1$, so that $\max|\Phi(z)|$ on $|z|=\rho$ is an increasing function of ρ in $0 \leqslant \rho < 1$. Since $Z(z)$ is continuous in $|z| \leqslant 1$, and since each of the expressions whose logarithms are considered in (2.111) is non-zero in $|z| \leqslant 1$, $\Phi(z)$ is continuous in $|z| \leqslant 1$ and $\max|\Phi(z)|$ on $|z|=\rho$ is a continuous function of ρ in $0 \leqslant \rho \leqslant 1$. We have

$$\begin{aligned}
&\sum_{n=1}^{N-1} |\arg[Z(\zeta_{2n+2})-Z(\zeta_{2n+1})]-\arg[Z(\zeta_{2n})-Z(\zeta_{2n-1})]| \\
&\quad = \Phi(r) \leqslant \max_{|z|=1} |\Phi(z)| \\
&\quad = \max_{|z|=1} \Big| \sum_{n=1}^{N-1} \eta_n \{\arg[Z(z\zeta_{2n+2}/r)-Z(z\zeta_{2n+1}/r)] \\
&\qquad\qquad -\arg[Z(z\zeta_{2n}/r)-Z(z\zeta_{2n-1}/r)]\Big| \\
&\quad \leqslant \max_{|z|=1} \sum_{n=1}^{N-1} |\arg[Z(z\zeta_{2n+2}/r)-Z(z\zeta_{2n+1}/r)] \\
&\qquad\qquad -\arg[Z(z\zeta_{2n}/r)-Z(z\zeta_{2n-1}/r)]| \\
&\quad \leqslant \Lambda\{\Gamma(1)\} = \Lambda\{\Gamma\},
\end{aligned}$$

since $|\eta_n| = 1$, $1 \leqslant n \leqslant N-1$, and the points $Z(z\zeta_\nu/r)$ $(\nu = 1, 2, \ldots, 2N)$ lie in order on $\Gamma = \Gamma(1)$ if $|z| = 1$.

Now the points $\{\zeta_\nu\}$ $(\nu = 1, 2, \ldots, 2N)$ may be any set of points in order on γ (r), so that the points $\{Z(\zeta_\nu)\}$ $(\nu = 1, 2, \ldots, 2N)$ may be any set of points in order on $\Gamma(r)$, and we have, therefore,

$$\Lambda\{\Gamma(r)\} \leqslant \Lambda\{\Gamma(1)\} = \Lambda\{\Gamma\}.$$

This establishes (2.11).

We now turn to the second inequality of our theorem. By continuity it will be sufficient if we deal with $d(\rho)$, the portion of d lying in $|z| \leqslant \rho$, in place of d the whole diameter, and in particular it will be sufficient to take d lying along the real axis, so that $d(\rho)$ is the line joining the points $\pm\rho$. It will be convenient to work in terms of a half-plane rather than of a circle and we consider the transformation $w = (\rho - z)/(\rho + z)$ which transforms the circle $|z| \leqslant \rho$ into the half-plane $\Re w \geqslant 0$. $d(\rho)$ transforms into the positive portion of the real axis of the w-plane.

Consider any finite set of points $\{w_\nu\}$ $(\nu = 1, 2, \ldots, N)$, on the real axis in $\Re w > 0$, in ascending order and write

$$\Phi(w) = \sum_{n=1}^{N-1} \eta_n \{\arg Z'(ww_{n+1}) - \arg Z'(ww_n)\},$$

where* Z' denotes $Z'(w) = dZ/dw$, $\eta_n = \pm 1$ $(1 \leqslant n \leqslant N-1)$, and the η's are chosen so that

$$\Phi(1) = \sum_{n=1}^{N-1} |\arg Z'(w_{n+1}) - \arg Z'(w_n)|.$$

Our argument runs upon the same lines as the preceding argument of this paragraph, but with this difference, that we may use the differentials of Z, dZ/dw and dZ/dz, because we are able to avoid boundary values of Z on Γ and the differentials exist at all internal points.

We need a uniform bound for $\Phi(w)$ in $\Re w \geqslant 0$ as $w \to \infty$, and it is perhaps simplest to prove that $\Phi(w) \to 0$ as $w \to \infty$ uniformly in $\Re w \geqslant 0$. We denote by $l(W)$ the straight line joining the points (Ww_1), (Ww_n), and it is easily seen that, for any W subject to $\Re W \geqslant 0$,

$$\begin{aligned} |\Phi(W)| &\leqslant \sum_{n=1}^{N-1} |\arg Z'(Ww_{n+1}) - \arg Z'(Ww_n)| \\ &\leqslant \text{total variation}_{l(W)} \{\arg Z'(w)\} \\ &= \text{total variation}_{l(W)} \left\{\arg \left[\frac{dZ}{dz}\frac{2\rho}{(1+w)^2}\right]\right\}, \end{aligned}$$

since $dz/dw = 2\rho/(1+w)^2$, so that

$$|\Phi(w)| \leqslant \text{total variation}_{l(W)} \left\{\arg \frac{dZ}{dz}\right\} + 2 \text{ total variation}_{l(W)} \{\arg (1+w)\}.$$

Since $z = z(w) \to -\rho$ uniformly as $w \to \infty$ in $\Re w \geqslant 0$, and since $dZ/dz \neq 0$ when $z = -\rho$ by the schlichtheit property,

$$\text{total variation}_{l(W)} \left\{\arg \frac{dZ}{dz}\right\} \to 0$$

uniformly in $\Re W \geqslant 0$ as $W \to \infty$.

Since $l(W)$ is a straight line and $w_N > w_1 > 0$,

$$\text{total variation}_{l(W)} \{\arg (1+w)\} = \left|\arg \left(\frac{1+Ww_N}{1+Ww_1}\right)\right| \to 0$$

* $Z' = Z'(w) = dZ/dw$ throughout the remainder of this paragraph.

uniformly in $\Re W \geqslant 0$ as $W \to \infty$. Thus $|\Phi(W)| \to 0$ uniformly in $\Re W \geqslant 0$ as $W \to \infty$.

Now, since $\arg Z'(w) = \Im \log Z'(w)$ and $Z'(w) \neq 0$ in the region considered, $\Phi(w)$ is a harmonic function in $\Re w \geqslant 0$ and, if $w = u+iv$,

$$\Phi_1(w) = \tfrac{1}{2}\{|\Phi(u+iv)|+|\Phi(u-iv)|\}$$

is subharmonic in $\Re w \geqslant 0$. Further, $\Phi_1(w) \to 0$ uniformly in $\Re w \geqslant 0$ as $w \to \infty$ and hence $\max \Phi_1(w)$ for w in $\Re w \geqslant 0$ is attained for $w = W_0$, where W_0 is a point on the boundary, that is the v-axis. If we write $W_0 = iv_0$, we have

$$\begin{aligned}
&\sum_{n=1}^{N-1} |\arg Z'(w_{n+1}) - \arg Z'(w_n)| = \Phi(1) \\
&\quad = \Phi_1(1) \leqslant \Phi_1(iv_0) = \tfrac{1}{2}\{|\Phi(iv_0)|+|\Phi(iv_0)|\} \\
&\quad = \tfrac{1}{2}\left|\sum_{n=1}^{N-1} \eta_n \{\arg Z'(iv_0 w_{n+1}) - \arg Z'(iv_0 w_n)\}\right| \\
&\qquad\qquad + \tfrac{1}{2}\left|\sum_{n=1}^{N-1} \eta_n \{\arg Z'(-iv_0 w_{n+1}) - \arg Z'(-iv_0 w_n)\}\right| \\
&\quad \leqslant \tfrac{1}{2}\sum_{n=1}^{N-1} |\arg Z'(iv_0 w_{n+1}) - \arg Z'(iv_0 w_n)| \\
&\qquad\qquad + \tfrac{1}{2}\sum_{n=1}^{N-1} |\arg Z'(-iv_0 w_{n+1}) - \arg Z'(-iv_0 w_n)|.
\end{aligned}$$

Now each of the sets of points $\{iv_0 w_n\}$ $(n = 1, 2, \ldots, N)$ and $\{-iv_0 w_n\}$ $(n = 1, 2, \ldots, N)$ lies in order upon a distinct half of the v-axis, since the w_n's are real, positive, and in ascending order. Further, $\Gamma(\rho)$ is the transform of the v-axis by the transformation $Z = Z(w)$, so that the points $\{Z(iv_0 w_n)\}$, $\{Z(-iv_0 w_n)\}$ lie in order on two distinct parts of $\Gamma(\rho)$, and since $\psi = \psi(w)$ is constant along the v-axis, we have

$$\begin{aligned}
&\tfrac{1}{2}\sum_{n=1}^{N-1} |\arg Z'(iv_0 w_{n+1}) - \arg Z'(iv_0 w_n)| \\
&\qquad\qquad + \tfrac{1}{2}\sum_{n=1}^{N-1} |\arg Z'(-iv_0 w_{n+1}) - \arg Z'(-iv_0 w_n)| \\
&\quad \leqslant \tfrac{1}{2} \text{ total variation}_{\Gamma(\rho)} \Psi = \tfrac{1}{2}\Lambda\{\Gamma(\rho)\} \leqslant \tfrac{1}{2}\Lambda\{\Gamma\}
\end{aligned}$$

by (2.11). Combining this with our previous inequality, we have

$$\sum_{n=1}^{N-1} |\arg Z'(w_{n+1}) - \arg Z'(w_n)| \leqslant \tfrac{1}{2}\Lambda\{\Gamma\},$$

and since this holds for any set of points $\{w_n\}$ in order on the u-axis in

$\Re w > 0$, along which ψ is constant, we have

$$\Lambda\{D(\rho)\} = \text{upper bound} \sum_{n=1}^{N-1} |\arg Z'(w_{n+1}) - \arg Z'(w_n)|$$
$$\leqslant \tfrac{1}{2}\Lambda\{\Gamma\}.$$

As we have already remarked, since this is true for $D(\rho)$, $0 \leqslant \rho < 1$, it is also true by continuity for $\rho = 1$, and we have

$$\Lambda\{D\} \leqslant \tfrac{1}{2}\Lambda\{\Gamma\},$$

so that (2.12) is established.

Although we have stated these inequalities in quasi-geometrical form, we shall use them stated in terms of the variations of certain complex functions, and we conclude this paragraph by setting out our results in this required form.

If $\Psi = \Psi(Z)$ is the angle of inclination at Z of the curve $\Gamma(r)$, $0 \leqslant r < 1$, to the real axis of the Z-plane,

$$\Psi = \tfrac{1}{2}\pi + \arg\{z\} + \arg\{dZ/dz\} = \tfrac{1}{2}\pi + \arg\{zZ'(z)\},$$

since $\psi(z)$, the inclination of $\gamma(r)$ to the real axis of the z-plane, is given by $\gamma = \tfrac{1}{2}\pi + \arg\{z\}$. Thus (2.11) becomes

$$\text{(2.13)} \qquad \text{total variation}_{\gamma(r)} \left[\arg\left\{z\frac{dZ}{dz}\right\}\right] = \text{total variation}_{\Gamma(r)} \Psi$$
$$= \Lambda\{\Gamma(r)\} \leqslant \Lambda\{\Gamma\}.$$

Along any diameter d of γ, $\psi(z)$ is constant and (2.12) becomes

$$\text{(2.14)} \qquad \text{total variation}_{d} [\arg\{Z'(z)\}] \leqslant \tfrac{1}{2}\Lambda\{\Gamma\}.$$

3. We now consider any simple polygon K, whose vertices all lie on Γ and which lies wholly inside Γ save for its vertices, which occur in the same order on K and Γ since both curves are simple. The particular case in which portions of K and Γ coincide may be obtained as a limiting case. If Γ is convex, so is K. The sides K_m ($m = 1, 2, \ldots, M$), say, of K are chords of Γ and each K_m divides the interior of Γ into two regions, one of which does not contain any point of K in its interior; we denote this region by Δ_m'. If K does not consist entirely of a single line taken possibly twice, there is a portion of the interior of Γ which belongs to no Δ_m' and the exceptional case in which K consists of one line only may be obtained as a limiting case of the general case. We consider a transformation $z = z(Z)$ which transforms the interior of Γ conformally on to the interior

of γ, and in particular we discuss certain properties of the simple curve k which is the transform of K. We may suppose that $z=0$ corresponds to a point Z_0 which lies inside no Δ_m'. Let k_m be that portion of k which is the transform of K_m $(m=1, 2, \ldots, M)$, and let Δ_m be the transform of Δ_m', so that Δ_m is bounded by k_m and a portion of γ, and contains no point of k in its interior $(m=1, 2, \ldots, M)$. There is a tangent to K at every point Z on K, save at the vertices, and hence there is a tangent to k at every point z on k which lies strictly inside γ and therefore does not correspond to a vertex of K. We suppose k and K to be described in corresponding senses, so that an appropriate sense is attached to every tangent considered. With our previous notation of ψ and Ψ, if z and Z are corresponding points, we have

$$\Psi(Z)=\psi(z)+\arg\{dZ/dz\}. \tag{3.1}$$

The points at which there is no tangent are finite in number and, since our purpose is to deal with integration, these may be ignored.

Consider the points $\{{}_mz_i\}$ $(i=1, 2, \ldots, I)$, in which k_m cuts a semi-diameter $d(\theta)$, say, joining the points 0 and $e^{\theta i}$, where $0\leqslant\theta<2\pi$. We may suppose that the points are enumerated by $i=1, 2, \ldots, I$ in order of increasing modulus and that points where k_m touches $d(\theta)$ are to be enumerated an appropriate number of times. $I=I(m)$ is necessarily finite, though this is immaterial to our argument. Clearly one of the intervals $({}_mz_{i-1}, {}_mz_i)$, $({}_mz_i, {}_mz_{i+1})$ on $d(\theta)$ lies wholly in Δ_m, since ${}_mz_i$ on k_m is a boundary point of Δ_m and these intervals contain no boundary point of Δ_m in their interiors. If $i=1$, then, since $z=0$ lies in no Δ_m, the portion of $d(\theta)$ between 0 and ${}_mz_i$ does not belong to Δ_m and hence the portion of $d(\theta)$ between ${}_mz_1$ and ${}_mz_2$ must lie inside Δ_m since ${}_mz_1$ is a boundary point of Δ_m. If $i=I$, we remark that if $({}_mz_I, e^{\theta i})$ on $d(\theta)$ does not belong to Δ_m, then $({}_mz_{I-1}, {}_mz_I)$ on $d(\theta)$ must belong to Δ_m and we notice that, for any particular θ, there is only one value of m such that $({}_mz_I, e^{\theta i})$ on $d(\theta)$ belongs to Δ_m, since the Δ_m's are mutually exclusive. Thus for every point ${}_mz_i$ $[m=1, 2, \ldots, M;$ $i=1, 2, \ldots, I(m)]$ of k on $d(\theta)$, save one, there is a corresponding interval $({}_mz_i, {}_mz_j)$ on $d(\theta)$ lying wholly in Δ_m, where $j=i\pm1$. We call this the interval on $d(\theta)$ "appropriate" to ${}_mz_i$. These "appropriate" intervals are clearly mutually exclusive for all m and i except in so far as it may happen that $({}_mz_i, {}_mz_j)$ is the interval appropriate to ${}_mz_j$ as well as to ${}_mz_i$. The exceptional point will be the point of k on $d(\theta)$ nearest γ.

A similar argument holds for the intersections of k with $\gamma(r)$, with the difference that there is no exceptional point which has no "appropriate" interval on $\gamma(r)$.

We are now in a position to prove a lemma.

LEMMA. *Let $\chi(re^{\theta i})$, $\chi'(re^{\theta i})$ be the smallest angles, regardless of sign, between k and the lines $d(\theta)$, $\gamma(r)$ respectively at a point $re^{\theta i}$ on k, and suppose that $\{z_p\}$ $(p=1, 2, \ldots, P)$ are all the points of k on $d(\theta)$ and that $\{z_q\}$ $(q=1, 2, \ldots, Q)$ are all the points of k on $\gamma(r)$, then, for all θ, $0 \leqslant \theta \leqslant 2\pi$, and all r, $0 \leqslant r < 1$,*

$$\sum_{p=1}^{P} \sin|\chi(z_p)| \leqslant \tfrac{1}{2}\Lambda\{\Gamma\}+1, \tag{3.21}$$

$$\sum_{q=1}^{q} \sin|\chi'(z_q)| \leqslant \Lambda\{\Gamma\}. \tag{3.22}$$

We deal first with (3.21). With the notation of this section, ${}_mz_i$, ${}_mz_j$ are consecutive points of intersection on $d(\theta)$ with k, so that k cuts $d(\theta)$, therefore, from opposite sides at these points. It follows at once that

$$|\chi({}_mz_i)|+|\chi({}_mz_j)| \leqslant |\psi({}_mz_i)-\psi({}_mz_j)| = |\arg Z'({}_mz_i)-\arg Z'({}_mz_j)|$$

by (3.1), since ${}_mz_i$, ${}_mz_j$ lie on the same k_m and the value of Ψ is the same at all points of K_m. Consider

$$\sum_m \sum_i{}' \{|\chi({}_mz_i)|+|\chi({}_mz_j)|\} \leqslant \sum_m \sum_i{}' |\arg Z({}_mz_i)-\arg Z({}_mz_j)|, \tag{3.23}$$

where the dash denotes that the summation is to be taken over such a range of m and i that every distinct interval of our "appropriate" type $({}_mz_i, {}_mz_j)$ on $d(\theta)$ is considered exactly once; that is to say, if $({}_mz_i, {}_mz_j)$ is an interval on $d(\theta)$ "appropriate" to each of its end points, it is nevertheless to be considered once only in the summation. Both $|\chi({}_mz_i)|$ and $|\chi({}_mz_j)|$ still appear in the summation on the left-hand side of (3.23), so that this summation will contain every $|\chi(z_p)|$ such that z_p is a point on $d(\theta)$ possessing an "appropriate" interval, and we have already seen that all points z_p save the one of largest modulus on $d(\theta)$ possess an "appropriate" interval. Since no interval $({}_mz_i, {}_mz_j)$ is considered twice on the right-hand side of (3.23), and since all the intervals are mutually exclusive, we have

$$\sum_m \sum_i{}' |\arg Z'({}_mz_i)-\arg Z'({}_mz_j)| \leqslant \text{total variation}_{D(\theta)} \Psi = \Lambda\{D(\theta)\} \leqslant \tfrac{1}{2}\Lambda\{\Gamma\},$$

by (2.14), where $D(\theta)$ is the transform in the Z-plane of $d(\theta)$. Thus, by (3.23),

$$\sum_m \sum_i{}' \{|\chi({}_mz_i)|+|\chi({}_mz_j)|\} \leqslant \tfrac{1}{2}\Lambda\{\Gamma\}. \tag{3.24}$$

Now we have already seen that all the points $\{z_p\}$ $(p=1, 2, \ldots, P)$ save one, z_p say, have appropriate intervals and hence the points $\{z_p\}$

($p=1, 2, \ldots, P-1$) all have "appropriate" intervals and are, therefore, enumerated among $\{{}_mz_i\}$ and $\{{}_mz_j\}$ in the summations of (3.23) and (3.24).

We have, therefore,

$$\sum_{p=1}^{P-1} |\chi(z_p)| \leqslant \tfrac{1}{2}\Lambda\{\Gamma\}. \tag{3.25}$$

Since all the χ's lie between 0 and $\frac{1}{2}\pi$, it can easily be shown that, subject to (3.25),

$$\max \sum_{p=1}^{P-1} \sin|\chi(z_p)|$$

occurs when all the χ's are equal, so that, subject to (3.25), we have

$$\sum_{p=1}^{P-1} \sin|\chi(z_p)| \leqslant \max_P (P-1) \sin\left\{\frac{\Lambda(\Gamma)}{2(P-1)}\right\} \leqslant \tfrac{1}{2}\Lambda\{\Gamma\},$$

and hence

$$\sum_{p=1}^{P} \sin|\chi(z_p)| \leqslant \tfrac{1}{2}\Lambda\{\Gamma\}+1.$$

This establishes (3.21) and we now deal with (3.22).

At a point $z=re^{\theta i}$ on k and on $\gamma(r)$, the angle $\phi=\phi(z)$ between k and $\gamma(r)$ is given by

$$\phi(z) = \tfrac{1}{2}\pi+\arg\{z\}-\psi(z)$$

and with our previous notation

$$\begin{aligned}\Psi = \Psi(Z) &= \psi(z)+\arg Z'(z)\\ &= \tfrac{1}{2}\pi+\arg\{z\}-\phi(z)+\arg Z'(z)\\ &= \tfrac{1}{2}\pi+\arg\{zZ'(z)\}-\phi(z).\end{aligned}$$

Now, if $\{{}_mz_i\}$ ($i=1, 2, \ldots, I$) are the points of k_m in order on $\gamma(r)$, and if $({}_mz_i, {}_mz_j)$ is the interval on $\gamma(r)$ "appropriate" to ${}_mz_i$, we have

$$\begin{aligned}\tfrac{1}{2}\pi+\arg\{{}_mz_i Z'({}_mz_i)\}-\phi({}_mz_i) = \Psi({}_mZ_i) &= \Psi({}_mZ_j)\\ &= \tfrac{1}{2}\pi+\arg\{{}_mz_j Z'({}_mz_j)\}-\phi({}_mz_j),\end{aligned}$$

since K_m is a straight line and Ψ is, therefore, constant along it. Thus

$$|\phi({}_mz_i)-\phi({}_mz_j)| = |\arg\{{}_mz_i Z'({}_mz_i)\}-\arg\{{}_mz_j Z'({}_mz_j)\}|.$$

Since ${}_mz_i, {}_mz_j$ on $\gamma(r)$ are consecutive points of intersection of $\gamma(r)$ with

k, k cuts $\gamma(r)$ from opposite sides at these points and we have

$$|\chi'({}_mz_i)|+|\chi'({}_mz_j)|\leqslant|\phi({}_mz_i)-\phi({}_mz_j)|$$
$$\leqslant|\arg\{{}_mz_i\,Z'({}_mz_i)\}-\arg\{{}_mz_j\,Z'({}_mz_j)\}|.$$

Consider

$$\sum_m\sum_i{}'\{|\chi'({}_mz_i)|+|\chi'({}_mz_j)|\} \tag{3.26}$$
$$\leqslant\sum_m\sum_i{}'|\arg\{{}_mz_i\,Z'({}_mz_i)\}-\arg\{{}_mz_j\,Z'({}_mz_j)\}|,$$

where the dash appended to the summation sign denotes that the summation is to be taken over such a range of m and i that every distinct "appropriate" interval $({}_mz_i,\ {}_mz_j)$, on $\gamma(r)$, is considered, as previously, exactly once. Both $|\chi'({}_mz_i)|$ and $\chi'({}_mz_j)|$ still appear in the summation on the left-hand side of (3.26), so that this summation will contain every $|\chi'(z_q)|$, since every point z_q of k on $\gamma(r)$ possesses an "appropriate" interval on $\gamma(r)$ without exception. Since no interval $({}_mz_i,\ {}_mz_j)$ is considered twice on the right-hand side of (3.26), and since all the intervals are mutually exclusive, we have, by (3.26),

$$\sum_{q=1}^{Q}|\chi'(z_q)|\leqslant\sum_m\sum_i{}'\{|\chi'({}_mz_i)|+|X'({}_mz_j)|\}$$
$$\leqslant\sum_m\sum_i{}'|\arg\{{}_mz_i\,Z'({}_mz_i)\}-\arg\{{}_mz_j\,Z'({}_mz_j)\}|$$
$$\leqslant\text{total variation}_{\gamma(r)}\{zZ'(z)\}$$
$$\leqslant\Lambda\{\Gamma\},$$

by (2.13). As before, this implies

$$\sum_{q=1}^{Q}\sin|\chi'(z_q)|\leqslant\Lambda\{\Gamma\}.$$

4. We can now prove

THEOREM 4.1. *If $F(Z)$ is regular inside and on Γ and K is any simple polygon with its vertices on Γ, then*

$$\int_K|F(Z)|^\lambda|dZ|\leqslant A_1[\Lambda\{\Gamma\}]\int_\Gamma|F(Z)|^\lambda|dZ|,$$

where $A_1[\Lambda\{\Gamma\}]=\frac{1}{2}(e+1)\Lambda\{\Gamma\}+e$.

We consider first a simple polygon $K(\rho)$ with its vertices on $\Gamma(\rho)$, where $0<\rho<1$. We shall denote by $k(\rho)$ the transform of $K(\rho)$ by $z=z(Z)/\rho$,

the transformation which represents $\Gamma(\rho)$ and its interior, conformally upon $\gamma(\rho)$ and its interior. If $k(\rho)$ cuts $d(\theta)$ in points z_p $(p=1, 2, \ldots, P)$, and $\gamma(r)$, $0 \leqslant r \leqslant \rho$, in points z_q $(q=1, 2, \ldots, Q)$, we have, by (3.21) and (3.22),

$$\sum_{p=1}^{P} \sin|\chi(z_p)| \leqslant \tfrac{1}{2}\Lambda\{\Gamma(\rho)\}+1 \leqslant \tfrac{1}{2}\Lambda\{\Gamma\}+1, \tag{4.11}$$

$$\sum_{q=1}^{Q} \sin|\chi'(z_q)| \leqslant \Lambda\{\Gamma(\rho)\} \leqslant \Lambda\{\Gamma\}. \tag{4.12}$$

We write $z_p = r_p e^{\theta_p i}$, $z_q = re^{\theta_q i}$ and we consider

$$f(z) = F(Z)\left\{\frac{1}{\rho}\frac{dZ}{dz}\right\}^{1/\lambda},$$

regular in $|z| \leqslant \rho$, since $Z = Z(z)$ is regular and $dZ/dz \neq 0$ in this region.

We have

$$\int_{K(\rho)} |F^\lambda(Z)||dZ| = \int_{k(\rho)} |F^\lambda(Z)|\left|\frac{dZ}{dz}\right||dz|$$

$$= \int_{k(\rho)} |f^\lambda(z)||dz| = \int_{k(\rho)} |f^\lambda(z)|\{|z|\sin|\chi(z)||d\theta|+\sin|\chi'(z)||dr|\}$$

$$\leqslant \int_0^{2\pi} \rho\left\{\sum_{p=1}^{P} \sin|\chi(r_p e^{\theta_p i})|\right\} \max_{0\leqslant r\leqslant\rho} |f^\lambda(re^{\theta i})|\,d\theta$$

$$+\int_0^{\rho}\left\{\sum_{q=1}^{Q} \sin|\chi'(re^{\theta_q i})|\right\} \max_{0\leqslant\theta\leqslant 2\pi} |f^\lambda(re^{\theta i})|\,dr$$

$$\leqslant [\tfrac{1}{2}\Lambda\{\Gamma\}+1]\int_0^{2\pi} \max_{0\leqslant r\leqslant\rho} |f^\lambda(re^{\theta i})|\,\rho\,d\theta+\Lambda\{\Gamma\}\int_0^{\rho} \max_{0\leqslant\theta\leqslant 2\pi} |f^\lambda(re^{\theta i})|\,dr$$

[by (4.11) and (4.12)]

$$\leqslant e[\tfrac{1}{2}\Lambda\{\Gamma\}+1]\int_0^{2\pi} |f^\lambda(\rho e^{\theta i})|\,\rho\,d\theta+\tfrac{1}{2}\Lambda\{\Gamma\}\int_0^{2\pi} |f^\lambda(\rho e^{\theta i})|\,\rho\,d\theta$$

[by Theorems 1.1 and 1.3]

$$= [\tfrac{1}{2}(e+1)\Lambda\{\Gamma\}+e]\int_{\gamma(\rho)} |f^\lambda(z)||dz|$$

$$= [\tfrac{1}{2}(e+1)\Lambda\{\Gamma\}+e]\int_{\gamma(\rho)} |F^\lambda(Z)|\left|\frac{dZ}{dz}\right||dz|$$

$$= [\tfrac{1}{2}(e+1)\Lambda\{\Gamma\}+e]\int_{\Gamma(\rho)} |F^\lambda(Z)||dZ|.$$

Theorem 4.1 now follows by continuity since $Z(\rho e^{\theta i}) \to Z(e^{\theta i})$ as $\rho \to 1$ uniformly in $0 \leqslant \theta \leqslant 2\pi$.

5. In obtaining results concerning curves inside Γ, but not of type K, it will be convenient to discuss first the case in which Γ is convex. We prove the following

THEOREM 5.1. *If Γ is any convex curve and C any convex curve inside Γ, and if $F(Z)$ is regular inside and on Γ,*

$$\int_C |F^\lambda(Z)||dZ| \leqslant A \int_\Gamma |F^\lambda(Z)||dZ|, \tag{5.1}$$

where $A = \pi(e+1)+e$.

By continuity it is sufficient to discuss the case in which C and Γ are both convex polygons. In this case it has been shown* that, for a given $F(Z)$,

$$\max \int_C |F^\lambda(Z)||dZ|$$

is obtained when all the vertices of C lie on Γ, and by Theorem 4.1 we know that (5.1) holds in these circumstances, since C is then of type K, and $\Lambda\{\Gamma\} = 2\pi$ if Γ is convex. This proves Theorem 5.1.

We now state and prove

THEOREM 5.2. *If $F(Z)$ is regular inside and on any simple closed curve Γ and C is any convex curve inside Γ,*

$$\int_C |F(Z)|^\lambda |dZ| \leqslant A_2[\Lambda\{\Gamma\}] \int_\Gamma |F^\lambda(Z)||dZ|,$$

where $A_2[\Lambda\{\Gamma\}] = 3A_1[\Lambda\{\Gamma\}]$.

By the usual continuity argument it is sufficient to prove this in the case where C and Γ are both polygons, C being convex. We deduce our theorem from Theorem 4.1, which deals with curves K all of whose vertices lie on Γ, in place of C. We construct a set of three curves of type K such that every portion of C is included among the three curves K. It will be convenient to write for any curve C whatever

$$I(C) = \int_C |F^\lambda(Z)||dZ|.$$

We shall require

THEOREM 5.3. *Suppose that A and B are two fixed points in a closed region Δ, such that if P is any point of Δ, the line segments PA, PB both lie*

* See R. M. Gabriel, *loc. cit.*, p. 126, Theorem 3.2.

wholly in Δ, *and suppose further that* $F(Z)$ *is regular in* Δ; *then, for* P *in* Δ, $\max I(AP+PB)$ *is attained when* P *is a boundary point of* Δ.

A less general form of this theorem is to be found in my paper* already quoted and the proof given there will suffice to establish this theorem. We proceed to apply the theorem to our problem.

If $\{P_\nu\}$ $(\nu = 1, 2, \ldots, M)$ are the vertices of C in order, consider $P_{\nu-1}$, $P_{\nu+1}$ as the two fixed points A and B of Theorem 5.3 and replace P_ν by a variable point which is to be confined to the closed region Δ consisting of all the points P inside and on Γ such that the modified polygon $C(P)$ with P replacing P_ν as a vertex in C, (i) is convex, (ii) lies wholly inside and on Γ. It will easily be seen that Δ satisfies the conditions of Theorem 5.3 for $P_{\nu-1}$ and $P_{\nu+1}$. Let P_ν' be the position of P which gives

$$\max I(P_{\nu-1}P_\nu' + P_\nu' P_{\nu+1}) \text{ for } P \text{ in } \Delta.$$

By Theorem 5.3, P_ν' is a boundary point of Δ, and hence for some P in any neighbourhood of P_ν' the conditions for P lying in Δ are infringed. If the convexity condition of $C(P)$ is infringed in the neighbourhood of P_ν', it follows at once that P_ν' lies on one of the lines $P_{\nu-2}P_{\nu-1}$, $P_{\nu-1}P_{\nu+1}$, $P_{\nu+2}P_{\nu+1}$, and in this case the substitution of P_ν' for P_ν has the effect of giving $C(P_\nu')$ one less side than $C(P_\nu) = C$. If the condition that $P_{\nu-1}P + PP_{\nu+1}$ must lie inside and on Γ is infringed in the neighbourhood of P_ν', it follows at once that $P_{\nu-1}P_\nu' + P_\nu' P_{\nu+1}$ has a point, Q say, in common with Γ. This process of replacing P_ν by P_ν' does not increase the number of sides of $I(C)$ and we apply it to each vertex in turn, possibly with a reduction in the number of sides of C. Clearly we arrive finally at a modified convex polygon† C such that any two adjacent sides $P_{\nu-1}P_\nu$, $P_\nu P_{\nu+1}$ possess at least one point in common with Γ and $I(C)$ has not been decreased in the process. It is sufficient to prove our theorem for the final modified polygon C.

Let Q_1, Q_2 be two points common to C and Γ such that there is no other point of type Q between them on one arc of C; since Γ is simple and C lies inside Γ there is no point of type Q between Q_1, Q_2 on one arc of Γ. Let us denote by D the closed region bounded by the arcs $Q_1 Q_2$ on Γ and C, the arcs being chosen so as to contain no other point Q. Any two adjacent sides of C contain at least one point Q common to C and Γ, so that the portion of C between Q_1 and Q_2 contains at most

* See R. M. Gabriel, *loc. cit.*, p. 124, Theorem 3.1.

† If the final C consists of a straight line, taken possibly twice, the line can be produced at each end to meet Γ.

two vertices* P_ν, $P_{\nu+1}$, say, of C, and we suppose therefore that Q_1, Q_2 lie on $P_{\nu-1}P_\nu$ and $P_{\nu+1}P_{\nu+2}$ respectively. Produce the lines Q_1P_ν, $Q_2P_{\nu+1}$ sufficiently for each to cut Γ once, in R_1 and R_2 say, respectively, and produce $P_\nu P_{\nu+1}$ at each end to cut Γ once at each end in S and T say. The line segments Q_1R_1, Q_2R_2, ST all lie inside the region D. Now the regions D are mutually exclusive and hence no two line segments of the type considered in different D's can intersect. We, therefore, can construct a simple polygon K, which shall contain all the line segments Q_1R_1 as parts of its perimeter, the remaining parts of K_1 being parts of Γ, since both Q_1 and R_1 lie on Γ. All the vertices of K_1 lie on Γ and K_1 is of type K. Similarly we can construct simple polygons K_2, K_3, all of whose vertices lie on Γ, and which contain all the line segments Q_2R_2, ST respectively, as parts of their perimeters. The whole of C is to be found among the perimeters of K_1, K_2, and K_3, and we have

$$\begin{aligned} I(C) &\leqslant I(K_1+K_2+K_3) = I(K_1)+I(K_2)+I(K_3) \\ &\leqslant 3A_1[\Lambda\{\Gamma\}]\,I(\Gamma) = A_2[\Lambda\{\Gamma\}]\,I(\Gamma). \end{aligned}$$

This establishes Theorem 5.2.

I close this paper with the remark that I know of no Gegenbeispiel which renders $A_1[\Lambda\{\Gamma\}] = 2$ impossible in our theorems.

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* Our arguments apply equally well if either or both of the vertices are missing, for these can be obtained as limiting cases.

ON THE EQUIVALENCE OF PENCILS OF HERMITIAN FORMS

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Introduction.

The purpose of this paper is to give necessary and sufficient conditions for two pairs of Hermitian forms to be equivalent under conjunctive transformation. Hitherto it has been supposed that such forms ϕ and ψ are respectively equivalent to ϕ' and ψ' if the elementary divisors of the pencil $r\phi-\psi$ coincide with those of $r\phi'-\psi'$. This, which may usefully be called the *divisor test*, is necessary, but not sufficient: it needs supplementing with the *signature test* § 2. The full conditions are virtually those governing real quadratic forms which were first investigated by Muth (**4**) in 1905. Indeed, the proof here given in §§ 3 and 4 is immediately applicable to the case of real quadratic forms, and was, in fact, suggested by the original argument of Muth. Certain contravariant matrices P are evaluated whose signatures provide the necessary test: in § 4 it is proved that an alternative set of matrices Q can be found which are those employed by Muth, being the matrices of certain contravariant Hermitian forms. The matrix notation seems well adapted to bring out the salient points of this somewhat complicated theory. In § 5 the singular case is fully discussed, since the methods applicable to quadratics are no longer valid for Hermitian pencils. At each stage care must be taken to maintain the conjunctive nature of the transformations.

The exposition in § 1 is an amplification of the original reduction given by Bromwich (**3**), who followed the method already used by Stickelberger (**2**) for quadratic forms. Bromwich is concerned with the actual reduction and not with the conditions of equivalence; the crucial step, subsequent neglect of which has caused the whole difficulty, is given only as a foot-note (**3**, 348), which determines the sign ($\epsilon_\sigma = \pm 1$) to

be attached to each real submatrix of the canonical form, and it is the starting point of the signature test. An alternative reduction, based on the theory of regular minors, is given by Logsdon (**5**, 254–260), but the final step (from x' to x on p. 259) needs the qualifying signs as anticipated by Bromwich.

The theory of complex (but *not* of real) quadratics has been based upon a theorem of Frobenius, who proved that a non-singular matrix H can always be found such that $H'AH = PAQ$, A being symmetric or skew, and P, Q being non-singular given matrices, while H' is the transposed of H. It has been erroneously supposed that the theorem also applies when A is Hermitian and H' is the transposed conjugate of H, and it is this error which has led to the omission of the signature test for equivalence. A warning that such an extension of the theorem was unjustified was actually given by Bromwich (**3**, 321), but it has remained unheeded until recently†. The error arose from confusing two polynomial expressions such as $\overline{g}(z)$ and $\overline{g}(\bar{z})$, the bar denoting the operation of changing i to $-i$ in all complex numbers. Does the bar over the g involve changing both variable and coefficients or changing only the latter? This error vitiates the argument of **6**, 131, where Lemma III is no longer true of Hermitian forms.

1. *Reduction to canonical form.*

Let $(r\phi - \psi)$ denote an ordinary pencil of Hermitian forms, where

$$\phi = \sum_{i,\,j=1}^{n} \bar{x}_i a_{ij} x_j \quad (a_{ij} = \bar{a}_{ji}), \qquad \psi = \sum_{i,\,j=1}^{n} \bar{x}_i b_{ij} x_j \quad (b_{ij} = \bar{b}_{ji}), \tag{1}$$

the bar over a letter indicating the conjugate complex of the corresponding unbarred letter. In matrix notation such forms may be written

$$\phi = \bar{x}Ax \quad (A = \bar{A}'), \qquad \psi = \bar{x}Bx \quad (B = \bar{B}'), \tag{2}$$

where x is a column, and $\bar{x}$ a row of n variables, A and B being Hermitian matrices. If, as usual, the accent denotes transposition of rows and columns, then $\phi = x'A'\bar{x}'$; also $\psi = x'B'\bar{x}'$. By an ordinary pencil is meant one for which the determinant is not identically zero for all r: otherwise the pencil is singular. As Bromwich has shown, the general methods of Stickelberger may be adapted to the Hermitian case, whereby the forms may be reduced by conjunctive linear transformations of the x and $\bar{x}$ to

† I am grateful to Dr. J. Williamson for first pointing this out, and for connecting the theory with that of real quadratic forms.

canonical forms Φ and Ψ, such as

$$\Phi = \sum_{\sigma=1}^{k} \epsilon_\sigma (x\bar{x})_{e_\sigma} + \sum_{\sigma=k+1}^{k+p} \{(x\bar{y})_{e_\sigma} + (\bar{x}y)_{e_\sigma}\},$$

$$\Psi = \sum_{\sigma-1}^{k} \epsilon_\sigma \{c_\sigma (x\bar{x})_{c_\sigma} + (x\bar{x})_{e_\sigma - 1}\} + \sum_{\sigma=k+1}^{k+p} \{a_\sigma (x\bar{y})_{e_\sigma} + (x\bar{y})_{e_{\sigma-1}} + \bar{a}_\sigma (\bar{x}y)_{e_\sigma} + (\bar{x}y)_{e_\sigma - 1}\}. \tag{3}$$

Here $$(x\bar{y})_{e_\sigma} = x_{\sigma 1}\bar{y}_{\sigma e_\sigma} + x_{\sigma 2}\bar{y}_{\sigma, e_\sigma - 1} + \cdots + x_{\sigma e_\sigma}\bar{y}_{\sigma 1}, \tag{4}$$

with a similar meaning for $(x\bar{x})$, $(\bar{x}y)$. The coefficient ϵ_σ denotes either $+1$ or -1, as determined by a rule to be presently explained. The indices k and p are such that there are exactly n distinct variables among the $x_{\sigma j}$, $y_{\sigma j}$ inclusive. Similarly there are n conjugates $\bar{x}_{\sigma j}$, $\bar{y}_{\sigma j}$. These are linear combinations of the original variables, given by the transformation

$$\xi = Hx, \quad \bar{\xi} = \bar{x}\bar{H}',$$

where H is the matrix of the transformation, and where the n elements ξ_i denote the $x_{\sigma k}$, $y_{\sigma k}$ suitably arranged. In fact,

$$n = \sum_{\sigma=1}^{k} e_\sigma + 2 \sum_{\sigma=k+1}^{k+p} e_\sigma.$$

The parameter r is assumed to be *real*, and the determinant $|A|$ of the form ϕ to be non-zero. The determinant w_0 of the pencil is therefore a binary n-ic in r, which may be resolved into linear factors, real $r-c_\sigma$ or complex $r-a_\sigma$, $r-\bar{a}_\sigma$. When these are grouped into *elementary divisors* (**6**, 118) of the pencil, they assume the form

$$w_0 = |rA-B| = |A| \prod_{\sigma=1}^{k} (r-c_\sigma)^{e_\sigma} \prod_{\sigma=k+1}^{k+p} \{(r-a_\sigma)^{e_\sigma}(r-\bar{a}_\sigma)^{e_\sigma}\}, \tag{5}$$

where $|A| \neq 0$, $c_\sigma = \bar{c}_\sigma$, $a_\sigma \neq \bar{a}_\sigma$. In this way all the coefficients c_σ, a_σ, $\bar{a}_\sigma$ and *exponents* e_σ contained in the above canonical forms are defined. There are thus k real, and p pairs of complex, elementary divisors of such a pencil.

That complex divisors occur in pairs follows readily from the fundamental property of Hermitian determinants epitomized in the relations

$$w_0 = |rA-B| = |r\bar{A}'-\bar{B}'| = |r\bar{A}-\bar{B}|' = |r\bar{A}-\bar{B}| \neq 0. \tag{6}$$

These show that w_0 is real; hence in particular $|A|$ is real, and complex factors, if they exist, must occur in pairs. The reality of certain bordered

Hermitian determinants then follows: let us say, of

$$w_0 = |C|, \quad w_1 = \begin{vmatrix} C & \bar{u}^1 \\ u^1 & . \end{vmatrix}, \quad w_2 = \begin{vmatrix} C & \bar{u}^1 & \bar{u}^2 \\ u^1 & . & . \\ u^2 & . & . \end{vmatrix}, \ \ldots, \ w_n, \tag{7}$$

where $C = rA - B$, and where $w_\nu (\nu = 1, 2, \ldots, n)$ is obtained from $w_{\nu-1}$ by attaching a row u^ν of n complex constants, independent of r, together with its complex conjugate column $\bar{u}^\nu$. The ν^2 elements common to the last ν rows and columns are zero. Since w_ν is real, to any factor $(r-a)^{\lambda_\nu}$ will correspond a conjugate $(r-\bar{a})^{\lambda_\nu}$: and by taking the highest value of λ_ν for each ν and then forming the differences $\lambda_0-\lambda_1$, $\lambda_1-\lambda_2$, ..., the exponents e_σ are obtained, belonging to the latent root a, or to $\bar{a}$. Hence complex elementary divisors occur in pairs with equal exponent e_σ.

Here it is essential that *none of the w_ν should vanish*: with this proviso the choice of the constants u^ν, $\bar{u}^\nu$, for all ν, is arbitrary. Bromwich (3, 346) states, without proof, that the argument of Stickelberger for the cases of general and of symmetric forms is applicable at once to Hermitian forms. Actually the proof involves an extra step, so that it is best to give it explicitly. In fact, let W_ν be the result of bordering w_ν with $2n$ variables u, $\bar{u}$ instead of constants, so that W_ν is a Hermitian form in u, $\bar{u}$. Also w_0 is assumed to be non-zero. If a denotes any latent root, real or complex, then W_ν may be expanded in ascending powers of $r-a$ and written

$$W_\nu = (r-a)^{\lambda_{\nu+1}}\{u H_0 \bar{u} + (r-a) u H_1 \bar{u} + \ldots\} \quad (H_0 \neq 0), \tag{8}$$

where each expression $u H_i \bar{u}$ is a bilinear, but not necessarily a Hermitian, form. All the above requirements are satisfied if, and only if, the initial term $u H_0 \bar{u}$ is not identically zero. For then we select as u^ν any value of the vector u for which this does not vanish, so that $u^\nu H_0 \bar{u}^\nu \not\equiv 0$. Were $u H_0 \bar{u}$ to vanish identically, then, on writing $u = \alpha + i\beta$, $\bar{u} = \alpha - i\beta$, and picking out the coefficient of $\alpha_j \beta_k$ for each independent term in $u H_0 \bar{u}$, we should have the elements h_{ij} of the matrix H_0 satisfying the conditions $h_{ii} = 0$, $h_{ij} \pm h_{ji} = 0$, that is, $H_0 = 0$, which is not the case. This proves, by induction on ν, that the requisite choice of $u^\nu, \bar{u}^\nu$ is justified. They must satisfy a finite number of inequalities such as $u^\nu H_0 \bar{u}^\nu \not\equiv 0$, one for each distinct latent root a, $\bar{a}$, or c.

The canonical form now follows by general Weierstrassian methods, as expounded by Stickelberger (**2**). It is necessary to expand the non-zero Hermitian form $-W_0/w_0$ in descending powers of r in two ways, and then to pick out the coefficients of r^{-1} and r^{-2}. The first expansion gives

(*cf.* **6**, 162), for suitably large values of r,

$$-W_0/w_0 = u(rA-B)^{-1}\bar{u} = u\left(\frac{A^{-1}}{r}+\frac{A^{-1}BA^{-1}}{r^2}+\ldots\right)\bar{u}$$

$$= \frac{\bar{x}Ax}{r}+\frac{\bar{x}Bx}{r^2}+\ldots = \frac{\phi}{r}+\frac{\psi}{r^2}+\ldots, \tag{9}$$

provided that $\bar{u} = Ax$, and $u = \bar{x}\bar{A}' = \bar{x}A$. That is, the column of variables $\bar{u}$ must be chosen as linear functions of the column x with matrix A, while the conjugate row u must satisfy the conjugate relation. The second expansion takes the form

$$-\frac{W_0}{w_0} = -\frac{\bar{U}_1U_1}{w_0w_1}-\frac{\bar{U}_2U_2}{w_1w_2}-\ldots-\frac{\bar{U}_nU_n}{w_{n-1}w_n}, \tag{10}$$

depending upon the repetition (for $\nu = 0, 1, \ldots, n$) of the determinantal identity $W_\nu w_{\nu-1} = W_{\nu-1}w_\nu - \bar{U}_\nu U_\nu$, where $W_{\nu-1}, w_\nu, \bar{U}_\nu, U_\nu$ are the co-factors in W_ν of the four elements belonging to its two final rows and columns. Evidently $\bar{U}_\nu$ is linear in the $\bar{u}_i$, and therefore in the x_i, while U_ν is the conjugate linear form in the u_i and the $\bar{x}_i$. As in the general theory Bromwich shows that the canonical forms Φ and Ψ are obtained, answering to the coefficients ϕ and ψ of r^{-1} and r^{-2} in (9), by expanding each term on the right of (10) in ascending powers of $r-c$, picking out the coefficients (i) of $(r-c)^{-1}$ and (ii) of $(r-c)^{-2}$, and adding the results (i) to obtain Φ and (ii) to obtain Ψ, for all distinct real and complex roots c.

In order to preserve the conjunctive nature of the transformation it is now necessary to introduce the sign ϵ_σ which appears in each term of (3) answering to a *real* root c (but not to a complex root). In fact, let the real determinant w_ν contain $r-c$ exactly to power λ_ν, for each value of ν. Then we can write

$$w_\nu w_{\nu-1} = -(r-c)^{\lambda_\nu+\lambda_{\nu-1}}\{\beta_1+\beta_2(r-c)+\beta_3(r-c)^2+\ldots\}, \tag{11}$$

where the β_i are real constants independent of r, with $\beta_1 \neq 0$. *Choose the sign* ϵ_σ *so that* $\epsilon_\sigma\beta_1 > 0$: that is

$$\epsilon_\sigma = 1, \quad \beta_1 > 0; \quad \epsilon_\sigma = -1, \quad \beta_1 < 0. \tag{12}$$

For if so, then a *real* square root of the β series in (11) exists such that

$$\sqrt{[\epsilon_\sigma\{\beta_1+\beta_2(r-c)+\ldots\}]} = \alpha_1+\alpha_2(r-c)+\ldots \quad (\alpha_1 > 0) \tag{13}$$

$$= (\alpha, t)$$

say, where $t = r - c$. But U_ν can be similarly written as $t^{\lambda_\nu}(\theta, t)$ with the exponent λ_ν exactly, since, when $u = w^\nu$, U_ν becomes w_ν. The coefficients $\theta_1, \theta_2, \ldots$ are then linear in the u_i, and therefore in the variables $\bar{x}_i$, by (9). Also, since t is real, $\bar{U}_\nu = t^{\lambda_\nu}(\bar{\theta}, t)$. When we substitute these expressions in the ν-th term of the series (10) it becomes

$$-\frac{\bar{U}_\nu U_\nu}{w_\nu w_{\nu-1}} = \frac{\epsilon_\sigma}{t^{e_\nu}} \frac{(\bar{\theta}, t)(\theta, t)}{(a, t)(a, t)} = \frac{\epsilon_\sigma}{t^{e_\nu}} (\bar{\xi}, t)(\xi, t), \tag{14}$$

where $(\bar{\xi}, t)$ denotes the ascending expansion of $(\bar{\theta}, t)/(a, t)$, and similarly for (ξ, t). When the coefficients of t^{-1} and t^{-2} are picked out of (14), they give at once the requisite terms of Φ and Ψ in (3), in view of the relations (9) and (10). The exponent e_ν, if non-zero, denotes the ν-th exponent belonging to the same real root c. If we revert to the original notation, it is the e_σ of (3).

Corresponding to a complex root a an expression analogous to (14),

$$-\frac{\bar{U}_\nu U_\nu}{w_\nu w_{\nu-1}} = \frac{1}{\tau^{e_\nu}} (\xi, \tau)(\bar{\eta}, \tau), \tag{15}$$

may be set up, where $\tau = r - a \neq \bar{\tau}$, while (ξ, τ) and $(\bar{\eta}, \tau)$ are power series in τ with non-zero initial terms. The (ξ, τ) series may be obtained from the expansion of $\bar{U}_\nu/(w_\nu w_{\nu-1})$, and the (η, τ) from U_ν. The coefficients of τ^{-1} and τ^{-2} in the expansion of (15) yield the appropriate terms in Φ and Ψ, answering to a value of σ between $k+1$ and $k+p$ inclusive. The alternative resolution

$$\frac{1}{\bar{\tau}^{e_\nu}} (\bar{\xi}, \bar{\tau})(\eta, \bar{\tau}) \quad (\bar{\tau} = r - \bar{a}) \tag{16}$$

of the *real* expression on the left of (15) supplies the conjugate part of the same terms in Φ and Ψ. The first e_ν coefficients of each series (ξ, τ), $(\bar{\eta}, \tau)$, $(\bar{\xi}, \bar{\tau})$, $(\eta, \bar{\tau})$ are the $4e_\nu$ variables denoted by $x, \bar{y}, \bar{x}, y$ in the typical term of the second series of Ψ in (3).

2. *The signature test.*

As in the case of real quadratic forms† under real transformations, necessary and sufficient conditions for equivalence depend not merely upon the elmentary divisors but also upon the *signatures* of certain con-

† Muth, **4.**

comitant forms. It is convenient to speak of the *divisor test*, and the *signature test* of equivalence.

Two pencils $r\phi-\psi$, $r\phi'-\psi'$ which satisfy the former test can therefore be brought to the forms $r\Phi-\Psi$, $r\Phi'-\Psi'$, where Φ' and Φ are identical except for possible differences among the signs $\epsilon_\sigma = \pm 1$ of § 1 (3): and similarly for Ψ' and Ψ. The signature test is needed to remove this difference.

Now the signature of a Hermitian form

$$\sum_{i=1}^{k} a_i \bar{x}_i x_i - \sum_{i=k+1}^{k+l} a_i \bar{x}_i x_i \tag{1}$$

is defined to be $k-l$ provided that each $a_i > 0$ and all the $k+l$ variables x_i are linearly independent. According to Sylvester's law of inertia (**6**, 89), the signature is a numerical invariant of conjunctive transformations. Thus the forms Φ and Φ' cannot be equivalent unless they have the same signature.

Consider the signature of Φ. Since the variables occurring in any one of its $k+p$ terms are distinct from those of the remaining terms, the signature of the whole is equal to the sum of the signatures of these several terms. The same argument applies in finding the signature of the term $(x\bar{x})_e$. After dropping the suffix σ for brevity [*cf.* § 1 (4)], we may rearrange this term as

$$\text{(i)} \quad \sum_{i=1}^{r} (x_i \bar{x}_{2r-i+1} + \bar{x}_i x_{2r-i+1}) \quad (e = 2r),$$

$$\text{(ii)} \quad \sum_{i=1}^{r-1} (x_i \bar{x}_{2r-i} + \bar{x}_i x_{2r-i}) + x_r \bar{x}_r \quad (e = 2r-1).$$

But the identity

$$x\bar{y} + \bar{x}y = \left(\frac{x+y}{2}\right)\left(\frac{\bar{x}+\bar{y}}{2}\right) - \left(\frac{x-y}{2}\right)\left(\frac{\bar{x}-\bar{y}}{2}\right) \tag{2}$$

between four non-zero complex numbers x, $\bar{x}$, y, $\bar{y}$ shows at once that each term in the above two series may *separately* be transformed into a term of signature zero. Also that of the isolated term $x_r \bar{x}_r$ is unity. Hence the signature of $(x\bar{x})_e$ is 1 or 0 according as e is odd or even. Or, let us say,

$$\operatorname{sig}\{\epsilon_\sigma (x\bar{x})_{e_\sigma}\} = \epsilon_\sigma \text{ or } 0, \quad \textit{as } e_\sigma \textit{ is odd or even.} \tag{3}$$

The same argument also shows that the signature of each of the last p terms of Φ is zero.

It is illuminating to throw the above theory of signature into matrix form, expressing the invariance over a transformation $x = H\xi$: thus

$$\operatorname{sig}(\bar{x}Ax) = \operatorname{sig}(\bar{\xi}\bar{H}'AH\xi).$$

But this property belongs essentially to the matrix, however the variables are changed. So without ambiguity we may write

$$\operatorname{sig}(A) = \operatorname{sig}(\bar{H}'AH). \tag{4}$$

In particular, when the matrix of $(x\bar{x})_{e_\sigma}$ is explicitly stated, we have

$$[1] = 1, \quad \operatorname{sig}\begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix} = 0, \quad \operatorname{sig}\begin{bmatrix} . & . & 1 \\ . & 1 & . \\ 1 & . & . \end{bmatrix} = 1, \quad \operatorname{sig}\begin{bmatrix} . & . & . & 1 \\ . & . & 1 & . \\ . & 1 & . & . \\ 1 & . & . & . \end{bmatrix} = 0, \tag{5}$$

and so on.

Next, let the k real elementary divisors $(r-c_\sigma)^{e_\sigma}$ be arranged according to repetitions, so that the first ρ roots c are equal,

$$c_1 = c_2 = \ldots = c_\rho, \tag{6}$$

and let their exponents be arranged in descending value,

$$e_1 = e_2 = \ldots = e_\alpha > e_{\alpha+1} = \ldots = e_{\alpha+\beta} > \ldots \geqslant e_\rho. \tag{7}$$

Suppose that all further $k-\rho$ elementary divisors involve roots distinct from c_1 and that they are dealt with in the same way. Now consider the following sums:

$$\begin{aligned} E_1 &= \epsilon_1 + \epsilon_2 + \ldots + \epsilon_\alpha, \\ E_2 &= \epsilon_{\alpha+1} + \epsilon_{\alpha+2} + \ldots + \epsilon_{\alpha+\beta}, \\ E_3 &= \epsilon_{\alpha+\beta+1} + \epsilon_{\alpha+\beta+2} + \ldots + \epsilon_{\alpha+\beta+\gamma}, \text{ etc.}, \end{aligned} \tag{8}$$

corresponding to the groups of α, β, γ, ... equal exponents among the e_σ. Since each ϵ_σ only takes the values ± 1 it follows that, when two values of E_1 are equal, they contain the same number of positive terms ϵ_σ (and of negative terms). They can differ merely by derangement of their terms. Likewise for each E_i. Let similar sets E', E'', ... be formed, one for each new distinct real root c', c'', ..., and let the aggregate of all the E's be denoted

by $[E]$ and contain m terms. Then evidently *two Hermitian forms* Φ *and* Φ', *possessing the same elementary divisors and the same set* $[E]$, *will differ merely by a derangement of their variables, and will therefore be equivalent.*

The signature test is precisely the invariance of the m integers $[E]$.

3. *Matrix form of the test.*

To express the E_i in terms of signatures of concomitants, it is simplest to proceed as follows. Let $rA-B$ denote the *reduced* form of the matrix pencil, so that A is the matrix of Φ and B is that of Ψ. When the matrices of § 1 (3) are picked out they can evidently be written as

$$rA-B = \operatorname{diag}(\epsilon_1 R_1,\ \epsilon_2 R_2,\ \ldots,\ \epsilon_k R_k,\ R_0), \tag{1}$$

where, by § 2(5),

$$R_\sigma = \begin{bmatrix} . & \ldots & -1 & r-c_\sigma \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -1 & r-c_\sigma & \ldots & . \\ r-c_\sigma & . & \ldots & . \end{bmatrix} \quad (\sigma = 1, 2, \ldots, k), \tag{2}$$

consisting of e_σ elements $r-c_\sigma$ upon the secondary diagonal, $e_\sigma-1$ elements -1 upon the adjacent upper parallel, and zeros elsewhere in the c_σ rows and columns. Also R_0 denotes the matrix of the final p terms in $r\Phi-\Psi$. The notation indicates that the R_i are separate submatrices of the whole, arranged upon the principal diagonal. It is readily verified that

$$(rA-B)^{-1} = \operatorname{diag}(\epsilon_1 S_1,\ \epsilon_2 S_2,\ \ldots,\ \epsilon_k S_k,\ S_0), \tag{3}$$

where $S_\sigma = R_\sigma^{-1}$, and

$$S_\sigma = \begin{bmatrix} . & . & . & \ldots & s \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ . & . & s & \ldots & s^{e_\sigma-2} \\ . & s & s^2 & \ldots & s^{e_\sigma-1} \\ s & s^2 & s^3 & \ldots & s^{e_\sigma} \end{bmatrix} \quad [s = (r-c_\sigma)^{-1},] \tag{4}$$

consisting of the secondary and lower diagonals, containing equal elements s, s^2, s^3, ... respectively. All the upper elements are zero. According to

the arrangement adopted in § 2 (8) the first ρ matrices S_σ fall into groups

$$S_1 = S_2 = \ldots = S_\alpha, \quad S_{\alpha+1} = S_{\alpha+2} = \ldots = S_{\alpha+\beta}, \ \ldots, \tag{5}$$

being matrices of orders e_1, $e_{\alpha+1}$, $e_{\alpha+\beta+1}$, ... respectively.

Again, the reciprocal matrix (3) may be expressed in the usual way as the adjoint matrix divided by the determinant $|rA-B|$. The highest power $(r-c)^{\lambda_1}$ may then be cancelled from numerator and denominator, and the result developed in partial fractions. We therefore first write $r-c_1 = t$, and

$$\operatorname{adj}(rA-B) = Q_0+Q_1t+Q_2t^2+\ldots+Q_{n-1}t^{n-1}, \tag{6}$$

where, owing to t^{λ_1} being a factor, the first λ_1 constant *matrix* coefficients Q_i vanish identically. Also let

$$|rA-B| = t^{\lambda_0}(\theta_0+\theta_1t+\ldots+\theta_{n-\lambda_0}t^{n-\lambda_0}), \tag{7}$$

where the θ_i are constants independent of t. Hence

$$\begin{aligned}\frac{1}{rA-B} &= \frac{Q_{\lambda_1}+Q_{\lambda_1+1}t+\ldots+Q_{n-\lambda_1-1}t^{n-\lambda_1-1}}{t^{e_1}(\theta_0+\theta_1t+\ldots+\theta_{n-\lambda_0}t^{n-\lambda_0})} \\ &= \frac{P_1}{t^{e_1}}+\frac{P_2}{t^{e_1-1}}+\ldots+\frac{P_{e_1}}{t}+P_0(t)\end{aligned} \tag{8}$$

say, where, as before, $\lambda_0-\lambda_1 = e_1$, and the P_σ denote the matrix numerators of the partial fractions, while $P_0(t)$ is a series in ascending non-negative powers of t. Also $st = 1$, $Q_{\lambda_1} \neq 0$, $\theta_0 \neq 0$.

On comparing (3), (4), and (8) each matrix P_σ can at once be given in terms of the diagonals of the S_σ. For example, take $\sigma = 1$; then P_1 is the coefficient of s^{e_1} in (3) expressed in descending powers of s and hence

$$P_1 = \operatorname{diag}(\epsilon_1\Sigma_1, \epsilon_2\Sigma_1, \ldots, \epsilon_\alpha\Sigma_1, 0, \ldots, 0), \tag{9}$$

where, by (4), each Σ_1 denotes a void matrix except for a single unit standing at the bottom right-hand corner. Hence, by § 2 (5), each has a signature unity; and since these submatrices are entirely independent when the variables are inserted, we may sum their signatures. Thus

$$\operatorname{sig} P_1 = \epsilon_1+\epsilon_2+\ldots+\epsilon_\alpha = E_1, \tag{10}$$

which gives the desired numerical invariant E_1 as the signature of P_1. Next, to find E_2, consider the coefficient of $s^{e_{\alpha+1}}$ in (3) and (8). In (8) it is P_{q_1}, where

$$q_1 = e_1-e_{\alpha+1}+1.$$

Hence

$$P_{q_1} = \text{diag}(\epsilon_1 \Sigma_{q_1}, \ldots, \epsilon_\alpha \Sigma_{q_1}, \epsilon_{\alpha+1} \Sigma_1', \ldots, \epsilon_{\alpha+\beta} \Sigma_1', 0, \ldots, 0), \tag{11}$$

where Σ_{q_1} denotes a matrix of q_1 units placed diagonally, and Σ_1' has one unit as before, but only $e_{\alpha+1}$ rows and columns. Hence, by § 2 (5),

$$\text{sig}\, P_{q_1} = \left.\begin{matrix} 0 \\ 1 \end{matrix}\right\} E_1 + E_2, \tag{12}$$

where the coefficient of E_1 is 1 or 0 according as q_1 is odd or even. Similarly, we may take

$$q_2 = e_1 - e_{\alpha+\beta+1} + 1,$$

$$\text{sig}\, P_{q_2} = \left.\begin{matrix} 0 \\ 1 \end{matrix}\right\} E_1 + \left.\begin{matrix} 0 \\ 1 \end{matrix}\right\} E_2 + E_3, \tag{13}$$

and so on. Each coefficient is determinate: that of E_i in the equation involving q_j being 1 or 0 according as the difference $e' - e''$ is even or odd, where e', e'' are the respective exponents to which E_i and E_{j+1} refer. The criterion is simply whether or not the Σ submatrix of P_{q_j} has a non-zero element upon the *principal* diagonal [§ 2 (5)].

Now the original pencil may be written $\bar{H}^{-1}(rA - B)H'^{-1}$, where H is a certain non-singular matrix independent of r. Hence the original form of the *reciprocal* pencil, and therefore of each P_i, will be found from (8) by multiplying each term fore and aft by H', $\bar{H}$ respectively. Hence *each P_i is a matrix of a contravariant Hermitian form, and therefore has a fixed signature for all conjunctive transformations.*

Combining these results we infer that *the set E_i is given if, and only if, the signatures of the contravariant matrices*

$$P_1, \quad P_{q_1}, \quad P_{q_2}, \quad \ldots, \quad P_{q_\mu} \tag{14}$$

are given, where $q_i = e_1 + 1 - e_j$ and $j = \alpha + \beta + \ldots + 1$, involving i letters $\alpha, \beta, \ldots$.

Such may be called the set $[P]$, corresponding to the real root c. Sets similar to (14) called $[P']$, $[P'']$ may now be formed corresponding to further distinct roots c', c'', and so on. The number m of these P's is exactly that of the whole set of $[E]$. Hence

The signature test is satisfied if, and only if, the signatures of the m contravariant matrices of the sets $[P]$, $[P']$, ... are fixed.

4. *Connection with Muth's theorem for real quadratic pencils.*

By equating coefficients of corresponding powers of t in § 3 (8) after cross multiplication, we obtain the relations

$$\left.\begin{aligned} Q_{\lambda_1} &= \theta_0 P_1, \\ Q_{\lambda_1+1} &= \theta_0 P_2 + \theta_1 P_1, \\ Q_{\lambda_1+2} &= \theta_0 P_3 + \theta_1 P_2 + \theta_2 P_1, \end{aligned}\right\} \tag{1}$$

and so on. If $\sigma < q_1$ we have, as in § 3 (9),

$$P_\sigma = \operatorname{diag}(\epsilon_1 \Sigma_\sigma,\ \epsilon_2 \Sigma_\sigma,\ \ldots,\ \epsilon_\alpha \Sigma_\sigma,\ 0,\ \ldots,\ 0), \tag{2}$$

where Σ_σ is a diagonal of σ units. The first α submatrices Σ are thus equal in P_σ, as also in P_{q_1} [by § 3 (11)]: and, since the coefficients θ are scalar, we can substitute for each P_σ and for P_{q_1} in (1) and obtain, in particular,

$$Q_{\lambda_1+q_1-1} = \operatorname{diag}(\epsilon_1 \Sigma,\ \ldots,\ \epsilon_\alpha \Sigma,\ \epsilon_{\alpha+1} \theta_0 \Sigma_1',\ \ldots,\ \epsilon_{\alpha+\beta} \theta_0 \Sigma_1',\ 0,\ \ldots,\ 0), \tag{3}$$

where each Σ is a linear combination of those already employed, while Σ_1' has a single unit element in the bottom right-hand corner of its $e_{\alpha+1}$ rows and columns. Now consider the signatures of Q_{λ_1} and of $Q_{\lambda_1+q_1-1}$. Let $\operatorname{sig} \theta_0 = \sigma_0 \neq 0$, since θ_0 is certainly non-zero. Then, by (1) and (3),

$$\operatorname{sig} Q_{\lambda_1} = \sigma_0 \operatorname{sig} P_1 = \sigma_0 E_1,$$

$$\operatorname{sig} Q_{\lambda_1+q_1-1} = E_1 \operatorname{sig} \Sigma + E_2 \sigma_0 \operatorname{sig} \Sigma_1' = E_1 \operatorname{sig} \Sigma + E_2 \sigma_0. \tag{4}$$

Hence the signatures of these Q matrices are determined by E_1 and E_2; and conversely. Exactly similar reasoning applies to each such member $Q_{\lambda_1+q_i-1}$; and this proves *Muth's theorem*†, namely that

The signature test is satisfied if, and only if, the signatures of the m contravariant matrices of the sets $[Q]$, $[Q']$, ... *are fixed, where*

$$[Q] = Q_{\eta_1},\ Q_{\eta_{\alpha+1}},\ Q_{\eta_{\alpha+\beta+1}},\ \ldots,\ Q_{\eta_{\alpha+\beta+\ldots+\omega+1}},$$

$$e_1 + e_2 + \ldots + e_\rho - e_\sigma = \eta_\sigma \quad (\sigma = 1, 2, \ldots, \rho). \tag{5}$$

It only remains to identify the suffixes. In fact, by (5), we have

$$\eta_1 = \lambda_1, \quad \eta_{\alpha+1} = \lambda_0 - e_{\alpha+1} = \lambda_1 + e_1 - e_{\alpha+1} = \lambda_1 + q_1 - 1, \quad \eta_{\alpha+\beta+1} = \lambda_1 + q_2 - 1,$$

and so on.

† 4, p. 312, Theorem (*d*). *Cf.* p. 308.

Muth remarks that if *all* the exponents e are even, then the signature of the matrix A is zero: but that if odd values of e appear then the non-zero signature of A may replace one of the m invariant signatures of the Q sets. This follows at once from § 3 (1) and (2), in that the signature of A is given by the sum of all the ϵ_σ corresponding to matrices R_σ of odd order: that is the sum of the E_i answering to odd values of the e.

If at the outset $|A| = |B| = 0$, but $|rA-B| \neq 0$, then new bases for the pencil may be chosen in the usual way, and they will give the same set of concomitants Q in § 3 (8).

Example. The quaternary forms in variables x, y, z, u,

$$\phi = \bar{x}y+x\bar{y}+\bar{z}u+z\bar{u},$$

$$\psi = \bar{x}x+\bar{x}y+x\bar{y}+i(\bar{z}u-z\bar{u}),$$

are Hermitian: so also are

$$\phi_1 = -\bar{x}y-x\bar{y}+\bar{z}u+z\bar{u},$$

$$\psi_1 = -\bar{x}x-\bar{x}y-x\bar{y}+i(\bar{z}u-z\bar{u}).$$

The pencils $r\phi-\psi$ and $r\phi_1-\psi_1$ possess the same elementary divisors $(r-1)^2$, $r+i$, $r-i$, but differ as to signatures. They cannot be transformed into each other conjunctively.

5. *Reduction of a singular Hermitian pencil.*

It is convenient to consider the pencil in homogeneous form $C = rA-sB$, where both r and s are real, and the determinant is identically zero, that is $|rA-sB| \equiv 0$. As in the general theory of singular pencils, certain *minimal* integral numbers $m\ (\geqslant 0)$ exist such that, for each m, $uC = 0$, where u is a vector whose elements are homogeneous in r, s and of degree m. Also, such a vector can be resolved into the form

$$u = \varpi H, \quad \varpi = [s^m, -rs^{m-1}, r^2 s^{m-2}, \ldots, (-)^m r^m, 0, \ldots, 0], \qquad (1)$$

where H is a non-singular constant square matrix independent of r and s. Hence $\varpi HC\bar{H}'$ vanishes also: which means that the vector ϖ annihilates a Hermitian pencil $H(rA-sB)\bar{H}' \equiv \Lambda$; and, by transposition, a column vector ϖ' does so too. Thus

$$\varpi\Lambda = 0, \quad \Lambda\varpi' = 0. \qquad (2)$$

It follows at once that *the leading square submatrix of* $(m+1)^2$ *elements in the first* $m+1$ *rows and columns of* Λ *is zero.* For if $ar+bs$ is the leading element of Λ, in the first row and column, it cannot be annihilated by ϖ

unless $b=0$, and unless the term as occurs in the second place of the first column. Since Λ is Hermitian this also implies a term $\bar{a}s$ in the second place of the first row, which in turn can only be annihilated by ϖ if $\bar{a}=0$. Hence the leading element in Λ is zero: and a repetition of the process, step by step, removes all elements within the scope of ϖ, that is, within the first $m+1$ rows and columns.

If $m=0$, the leading row and column of Λ are null: and the reduction is complete as far as this m is concerned. If $m>0$, no term in s can occur in the top row, owing to condition (2), while all terms in r can be concentrated in position $m+2$ of the top row, with unit coefficient. This is done by a suitable postmultiplier $\bar{H}_1'$, of type H above. For (2) to apply, a term s must now occur immediately below r, that is in the position $(2, m+2)$. If $m>1$, the rank of the first two rows as regards terms in r must also exceed unity. Hence terms in r must not only occur on the second row, but can be concentrated by postmultiplication at position $(2, m+3)$, again implying a term s at $(3, m+3)$, and so on as in the general case. Combining these processes, we infer that a matrix H_m exists for which

$$\Lambda H_m = \begin{bmatrix} . & L & . \\ X_1 & X_2 & X_3 \end{bmatrix}, \tag{3}$$

where

$$L = \begin{bmatrix} r & . & \cdots & . \\ s & r & \cdots & . \\ . & s & \cdots & . \\ \cdots & \cdots & \cdots & \\ . & . & \cdots & r \\ . & . & \cdots & s \end{bmatrix}. \tag{4}$$

If $m=0$, L is absent, there being a single top row of zeros in (3). If $m>0$, L has m elements r, and m elements s, arranged diagonally on $m+1$ rows in m columns. The submatrix X_1 denotes the original $m+1$ columns of Λ which have been undisturbed by the process H_m. A premultiplier $\bar{H}_m'$ will now affect X_1 exactly as H_m affected the corresponding rows and yielded L. Since X_1 was undisturbed by H_m, so also will L be undisturbed by $\bar{H}_m'$. Thus, let us say,

$$\bar{H}_m'\Lambda H_m = \begin{bmatrix} . & L & . \\ L' & X & Y \\ . & \bar{Y}' & T \end{bmatrix} \quad (\bar{X}'=X,\ \bar{T}'=T),$$

where L' is necessarily the transposed of the real submatrix L, and the X, Y, T are still to be considered.

Repetition of the process upon T will yield all further pairs of minimal matrices L_i, L_i', with a possible non-singular core X_0, just as in the general case. This core may be reduced by the methods of § 1 after a change of basis, if such be necessary (**6**, 127). The final result is typified by the following matrix:

$$\begin{bmatrix} . & L & . & . & . & \dots & . \\ L' & X_1 & Y_1 & X_2 & Y_2 & \dots & Y_h \\ . & \bar{Y}_1' & . & L_1 & . & \dots & . \\ . & \bar{X}_2' & L_1' & * & * & \dots & * \\ . & \bar{Y}_2' & . & * & . & \dots & * \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ . & \bar{Y}_h' & . & * & \dots & \dots & X_0 \end{bmatrix},$$

where the submatrices *, together with X_1, Y_1, X_2, Y_2, etc., can be annihilated by a further conjunctive transformation, leaving the desired canonical form containing only the minimal submatrices L_i, L_i' and the non-singular core X_0. The annihilation is a step by step process (**6**, 127), *but must be kept conjunctive throughout.* It will be shown that the process makes use, in each case, of that pair of matrices L_i, L_j', X_0 which stand in the same row and in the same column as the annihilated submatrix. Thus Y_1 is annihilated by L_1' and L'; and the use of L_1' implies that X_1, X_2, Y_2 are possibly modified by the submatrices in the *row* of L_1', whereas L', standing free in its column, acts without modifying L_1' in the column of Y_1. If the matrices of the second row are dealt with in the order

$$Y_1,\ Y_2,\ Y_3,\ \dots,\ X_3,\ X_2,\ X_1,$$

inspection shows that the removal of Y_i never alters a blank Y space on its left (but may affect an X), while the removal of an X never alters the blank spaces on its right (but may affect an X on its left). Hence, by clearing the Y's from left to right and then the X's from right to left, the whole gnomon is cleared. By repeating the process for each subsequent gnomon in turn, the complete reduction is effected.

It remains to justify the annihilating process. Let $x = ar + bs$ be an element of X_i or Y_i. Without altering any other element of its own submatrix, this x may be removed one step "north-east" or "south-west" if it lies in an X_i. For, owing to a term r of L_{i-1} in the column of x, ar may be

replaced by $-as$ one step leftwards, which in turn may be replaced by ar one step downwards, owing to the term s of L' in the same row. The resultant shift is south-west. Had L' operated before L_{i-1} it would be north-east. The same is true of the term bs, but with the rôles of L_{i-1} and L' interchanged. Now, if x belongs to X_1 and lies on the principal diagonal, it is real, with $a = \bar{a}$, $\bar{b} = b$. The above process is conjunctive if $\frac{1}{2}x$ is moved north-east, and the other half south-west. When this is done successively for each such element the principal diagonal is cleared. Next let each element of X_1 above this diagonal be moved north-east, and simultaneously at each step let the conjugate $\bar{x}$ (which is below the diagonal) be moved south-west. This clears the whole of X_1, for, when the boundary is reached, at the next step the element passes off the figure. Likewise it clears X_2 entirely north-east, and $\bar{X}_2'$ south-west, and so on.

Apart from the Y of the last column similar remarks apply to each Y. Such a Y is aligned with L' and L_i'. An element can move north-west or south-east: those of the first column together with all in their path must move south-east: the remainder move north-west. (*Cf.* 6, 127.) The whole process clears Y and $\bar{Y}'$ conjunctively.

The final Y_h, aligned vertically with X_0, requires special consideration. Let all the terms bs be cleared from Y_h by means of the s terms of L' (and, of course, conjunctively, let $\bar{Y}_h'$ be cleared of s by means of L). This process replaces any row of s terms by a row of r terms one place lower. Now the typical submatrix of X_0, by § 1, is such as

$$X_{0i} = \begin{bmatrix} . & -s & r-cs \\ -s & r-cs & . \\ r-cs & . & . \end{bmatrix},$$

containing zeros otherwise the whole row of X_0.

The r terms of a row of Y_h can be replaced from this X_0 by s terms in the same row, which in turn can be reduced by L' to r terms of the next lower row. Clearance by rows is therefore effected by proceeding downwards. Combining these results, we have

$$H'(rA-sB)H = \operatorname{diag}\left(\begin{bmatrix} . & L \\ L' & . \end{bmatrix}, \ \ldots, \ X_{01}, \ \ldots\right).$$

This last step is easily adapted to the case when a non-singular core is taken in the rational form of **6**, 134. Hence, finally,

Two singular Hermitian pencils are equivalent if, and only if, they possess the same minimal numbers, the same elementary divisors, and the same signatures to their sets $[P]$ (*or* $[Q]$).

Also, since the methods for dealing with the singular case have been rational, they apply immediately to that of real quadratics.

[*Added* 17 *April*, 1935.—Since the above was written various other treatments of the same subject have appeared. One was by G. R. Trott, "On the canonical form of a non-singular pencil of Hermitian matrices", *Amer. J. of Math.*, 56 (1934), 359–371. Then K. W. Wegner, *Bull. Amer. Math. Soc.*, 40 (1934), 533, stated the signature conditions with reference to the signature of the matrix $B(B^{-1}A+\lambda I)^m$ for every real λ and positive integral value of m. These conditions are sufficient and can be related to those of Muth by the method of page 240 *supra*. A letter from Prof. M. H. Ingraham in May, 1934, stated that the writer had completed the work of his collaborator, Mr. Wegner, by discussing also the singular case. The methods employed were an attractive alternative to the investigation given above, which has followed more closely the historical development.]

References.

1. Weierstrass, "Zur Theorie der bilinearen und quadratischen Formen", *Berliner Monatsbericht* (1868), 310; *Werke* 2 (1895), 19–44.
2. Stickelberger, "Über Scharen von bilinearen und quadratischen Formen", *J. für Math.*, 86 (1879), 20–43.
3. Bromwich, "On a canonical reduction of bilinear forms", *Proc. London Math. Soc.* (1), 30 (1900), 321–352.
4. Muth, "Über reelle Äquivalenz von Scharen reeller quadratischer Formen", *J. für Math.*, 128 (1905), 302–321.
5. Logsdon, "Pairs of Hermitian forms", *Amer. J. of Math.*, 44 (1922), 247–260.
6. Turnbull and Aitken, *Canonical matrices* (Blackie, 1932).

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HARMONIC INTEGRALS ASSOCIATED WITH ALGEBRAIC VARIETIES

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In two papers† I have defined integrals, which I have called harmonic integrals, which are associated with an analytic variety to which a metric is attached, and have established an existence theorem for them. More recently I have applied the theory of these integrals to the Riemannian manifold of an algebraic surface, in order to connect the geometric genus of the surface with the topological invariants of the manifold‡. There is reason to believe that this method of considering the Abelian integrals attached to an algebraic variety will prove a powerful one, and I have thought it advisable to set out in the following pages an account of the principles on which the method is based.

It is first necessary to construct the Riemannian manifold of an algebraic variety, and to attach to it a suitable metric. A method of doing this was suggested by a paper by G. Mannoury§, and has proved satisfactory. A detailed consideration is made of this metric and of the associated curvature tensor. The harmonic integrals are then introduced, and it is found that, because of the special properties of the metric, numerous theorems can be proved which are not true for harmonic integrals in general. These are set out in the paper with the purpose of arranging the harmonic integrals of given multiplicity into groups which render the discussion of their properties somewhat easier, and I believe that the theorems here proved will form the foundation on which subsequent investigation will be based.

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† W. V. D. Hodge, *Proc. London Math. Soc.* (2), 36 (1933), 257, and (2), 38 (1934), 72.

‡ W. V. D. Hodge, *Journal London Math. Soc.*, 8 (1933), 312.

§ G. Mannoury, *Nieuw Archief voor Wiskunde* (2), 4 (1898), 112.

It is not my purpose in this paper to make application of the general theory to deduce properties of algebraic varieties and their associated Abelian integrals. These, I hope, will form the subject of later papers. I have, however, in the last paragraph, mentioned a few of the immediate consequences of the general theory which may be of interest to geometers.

1. Let $(z_1, z_2, \ldots, z_{r+1})$ be homogeneous coordinates in a projective space S_r. Mannoury† has made an interesting study of the Riemannian of S_r, in the case $r = 2$, using a method which can be extended immediately to any value of r. The image of complex points of S_r is then obtained as follows. It is always possible to multiply the coordinates in S_r by a constant so that the condition

$$|z_1|^2+|z_2|^2+\ldots+|z_{r+1}|^2 = 1 \tag{1}$$

is always satisfied. Denote by $\bar{x}$ the conjugate of any complex number x, and write

$$\left.\begin{aligned} X_h &= \sqrt{2}\, z_h \bar{z}_h \qquad (h = 1, \ldots, r+1), \\ X_{hk} &= (z_h \bar{z}_k + \bar{z}_h z_k) = X_{kh} \quad (h, k = 1, \ldots, r+1;\ h \neq k), \\ Y_{hk} &= i(z_h \bar{z}_k - z_k \bar{z}_h) = -Y_{kh}. \end{aligned}\right\} \tag{2}$$

Regarding (X_h, X_{hk}, Y_{hk}) as rectangular Cartesian coordinates in a real Euclidean space E of $(r+1)^2$ dimensions, we consider the real locus R defined by the equations (1), (2). Since the coordinates of the points defined by these equations are unaltered when the coordinates of the points of S_r are multiplied by $e^{i\theta}$, we see that to each point of S_r there corresponds one point of R. Conversely, to the point (X_h, X_{hk}, Y_{hk}) of R there corresponds one point of S_r, since

$$\frac{z_h}{z_k} = \frac{X_{hk} - i\, Y_{hk}}{\sqrt{2}\, X_k}.$$

Therefore the locus R is in (1-1) correspondence with the complex points of S_r, and the correspondence is clearly continuous both ways. Moreover, the locus R lies entirely at a finite distance in E. For, generalising the formula of Mannoury, we find that the Euclidean distance between the images of the points $(z_1, \ldots, z_{r+1})$ and $(\zeta_1, \ldots, \zeta_{r+1})$ of S_r, whose coordinates are subject to (1), is

$$2[1-(z_1\bar{\zeta}_1+z_2\bar{\zeta}_2+\ldots+z_{r+1}\bar{\zeta}_{r+1})(\bar{z}_1\zeta_1+\bar{z}_2\zeta_2+\ldots+\bar{z}_{r+1}\zeta_{r+1})]^{\frac{1}{2}},$$

† G. Mannoury, *Nieuw Archief voor Wiskunde* (2), 4 (1898), 112.

and this never exceeds 2. R is, therefore, the Riemannian of S_r, and when we speak of the Riemannian in the sequel we shall mean this particular representation of it. Van der Waerden† has shown that it is an orientable manifold in the topological sense.

We shall also have to regard R as an analytic locus of the type discussed in differential geometry. For this purpose we shall use non-homogeneous coordinates which are valid in the portion of S_r under discussion. Suppose, for instance, that $z_{r+1} \neq 0$ in the neighbourhood of the point in which we are interested. We can then take as coordinates

$$\left(\frac{z_1}{z_{r+1}}, \quad \frac{z_2}{z_{r+1}}, \quad \ldots, \quad \frac{z_r}{z_{r+1}}\right),$$

and write

$$\frac{z_s}{z_{r+1}} = x_s + i x_{r+s}.$$

$(x_1, x_2, \ldots, x_{2r})$ can then be taken as parameters on R. The points of R can be given parametrically as follows:

$$X_h = \sqrt{2}\left(\frac{x_h^2 + x_{r+h}^2}{1 + \sum_1^{2r} x_s^2}\right) \quad (h = 1, \ldots, r),$$

$$X_{r+1} = \frac{\sqrt{2}}{1 + \sum_1^{2r} x_s^2},$$

$$X_{hk} = 2\left(\frac{x_h x_k + x_{r+h} x_{r+k}}{1 + \sum_1^{2r} x_s^2}\right) \quad (h,\ k \leqslant r),$$

$$X_{hr+1} = \frac{2x_h}{1 + \sum x_h^2},$$

$$Y_{hk} = 2\left(\frac{x_h x_{r+k} - x_k x_{r+h}}{1 + \sum_1^{2r} x_s^2}\right) \quad (h,\ k \leqslant r),$$

$$Y_{hr+1} = \frac{-2x_{r+h}}{1 + \sum_1^{2r} x_s^2}.$$

† B. L. van der Waerden, *Math. Annalen*, 102 (1930), 337.

From these we can calculate the first fundamental quadratic differential form, giving the element of length. Let this be

$$ds^2 = g_{ij}\,dx^i\,dx^j\dagger.$$

In giving a formula for the calculation of the components of the tensor g_{ij} we make the following convention. We allow the indices of the coordinates x_h to take any integral values and connect the value of a general x_h with those already given ($h = 1, \ldots, 2r$) by the formula

$$x_{2r+h} = -x_h. \tag{3}$$

Then, if we put

$$\psi = \log\left(1+\sum_1^{2r} x_s{}^2\right), \tag{4}$$

we find that

$$g_{ij} = \frac{\partial^2\psi}{\partial x_i\partial x_j} + \frac{\partial^2\psi}{\partial x_{r+i}\partial x_{r+j}} = \psi_{ij}+\psi_{m+i\,m+j}. \tag{5}$$

The function ψ is not a unique function of the points of R, but depends on the choice of parameters. Nevertheless, we can use (4) to define a function of the points of R, and (5) will define the metrical tensor g_{ij}, provided that ψ is finite at the point considered. For consider a projective transformation of S_r, or, more generally, consider any birational transformation

$$x_h+ix_{m+h} = f_h(y_1+iy_{r+1}, \ldots, y_r+iy_{2r}).$$

If the y's are used as parameters on S_r, the metric is given by

$$g'_{ij}\,dy^i\,dy^j,$$

where

$$g'_{ij} = g_{hk}\frac{\partial x_h}{\partial y_i}\frac{\partial x_k}{\partial y_j} = \psi_{hk}\frac{\partial x_h}{\partial y_i}\frac{\partial x_k}{\partial y_j}+\psi_{r+h\,r+k}\frac{\partial x_{r+h}}{\partial y_{r+i}}\frac{\partial x_{r+k}}{\partial y_{r+j}},$$

since, by the Cauchy-Riemann equations and (3),

$$\frac{\partial x_h}{\partial y_k} = \frac{\partial x_{r+h}}{\partial y_{r+k}}.$$

† We use the notation and conventions of tensor calculus, and since the differentials dx form a contravariant vector we write the index at the top, although we write the coordinates themselves with the indices at the bottom, to avoid confusion with powers. In quoting results from the "Theory of quadratic differential forms", we use the notation of O. Veblen's *Cambridge Mathematical Tract*, No. 24, of this title.

Further, since x_h is a harmonic function of y_k, y_{r+k}, we deduce that

$$g'_{ij} = \frac{\partial^2 \psi}{\partial y_i \partial y_j} + \frac{\partial^2 \psi}{\partial y_{r+i} \partial y_{r+j}}.$$

In particular, let $$y_k + i y_{r+h} = \frac{z_{h+1}}{z_1}.$$

Then, starting *ab initio*, we should have obtained

$$g'_{ij} = \frac{\partial^2 \psi'}{\partial y_i \partial y_j} + \frac{\partial^2 \psi'}{\partial y_{r+i} \partial y_{r+j}}; \qquad (5)'$$

hence we see that the function $\phi = \psi - \psi'$ satisfies the equation

$$\frac{\partial^2 \phi}{\partial y_i \partial y_j} + \frac{\partial^2 \phi}{\partial y_{r+i} \partial y_{r+j}} = 0,$$

that is, ϕ is an r-ply harmonic function of y_1, y_{r+1}; y_2, y_{r+2}, ... In the particular example given, it is easily verified that

$$\phi = \log(x_1^2 + x_{r+1}^2).$$

The function ψ' will therefore serve in place of ψ on $z_{r+1} = 0$, *i.e.* when ψ becomes infinite. Thus we readily come to the conclusion that the metric on R is defined by equation (4) and (5)′ whatever parameters are used in (5)′, it being understood that when ψ becomes infinite it is modified by the subtraction of an r-ply harmonic function, such as ϕ, so that the resulting function is finite.

In this it is to be remembered that we are dealing with a unique representation of the Riemannian of S_r. If we had begun with a different set of homogeneous coordinates we should have obtained, in general, a different Riemannian R', which, though homeomorphic with R, is not congruent to it in the sense of Euclidean geometry, and so we should have obtained a different metric. The coordinate systems which give the same metric are shown by Mannoury† to be obtained from the system $(z_1, \ldots, z_{r+1})$ by the transformations of the type

$$Z_i = \sum_j L_{ij} z_j,$$

where the matrix L satisfies the equation

$$L \bar{L}' = 1$$

† *Loc. cit.*

(a' being the transpose of the matrix a). Such a transformation is called by Mannoury a conjugate-orthogonal transformation. It is only from such systems of coordinates that we can define ψ as

$$\psi = \log\left(\sum_1^{r+1} Z_i \bar{Z}_i\right) - \log Z_k \bar{Z}_k.$$

Finally, since R is an orientable manifold, we must, in order to study its topology, fix the orientation. For this purpose the coordinate system $(x_1, x_2, \ldots, x_{2r})$ will serve as indicatrix, and it is easily proved that its orientation is independent of the choice of non-homogeneous coordinates in S_r.

2. We now consider any algebraic variety V of m dimensions. It has been proved that, for $m = 1, 2$, V can be transformed into a variety in a space of sufficiently high number of dimensions which has no singularity. The general theorem has not yet been proved, so we must limit our consideration to those varieties which can be transformed in this way. (This may, and it is to be hoped will, prove to be no limitation.) We shall therefore suppose that V is given as a variety of m dimensions without singularities, lying in a projective space S_r of r $(> m)$ dimensions. We construct the Riemannian manifold R of S_r as in No. 1. In this there is a sub-locus M, corresponding to the points of V, which is the Riemannian of V. Van der Waerden† has proved that M, like R, is an orientable manifold. Now let $(u_1+iu_{m+1}, \ldots, u_m+iu_{2m})$ be complex parameters on V valid in the neighbourhood of the points of V which we wish to discuss. Just as in § 1, we allow the suffixes of the u's to take any integral values, and write

$$u_{2m+h} = -u_h. \qquad (3)'$$

We can now express the x's as functions of the u's on M. We have now, from (3), (3)', and the Cauchy-Riemann equations,

$$\frac{\partial x_h}{\partial u_k} = \frac{\partial x_{r+h}}{\partial x_{m+k}}.$$

Hence we can repeat the analysis of § 1, and we find that the metric on M is given by

$$g_{ij}\, du^i\, du^j,$$

where $$g_{ij} = \frac{\partial^2 \psi}{\partial u_i \partial u_j} + \frac{\partial^2 \psi}{\partial u_{m+1} \partial u_{m+j}} = \psi_{ij} + \psi_{m+i\, m+j}.$$

† *Loc. cit.*

In future we shall deal exclusively with M, so we shall replace the u's by x's, to conform with the usual notation. Moreover, we see that, just as in § 1, the complex parameters lead to a definite orientation of M.

In the remainder of this paper, the only changes of parameters on M considered result from a change of the complex parameters on V, and hence satisfy Cauchy-Riemann conditions.

3. We must now study the properties of the quadratic differential form

$$g_{ij}\,dx^i\,dx^j,$$

where
$$g_{ij} = \psi_{ij} + \psi_{m+i\,m+j}. \tag{5}$$

We can allow the indices to take any integral values, by replacing $2m+a$ by a wherever it appears, and changing the sign of the term in question. All the results to be proved in this number are then valid for any integral values of the indices.

Write
$$h_{ij} = g_{im+j}.$$

For a change of parameters,

$$h_{ij}\frac{\partial x_i}{\partial y_h}\frac{\partial x_j}{\partial y_k} = g_{im+j}\frac{\partial x_i}{\partial y_h}\frac{\partial x_{m+j}}{\partial y_{m+k}} = g'_{hm+k} = h'_{hk};$$

hence h_{ij} is a covariant tensor. Similarly $h^{ij} = g^{m+ij}$ is a contravariant tensor such that

$$h_{ij}h^{jk} = \delta_i{}^k.$$

Now
$$h_{ij} = \psi_{im+j} - \psi_{jm+i} = -h_{ji},$$

so h_{ij} is anti-symmetric. The matrices $\mathbf{g} = (g_{ij})$, $\mathbf{h} = (h_{ij})$ are connected by the equation

$$\mathbf{g}\begin{pmatrix} \cdot & -1 \\ 1 & \cdot \end{pmatrix} = \mathbf{h};$$

hence
$$|g_{ij}| = |h_{ij}|.$$

Since h_{ij} is anti-symmetric, $|h_{ij}|$ is the square of a polynomial in the elements of the determinant, hence $g = |g_{ij}|$ is the square of a polynomial in the functions g_{ij}. Thus the function $\sqrt{g}$, which will appear

in subsequent formulae, is a polynomial in the g's. Moreover, (g_{ij}) and (g^{ij}) are both matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

where a is a symmetric matrix of order m and b a skew matrix of order m. We shall often use g_{im+j} in place of h_{ij} in the sequel, and it is to be remembered that it is then the (i, j) component of a covariant tensor and not the $(i, m+j)$ component. No confusion will result. A similar remark applies to $g^{im+j} = -h^{ij}$.

Now consider the Christoffel symbols of the first kind

$$[k, ij] = \tfrac{1}{2}[\psi_{ijk} + \psi_{im+jm+k} + \psi_{m+ijm+k} - \psi_{m+im+jk}].$$

Then $$[k, m+ij] = [k, im+j] = -[m+k, ij].$$

For the Christoffel symbols of the second kind we have

$$\Gamma^k_{ij} = g^{ka}[a, ij];$$

hence
$$\begin{aligned} \Gamma^{m+k}_{ij} &= g^{m+ka}[a, ij] = -g^{km+a}[a, ij] \\ &= g^{ka}[m+a, ij] = -g^{ka}[a, m+ij] \\ &= -\Gamma^k_{m+ij} = -\Gamma^k_{im+j}. \end{aligned} \tag{6}$$

Similar relations hold when we consider the curvature tensor†. We have

$$\begin{aligned} B^i_{aj\,m+k} &= \frac{\partial \Gamma^i_{aj}}{\partial x_{m+k}} - \frac{\partial \Gamma^i_{a\,m+k}}{\partial x_j} + \Gamma^b_{aj}\,\Gamma^i_{b\,m+k} - \Gamma^b_{a\,m+k}\,\Gamma^i_{bj} \\ &= \frac{\partial \Gamma^i_{aj}}{\partial x_{m+k}} - \frac{\partial \Gamma^i_{m+a\,k}}{\partial x_j} + \Gamma^b_{m+a\,j}\,\Gamma^i_{bk} - \Gamma^b_{m+a\,k}\,\Gamma^i_{bj} \\ &= B^i_{m+ajk} + \frac{\partial \Gamma^i_{aj}}{\partial x_{m+k}} - \frac{\partial \Gamma^i_{m+aj}}{\partial x_k}. \end{aligned}$$

Similarly,

$$B^i_{a\,m+jk} = B^i_{m+ajk} - \frac{\partial \Gamma^i_{ak}}{\partial x_{m+j}} + \frac{\partial \Gamma^i_{m+ak}}{\partial x_j}.$$

† Veblen, p. 41.

Now

$$\frac{\partial \Gamma^i_{a\,j}}{\partial x_{m+k}} - \frac{\partial \Gamma^i_{a\,k}}{\partial x_{m+j}}$$

$$= \frac{\partial}{\partial x_{m+k}} \{g^{ib}[b, a\,j]\} - \frac{\partial}{\partial x_{m+j}} \{g^{ib}[b, a\,k]\}$$

$$= -\Gamma^a_{a\,j}\,\Gamma^i_{a\,m+k} - g^{i\alpha}\,\Gamma^b_{\alpha\,m+k}[b, a\,j] + g^{ib}[b, a\,j]_{m+k}$$

$$+ \Gamma^a_{a\,k}\,\Gamma^i_{a\,m+j} + g^{i\alpha}\,\Gamma^b_{\alpha\,m+j}[b, a\,k] - g^{ib}[b, a\,k]_{m+j}$$

$$= -\Gamma^a_{m+a\,j}\,\Gamma^i_{a\,k} + \Gamma^a_{m+a\,k}\,\Gamma^i_{a\,j} + g^{i\alpha}\Gamma^b_{a\,k}[b, m+a\,j] - g^{i\alpha}\,\Gamma^b_{a\,j}[b, m+a\,k]$$

$$+ g^{ib}\{[b, a\,j]_{m+k} - [b, a\,k]_{m+j}\}.$$

Using the formula for the first Christoffel symbols, we verify that

$$[b, a\,j]_{m+k} - [b, a\,k]_{m+j} = [b, m+a\,k]_j - [b, m+a\,j]_k.$$

Therefore

$$\frac{\partial \Gamma^i_{a\,j}}{\partial x_{m+k}} - \frac{\partial \Gamma^i_{a\,k}}{\partial x_{m+j}} = \frac{\partial \Gamma^i_{m+a\,k}}{\partial x_j} - \frac{\partial \Gamma^i_{m+a\,j}}{\partial x_k} + 2\Gamma^a_{m+a\,k}\,\Gamma^i_{a\,j} - 2\Gamma^a_{m+a\,j}\,\Gamma^i_{a\,k}.$$

Therefore

$$B^i_{a\,m+j\,k} + B^i_{a\,j\,m+k} = 2B^i_{m+a\,j\,k} - 2\left\{\frac{\partial \Gamma^i_{m+a\,j}}{\partial x_k} - \frac{\partial \Gamma^i_{m+a\,k}}{\partial x_j} + \Gamma^a_{m+a\,j}\,\Gamma^i_{a\,k} - \Gamma^a_{m+a\,k}\,\Gamma^i_{a\,j}\right\}$$

$$= 0,$$

from the formula for $B^i_{m+a\,j\,k}$.

Moreover, from

$$B^i_{a\,m+j\,k} = -B^i_{a\,j\,m+k}, \tag{7}$$

we have

$$R_{i\,j\,m+k\,l} = g_{i\alpha}\,B^\alpha_{j\,m+k\,l} = -g_{i\alpha}\,B^\alpha_{j\,k\,m+l} = -R_{ij\,k\,m+l}.$$

Also

$$R_{m+i\,j\,k\,l} = R_{lk\,j\,m+i} = -R_{lk\,m+j\,i} = -R_{i\,m+j\,kl}.$$

from which we deduce immediately

$$B^i_{m+a\,j\,k} = -B^{m+i}_{a\,j\,k}. \tag{8}$$

4. Normal coordinate systems are defined by Veblen†. While it is not easy to show that it is possible to make a change of the complex parameters on V so that the resulting real parameters on M are normal, if indeed this is the case, it is possible to go part of the way and obtain a system which will serve our purpose just as well. The coordinates which we obtain satisfy the first conditions for normal coordinates, viz. at the origin,

$$(a) \qquad g_{ij} = \delta_{ij},$$

$$(b) \qquad \Gamma^i_{jk} = 0 \quad (\text{all } i, j, k).$$

The effect of using these coordinates is to reduce covariant derivatives at the origin to ordinary derivatives, thus simplifying the proofs of some identities. The name "semi-normal" suggests itself for these coordinates, but since they are the only coordinates of this sort which we shall use, no confusion will arise if for brevity we merely use the adjective "normal."

Let P on V have coordinates $(a_1, a_2, \ldots, a_{r+1})$ in the system $(z_1, z_2, \ldots, z_{r+1})$, where

$$|a_1|^2 + \ldots + |a_{r+1}|^2 = 1.$$

Let the tangent m-fold to V at P be

$$\sum_j \beta_{ij} z_j = 0 \quad (i = 1, \ldots, r-m).$$

We replace these equations by an equivalent set.

(1) Let $\gamma_{1j} = \lambda_1 \beta_{1j}$,

where $\lambda_1 \bar{\lambda}_1 \sum_j \beta_{1j} \bar{\beta}_{1j} = 1$, *i.e.* $\sum_j \gamma_{1j} \bar{\gamma}_{1j} = 1$.

(2) Let $\beta'_{2j} = \beta_{2j} + \mu \gamma_{1j}$,

where $\sum_j \beta_{2j} \bar{\gamma}_{1j} + \mu = 0$, *i.e.* $\sum_j \beta'_{2j} \bar{\gamma}_{1j} = 0$,

and then put $\gamma_{2j} = \lambda_2 \beta'_{2j}$,

where $\sum_j \gamma_{2j} \bar{\gamma}_{2j} = \lambda_2 \bar{\lambda}_2 \sum_j \beta'_{2j} \bar{\beta}'_{2j} = 1$.

† *Loc. cit.*, chap. 6.

Proceeding in this way we can take the equations of the tangent m-fold as

$$\sum_j \gamma_{ij} z_j = 0 \quad (i = 1, \ldots, r-m),$$

where

$$\sum_k \gamma_{ik} \bar{\gamma}_{jk} = \delta_{ij}.$$

It is then possible to add other m columns to the matrix

$$\begin{pmatrix} a_1 & \bar{\gamma}_{11} & \cdots & \bar{\gamma}_{(r-m)1} \\ a_2 & \bar{\gamma}_{12} & \cdots & \bar{\gamma}_{(r-m)2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{r+1} & \bar{\gamma}_{1r+1} & \cdots & \bar{\gamma}_{(r-m)r+1} \end{pmatrix}$$

so that the resulting matrix

$$(a_j, \bar{\gamma}_{1j} \cdots \bar{\gamma}_{(r-m)j} \epsilon_{1j} \cdots \epsilon_{mj})$$

is a conjugate-orthogonal matrix L. Then

$$z = \sum_j L_{ij} Z_j$$

is a conjugate-orthogonal transformation such that in the Z-coordinates P is $(1, 0, \ldots, 0)$ and the tangent m-fold at P is given by

$$Z_2 = 0, \ \ldots, \ Z_{r-m+1} = 0.$$

Take, as local parameters on M,

$$x_j + i x_{m+j} = 2 \frac{Z_{r-m+1+j}}{Z_1} \quad (j = 1 \ldots m).$$

Then, near P,

$$\psi = \log\left(1 + \tfrac{1}{4} \sum_1^{2m} x_s^2 + \text{terms of fourth and higher degree}\right).$$

Therefore, at the origin,

$$g_{ij} = \delta_{ij},$$

$$\Gamma_{ij}^k = 0 \quad (\text{all } i, j, k),$$

so that the coordinate system is normal in the sense defined.

5. The remainder of this paper will be devoted to the study of certain tensors on M, and the following lemma will be useful. *Let* $P_{i_1 i_2 \ldots i_p}$ *be any*

tensor, and let $Q_{i_1 i_2 \ldots i_p}$ *be an array of* m^p *quantities connected with the components of* $P_{i_1 i_2 \ldots i_p}$ *by changing one suffix, say* i_1, *into* $m+i_1$. Thus

$$Q_{i_1 i_2 \ldots i_p} = P_{m+i_1 i_2 \ldots i_p}, \quad Q_{m+i_1 i_2 \ldots i_p} = -P_{i_1 i_2 \ldots i_p}.$$

Then $Q_{i_1 i_2 \ldots i_p}$ *is a tensor.* For

$$Q_{i_1 i_2 \ldots i_p} \frac{\partial x_{i_1}}{\partial y_{j_1}} \frac{\partial x_{i_2}}{\partial y_{j_2}} \cdots \frac{\partial x_{i_p}}{\partial y_{j_p}} = P_{m+i_1 i_2 \ldots i_p} \frac{\partial x_{m+i_1}}{\partial y_{m+j_1}} \frac{\partial x_{i_2}}{\partial y_{j_2}} \cdots \frac{\partial x_{i_p}}{\partial y_{j_p}} = P'_{m+j_1 \ldots j_p},$$

and hence $Q_{i_1 i_2 \ldots i_p}$ transforms according to the tensor law. Further, for covariant differentiation we have the formula

$$Q_{i_1 i_2 \ldots i_p,\ k} = P_{m+i_1 i_2 \ldots i_p,\ k}.$$

For

$$\begin{aligned} Q_{i_1 i_2 \ldots i_p,\ k} &= \frac{\partial}{\partial x_k} Q_{i_1 i_2 \ldots i_p} - Q_{a i_2 \ldots i_p} \Gamma^a_{i_1 k} - \sum_2^p Q_{i_1 i_s \ldots i_{s-1} a i_{s+1} \ldots i_p} \Gamma^a_{i_s k} \\ &= \frac{\partial}{\partial x_k} P_{m+i_1 i_2 \ldots i_p} - P_{m+a i_2 \ldots i_p} \Gamma^a_{i_1 k} - \sum_2^p P_{m+i_1 i_2 \ldots i_{s-1} a i_{s+1} \ldots i_p} \Gamma^a_{i_s k} \\ &= P_{m+i_1 i_2 \ldots i_p,\ k} + P_{a i_2 \ldots i_p} \Gamma^a_{m+i_1 k} - P_{m+a i_2 \ldots i_p} \Gamma^a_{i_1 k}. \end{aligned}$$

But $$P_{m+a i_2 \ldots i_p} \Gamma^a_{i_1 k} = -P_{a i_2 \ldots i_p} \Gamma^{m+a}_{i_1 k} = P_{a i_2 \ldots i_p} \Gamma^a_{m+i_1 k},$$

and hence the result follows. It is readily seen that there is no need to restrict ourselves to a change in a single suffix, or to one covariant differentiation, and that a similar theorem holds for contravariant indices.

6. A tensor $P_{i_1 \ldots i_p}$ is said to be harmonic on M if it, and its covariant derivatives of any order, exist everywhere on M, and if it is anti-symmetric and, further, satisfies the equations

$$\sum_{r=1}^{p+1} (-1)^{r-1} P_{i_1 \ldots i_{r-1} i_{r+1} \ldots i_{p+1},\ i_r} = 0, \tag{9}$$

$$g^{rs} P_{i_1 \ldots i_{p-1} r,\ s} = 0. \tag{10}$$

Equation (9) expresses the condition that the integral

$$\phi = \frac{1}{p!} \int_{(p)} P_{i_1 \ldots i_p} da^{i_1} \ldots dx^{i_p}$$

is the integral of a total differential, and (10) is equivalent to the following condition. Let

$$P^*_{i_{p+1} \ldots i_{2m}} = \sqrt{g}\, g^{i_1 j_1} g^{i_2 j_2} \ldots g^{i_p j_p} P_{j_1 \ldots j_p}.$$

It is known that $P^*_{i_1 \dots i_{2m-p}}$ is an anti-symmetric covariant tensor. Equation (10) states that

$$\phi^* = \frac{1}{(2m-p)!} \int_{(2m-p)} P^*_{i_1 \dots i_{2m-p}} dx^{i_1} \dots dx^{i_{2m-p}}$$

is also the integral of a total differential, and the two integrability conditions are equivalent to (9) and (10). Now it is easily verified that

$$(\phi^*)^* = (-1)^p \phi,$$

hence $P^*_{i_1 \dots i_{2m-p}}$ is also a harmonic tensor. ϕ and ϕ^* are then said to be associated harmonic integrals.

Properties of harmonic integrals have been discussed by me in two papers†. The principal results which we shall require are that, if R_p is the p-th Betti number of the manifold M, there are exactly R_p linearly independent harmonic integrals attached to M, and that no combination of them can be without periods on all the R_p p-cycles of M. With the particular metric on M already examined there are, however, many properties of the integrals, and these we proceed to obtain.

7. THEOREM I. *If*

$$\frac{1}{p!} \int A_{i_1 \dots i_p} dz^{i_1} \dots dz^{i_p}$$

is an Abelian integral of the first kind attached to V, the real and imaginary parts of this integral are harmonic integrals.

Here we have written

$$z_h = x_h + i x_{m+h}.$$

We need only consider the real part. Since

$$\sum_1^{p+1} (-1)^{r-1} \frac{\partial A_{i_1 \dots i_{r-1} i_{r+1} \dots i_{p+1}}}{\partial z_{i_r}} = 0,$$

the real part, which we shall write as

$$\frac{1}{p!} \int P_{i_1 \dots i_p} dx^{i_1} \dots dx^{i_p},$$

† W. V. D. Hodge, *Proc. London Math. Soc.* (2), 36 (1933), 257, and (2), 38 (1934), 72.

is the integral of a total differential. Hence all the conditions are satisfied, except (10). Now the P's are connected by the relations

$$P_{i_1 \dots i_{p-2}\, j\, m+k} = P_{i_1 \dots i_{p-2}\, m+j\, k}, \tag{11}$$

$$-\frac{\partial P_{i_1 \dots i_{p-1}\, j}}{\partial x_k} = \frac{\partial P_{i_1 \dots i_{p-1}\, m+j}}{\partial x_{m+k}}, \tag{12}$$

and relations which can be deduced from these by using condition (3)′. The theorem is most easily proved by using normal coordinates. The condition (10) then becomes

$$\sum_k \frac{\partial P_{i_1 \dots i_{p-1}\, k}}{\partial x_k} = 0.$$

From (11) we observe that in the summation we have zero terms whenever

$$k \equiv i_s (\text{mod } m) \quad \text{or} \quad i_r \equiv i_s (\text{mod } m).$$

The remaining terms cancel out in virtue of (12) and the theorem is proved.

THEOREM II. *If $P_{i_1 \dots i_p}$ is a harmonic tensor of rank p, then*

$$g^{r\, m+s} P_{r\, s\, i_1 \dots i_{p-2}} = Q_{i_1 \dots i_{p-2}}$$

is a harmonic tensor of rank $p-2$.

Let

$$L_{i_1 \dots i_{p-1}} = \sum_1^{p-1} (-1)^{r-1} Q_{i_1 \dots i_{r-1}\, i_{r+1} \dots i_{p-1},\, i_r} = 2 g^{r\, m+s} P_{r\, i_1 \dots i_{p-2},\, s} \tag{13}$$

by the lemma of No. 5. Then

$$\sum_1^p (-1)^{r-1} L_{i_1 \dots i_{r-1}\, i_{r+1} \dots i_p,\, i_r} = 0.$$

Also

$$\begin{aligned} g^{ij} L_{i_1 \dots i_{p-2}\, i,\, j} &= 2 g^{ij} g^{r\, m+s} P_{r\, i_1 \dots i_{p-2}\, i,\, s,\, j} \\ &= -2 g^{ij} g^{r\, m+s} [P_{a\, i_1 \dots i_{p-2}\, i} B^a_{r\, s\, j} + P_{r\, i_1 \dots i_{p-2}\, a} B^a_{i\, s\, j}] \\ &\quad - 2 g^{ij} g^{r\, m+s} \sum_{t=1}^{p-2} P_{r\, i_1 \dots i_{t-1}\, a\, i_{t+1} \dots i_{p-2}\, i} B^a_{i_t\, s\, j}. \end{aligned}$$

This is proved by changing the order of differentiation and using the fact that $P_{i_1 \dots i_p}$ is harmonic.

Now

$$g^{ij}\,g^{r\,m+s}\,P_{\alpha\,i_1\ldots i_{p-2}\,i}\,B^{\alpha}_{r\,s\,j} = g^{rs}\,g^{i\,m+j}\,P_{\alpha\,i_1\ldots i_{p-2}\,r}\,B^{\alpha}_{ij\,s}$$

$$= -g^{rs}\,g^{ij}\,P_{\alpha\,i_1\ldots i_{p-2}\,r}\,B^{\alpha}_{i\,m+j\,s}$$

$$= -g^{rs}\,g^{ij}\,P_{\alpha\,i_1\ldots i_{p-2}\,r}\,B^{\alpha}_{i\,m+s\,j}$$

$$= -g^{r\,m+s}\,g^{ij}\,P_{r\,i_1\ldots i_{p-2}\,\alpha}\,B^{\alpha}_{i\,s\,j}.$$

Also $$g^{rj}\,g^{is}\,B^{\alpha}_{\beta\,m+s\,j} = g^{rj}\,g^{is}\,B^{\alpha}_{\beta\,m+j\,s} = g^{ij}\,g^{rs}\,B^{\alpha}_{\beta\,m+s\,j}.$$

Hence $$g^{ij}\,g^{r\,m+s}\,B^{\alpha}_{\beta\,s\,j}$$

is symmetrical in r and i. Since $P_{rj_1\ldots j_{p-2}\,i}$ is anti-symmetric, we conclude that

$$g^{ij}\,L_{i_1\ldots i_{p-2}\,i,\;j} = 0,$$

or $L_{i_1\ldots i_{p-1}}$ is a harmonic tensor. But, in virtue of (13),

$$\frac{1}{(p-1)\,!}\int L_{i_1\ldots i_{p+1}}\,dx^{i_1}\ldots dx^{i_{p-1}}$$

has zero periods on all cycles†, and hence, by a theorem on harmonic integrals quoted above,

$$L_{i_1\ldots i_{p-1}} = 0.$$

Thus $$\sum_{1}^{p-1}(-1)^{r-1}\,Q_{i_1\ldots i_{r-1}\,i_{r+1}\ldots i_{p+1},\;i_r} = 0.$$

Also $$g^{ij}\,Q_{i_1\ldots i_{p-3}\,i,\;j} = g^{ij}\,g^{r\,m+s}\,P_{r\,s\,i_1\ldots i_{p-3}\,i,\,j} = 0.$$

Hence $g^{r\,m+s}\,P_{r\,s\,i_1\ldots i_{p-2}}$ is a harmonic tensor. Incidentally we have proved

THEOREM III. *If $P_{i_1\ldots i_p}$ is harmonic, then*

$$g^{r\,m+s}\,P_{i_1\ldots i_{p-1}\,r,\,s} = 0.$$

THEOREM IV. *If $P_{i_1\ldots i_p}$ is harmonic, so is*

$$Q_{i_1\ldots i_p} = P_{m+i_1\ldots m+i_p}.$$

Let $$\sum_{1}^{p+1}(-1)^{r-1}\,Q_{i_1\ldots i_{r-1}\,i_{r+1}\ldots i_{p+1},\;i_r} = L_{i_1\ldots i_{p+1}}.$$

† G. de Rham, *Journal de Math.* (9), 10 (1931), 176.

Then $$\sum_{1}^{p+2}(-1)^{r-1}L_{i_1\dots i_{r-1}i_{r+1}\dots i_{p+2},\,i_r}=0.$$

Also

$$g^{ij}L_{i_1\dots i_p i,\,j}=(-1)^p g^{ij}Q_{i_1\dots i_p,\,i,\,j}+g^{ij}\sum_{1}^{p}(-1)^{r-1}Q_{i_1\dots i_{r-1}i_{r+1}\dots i_p i,\,i_r,\,j}$$

$$=(-1)^p\sum_{1}^{p}g^{ij}P_{m+i_1\dots m+i_{r-1}\,i\,m+i_{r+1}\dots m+i_p,\;m+i_r,\;j}$$

$$+(-1)^p\sum_{1}^{p}g^{ij}P_{m+i_1\dots m+i_{r-1}\,i\,m+i_{r+1}\dots m+i_p,\;i_r,\;m+j}$$

$$=(-1)^p\sum_{1}^{p}g^{ij}P_{m+i_1\dots m+i_{r-1}\,i\,m+i_{r+1}\dots m+i_p,\;m+j,\;i_r}.$$

This last step is achieved by interchanging the order of differentiation in each term, using (7) and the fact that $P_{i_1\dots i_p}$ is harmonic. Hence

$$g^{ij}L_{i_1\dots i_p i,\,j}=(-1)^{p-1}\sum_{1}^{p}g^{i\,m+j}P_{m+i_1\dots m+i_{r-1}\,i\,m+i_{r+1}\dots m+i_p,\;j,\;i_r}=0,$$

by Theorem III.

We now argue as in Theorem II and see that

$$\sum_{1}^{p+1}(-1)^{r-1}Q_{i_1\dots i_{r-1}i_{r+1}\dots i_{p+1},\,i_r}=0.$$

Also $$g^{ij}Q_{i_1\dots i_{p-1}i,\,j}=-g^{i\,m+j}P_{m+i_1\dots m+i_{p-1}i,\,j}=0,$$

and the theorem follows.

THEOREM V is the converse of Theorems II and IV. *Let $P_{i_1\dots i_p}$ be an anti-symmetric tensor, and let*

$$Q_{i_1\dots i_p}=P_{m+i_1\dots m+i_p},$$

$$L_{i_1\dots i_{p-2}}=g^{r\,m+s}P_{r\,s\,i_1\dots i_{p-2}}.$$

Then, if

$$\frac{1}{p!}\int P_{i_1\dots i_p}dx^{i_1}\dots dx^{i_p},$$

$$\frac{1}{p!}\int Q_{i_1\dots i_p}dx^{i_1}\dots dx^{i_p},$$

$$\frac{1}{(p-2)!}\int L_{i_1\dots i_{p-2}}dx^{i_1}\dots dx^{i_{p-2}}$$

are the integrals of total differentials, $P_{i_1 \dots i_p}$, $Q_{i_1 \dots i_p}$, $L_{i_1 \dots i_{p-2}}$ *are harmonic tensors.*

If we prove the result for $Q_{i_1 \dots i_p}$, it will follow for the others by Theorems II and IV. It is only necessary to prove that

$$g^{rs} Q_{i_1 \dots i_{p-1} r,\ s} = 0.$$

Now

$$\begin{aligned} & g^{rs} Q_{i_1 \dots i_{p-1} r,\ s} \\ & = g^{rs} P_{m+i_1 \dots m+i_{p-1} m+r,\ s} \\ & = g^{rs} P_{m+i_1 \dots m+i_{p-1} s,\ m+r} + \sum_1^{p-1} (-1)^{j-1} g^{rs} P_{s\, m+r\, m+i_1 \dots n+i_{j-1}\, m+i_{j+1} \dots m+i_{p-1},\ m+i_j} \\ & = -\tfrac{1}{2} \sum_1^{p-1} (-1)^{j-1} L_{m+i_1 \dots m+i_{j-1}\, m+i_{j+1} \dots m+i_{p-1},\ m+i_j} \\ & = 0. \end{aligned}$$

The conditions of this theorem are necessary and sufficient in order that $P_{i_1 \dots i_p}$ should be harmonic. In general, however, they are not independent. In the case $p = 1$ we merely have to omit the third condition.

8. There is a particular tensor which is harmonic on M, and which will play an important part in the sequel. It is the tensor $h_{ij} = g_{im+j}$. For, from the lemma of § 5,

$$h_{ij,\,k} + k_{jk,\,i} + h_{ki,\,j} = 0,$$

$$g^{ij} h_{ki,\,j} = 0.$$

We observe that the associated harmonic integral

$$\iint \omega = \tfrac{1}{2} \iint g_{im+j}\, dx^i\, dx^j$$

is really defined in the whole space R, and not merely on M, by using r complex coordinates in the projective space S_r, and then defining the integral by the above formula in the derived real parameters in R. If we then limit ourselves to fields of integration lying on M, the integral reduces to the integral

$$\tfrac{1}{2} \iint h_{ij}\, dx^i\, dx^j$$

defined on M.

There is only one 2-cycle Γ on R, corresponding to the straight line in S_r. Hence the integral which we have found is the only one which exists.

It is easy to find its period on Γ, by integrating directly over the cycle $z_2 = 0, \ldots, z_r = 0$; we have

$$\iint_\Gamma \omega = 4 \iint_\Gamma \frac{dx^1 dx^{r+1}}{(1+x_1{}^2+x_{r+1})^2} = 4\pi.$$

The product of two differential forms

$$P_{i_1 \ldots i_p} dx^{i_1} \ldots dx^{i_q}, \quad Q_{i_1 \ldots i_q} dx^{i_1} \ldots dx^{i_q},$$

taken in a given order, is found by multiplying the forms together in the given order, taking into account the order of the differentials, and omitting terms in which the same differential appears twice. Elementary properties of the products of forms, such as are used here, are given in my paper of 1933†. It can be verified easily that the forms

$$\omega_1 = \omega, \quad \omega_2 = \frac{\omega \times \omega}{2}, \quad \ldots, \quad \omega_q = \frac{\omega_{q-1} \times \omega}{q}, \quad \ldots, \quad \omega_m = \sqrt{g}\, dx^1 \ldots dx^n$$

are all harmonic. The proofs of the following properties of these particular forms come immediately by direct calculation.

(1) $$\omega_q = \frac{1}{(2q)!} N^{(q)}_{i_1 \ldots i_{2q}} dx^{i_1} \ldots dx^{i_q},$$

where $N^{(q)}_{i_1 \ldots i_q}$ is the Pfaffian of the skew determinant

$$\begin{vmatrix} g_{i_1\, m+i_1} \cdots g_{i_1\, m+i_p} \\ g_{i_p\, m+i_1} \cdots g_{i_p\, m+i_p} \end{vmatrix}.$$

(2) The $2(m-q)$-ple integral associated with $\int_{(2q)} \omega_q$ is

$$(-1)^{\frac{1}{2}m(m-1)} \int_{2(m-q)} \omega_{m-q}.$$

We now prove that, if

$$\phi = \int_{(p)} d\phi = \frac{1}{p!} \int_{(p)} P_{i_1 \ldots i_p} dx^{i_1} \ldots dx^{i_p}$$

is harmonic, so is

$$\int_{(p+2)} d\phi \times \omega = \frac{1}{(p+2)!} \int_{(p+2)} Q_{i_1 \ldots i_{p+2}} dx^{i_1} \ldots dx^{i_{p+2}}.$$

† *Loc. cit.*

It is known from the theory of products of forms that this is the integral of a total differential. It therefore only remains to show that

$$g^{rs} Q_{i_1 \dots i_{p+1} r,\, s} = 0.$$

Now $$Q_{i_1 \dots i_{p+2}} = \tfrac{1}{2} \Sigma \pm g_{i_1 m+i_2} P_{i_3 \dots i_{p+2}}$$

(summed for all derangements of $i_1 \dots i_{p+2}$), therefore

$$g^{rs} Q_{i_1 \dots i_{p+1} r,\, s} = \tfrac{1}{2} g^{rs} \Sigma \pm g_{i_1 m+i_2} P_{i_3 \dots i_{p+1} r,\, s} + (-1)^p g^{rs} \Sigma g_{i_1 m+r} P_{i_2 \dots i_{p+1},\, s} = 0.$$

9. For many purposes it is convenient to make an analysis of the harmonic integrals of rank p into subsets which are more easily manipulated in calculations. We require the following lemma:

If $P_{i_1 \dots i_{2q+r-2}}$ *is a tensor such that*

$$g^{i_1 m+j_1} \dots g^{i_q m+j_q} P_{i_1 j_1 \dots i_q j_q k_1 \dots k_{r-2}} = 0,$$

and if $$Q_{i_1 \dots i_{2q+r}} = \tfrac{1}{2} \Sigma \pm g_{i_1 m+i_2} P_{i_3 \dots i_{2q+r}},$$

then

$$\begin{aligned} g^{i_1 m+j_1} \dots g^{i_q m+j_q} & Q_{i_1 j_1 \dots i_q j_q k_1 \dots k_r} \\ &= 2q(m-r-q+1)\, g^{i_1 m+j_1} \dots g^{i_{q-1} m+j_{q-1}} P_{i_1 j_1 \dots i_{q-1} j_{q-1} k_1 \dots k_r} \\ &= 2q(m-r-q+1)\, L_{k_1 \dots k_r}. \end{aligned}$$

The result is obtained by considering the various ways in which $k_1 \dots k_r$ can be distributed between the suffixes of $g_{i\, m+j}$ and $P_{i_1 \dots i_{2q+r-2}}$. Proceeding in this way, we see that

(1) from terms of the type

$$g^{i_1 m+j_1} \dots g^{i_q m+j_q} P_{i_1 j_1 \dots i_q j_q k_1 \dots k_{r-2}} g_{k_{r-1} m+k_r},$$

we get zero;

(2) from terms of the type

$$-g^{i_1 m+j_1} \dots g^{i_q m+j_q} P_{i_1 j_1 \dots i_s k_{r-1} i_{s+1} j_{s+1} i_q j_q k_1 \dots k_{r-2}} g_{j_s m+k_r},$$

we get $$-qr\, L_{k_1 \dots k_r};$$

(3) from terms of a similar form with i_s and j_s interchanged we again have

$$-qr\, L_{k_1 \dots k_r};$$

(4) from terms of the type

$$g^{i_1 m+j_1} \dots g^{i_q m+j_q} P_{i_1 j_1 \dots i_{q-1} j_{q-1} k_1 \dots k_r} g_{i_q j_q + m},$$

we obtain $2mq\, L_{k_1 \dots k_r}$;

and from terms of types (5), (6), (7), where $g_{i\, m+j}$ is equal to $g_{i_s\, m+j_t}$, $g_{i_s\, m+i_t}$, $g_{j_s\, m+j_t}$ $(s \neq t)$, we obtain respectively

$$-q(q-1)\, L_{k_1 \dots k_r}, \quad -\tfrac{1}{2}q(q-1)\, L_{k_1 \dots k_r}, \quad -\tfrac{1}{2}q(q-1)\, L_{k_1 \dots k_r}.$$

We thus account for all the terms. Adding up, we obtain the formula stated.

From this lemma we obtain the following result. *Let* $P_{i_1 \dots i_r}$ *be a tensor such that*

$$g^{i\, m+j} P_{ij\, i_1 \dots i_{r-2}} = 0,$$

and let

$$\frac{1}{p!} P_{i_1 \dots i_r} dx^{i_1} \dots dx^{i_r} \times \omega_q = \frac{1}{(p+2q)!} P^{(q)}_{i_1 \dots i_{r+2q}} dx^{i_1} \dots dx^{i_{r+2q}}.$$

Then

$$g^{i_1 m+j_1} \dots g^{i_q m+j_q} P^{(q)}_{i_1 j_1 \dots i_q j_q k_1 \dots k_r} = \frac{2^q (m-r)!}{(m-r-q)!} P_{k_1 \dots k_r}. \tag{15}$$

We first observe that

$$g^{i_1 m+j_1} \dots g^{i_q m+j_q} P^{(q-1)}_{i_1 j_1 \dots i_q j_q k_1 \dots k_{r-2}} = 0.$$

Therefore, by our lemma,

$$g^{i_1 m+j_1} \dots g^{i_q m+j_q} P^{q}_{i_1 j_1 \dots i_q j_q k_1 \dots k_r}$$

$$= 2(m-r-q+1)\, g^{i_1 m+j_1} \dots g^{i_{q-1} m+j_{q-1}} P^{(q-1)}_{i_1 j_1 \dots i_{q-1} j_{q-1} k_1 \dots k_r},$$

and the general formula follows by induction.

Now let $P_{i_1 \dots i_q}$ be any harmonic tensor and suppose p to be even and equal to $2q$. Let

$$\alpha = \frac{1}{p!} P_{i_1 \dots i_q} dx^{i_1} \dots dx^{i_p},$$

$$\alpha_1 = \frac{1}{p!} P^{(1)}_{i_1 \dots i_p} dx^{i_1} \dots dx^{i_1},$$

$$= \alpha - \frac{(m-q)!}{2^q m!} g^{i_1 m+j_1} \dots g^{i_q m+j_q} P_{i_1 j_1 \dots i_q j_q} \times \omega_q = \alpha - \beta_0 \omega_q.$$

By (15), $$g^{i_1 m+j_1} \dots g^{i_q m+j_q} P^{(1)}_{i_1 j_1 \dots i_q j_q} = 0.$$

Next write

$$\alpha_2 = \frac{1}{p!} P^{(2)}_{i_1 \dots i_p} dx^{i_1} \dots dx^{i_r}$$

$$= \alpha_1 - \frac{1}{2!} \frac{(m-q-1)!}{2^{q-1}(m-2)!} g^{i_1 m+j_1} \dots g^{i_{q-1} m+j_{q-1}} P^{(1)}_{i_1 j_1 \dots i_{q-1} j_{q-1} k_1 k_2} dx^{k_1} dx^{k_2} \omega_{q-1}$$

$$= \alpha_1 - \beta_1 \omega_{q-1}.$$

Again, by (15),

$$g^{i_1 m+j_1} \dots g^{i_{q-1} m+j_{q-1}} P^{(2)}_{i_1 j_1 \dots i_{q-1} j_{q-1} k_1 k_2} = 0.$$

Proceeding in this way, we get a series of equations

$$\alpha_1 = \alpha - \beta_0 \omega_q,$$

$$\alpha_2 = \alpha_1 - \beta_1 \omega_{q-1},$$

$$\dots \quad \dots \quad \dots \quad \dots,$$

$$\alpha_q = \alpha_{q-1} - \beta_{q-1} \omega,$$

$$0 = \alpha_q - \beta_q.$$

Hence $$\alpha = \sum_{k=0}^{q} \beta_k \omega_{q-k}.$$

Now $$\beta_k = \frac{1}{(2k)!} Q^{(k)}_{i_1 \dots i_{2k}} dx^{i_1} \dots dx^{i_{2k}},$$

where $$g^{i m+j} Q^{(k)}_{ij i_1 \dots i_{2k-2}} = 0.$$

Moreover,, if $P_{i_1 \dots i_p}$ is a harmonic tensor, it follows from Theorem II of No. 7 that $Q^k_{i_1 \dots i_{2k}}$ is a harmonic tensor (in particular β_0 is a constant), and therefore that

$$\beta_k \omega_{q-k}$$

is a harmonic form. A similar argument enables us to show that, if $p = 2q+1$, we can write any harmonic form

$$\frac{1}{p!} P_{i_1 \dots i_p} dx^{i_1} \dots dx^{i_p} = \sum_{0}^{q} \gamma_k \omega_{q-k},$$

where $$\gamma_k = \frac{1}{(k+1)!} R^{(k)}_{i_1 \dots i_{2q+1}} dx^{i_1} \dots dx^{i_{2q+1}}$$

is a harmonic form, and

$$g^{i\,m+j} R^{(k)}_{ij\,i_1 \dots i_{2q-1}} = 0.$$

10. Let us determine the form of the $(2m-p)$-ple integral associated with the harmonic integral

$$\frac{1}{p!} \int_{(p)} P_{i_1 \dots i_p} dx^{i_1} \dots dx^{i_p} = \frac{1}{(p-2q)!} \int_{(p)} Q_{i_1 \dots i_{p-2q}} dx^{i_1} \dots dx^{i_{p-2q}} \times \omega_q,$$

where $$g^{i\,m+j} Q_{ij\,k_1 \dots k_{p-2q-2}} = 0.$$

We only require to consider the case $p \leqslant m$, for the cases $p > m$ can be deduced by the duality properties of harmonic integrals.

$$\begin{aligned}
\sqrt{g}\, g^{i_1 j_1} \dots g^{i_p j_p} P_{j_1 \dots j_p} &= \sqrt{g}\, g^{i_1 j_1} \dots g^{i_p j_p} \Sigma\, Q_{j_1 \dots j_{p-2q}} N^{(q)}_{j_{p-2q+1} \dots j_p} \\
&= \Sigma\, g^{i_1 j_1} \dots g^{i_{p-2q} j_{p-2q}} Q_{j_1 \dots j_{p-2q}} \sqrt{g}\, g^{i_{p-2q+1} j_{p-2q+1}} \dots g^{i_p j_p} N^{(q)}_{j_{p-2q+1} \dots j_p} \\
&= (-1)^{\frac{1}{2}m(m-1)} \Sigma\, g^{i_1 j_1} \dots g^{i_{p-2q} j_{p-2q}} Q_{j_1 \dots j_{p-2q}} N^{(m-q)}_{i_1 \dots i_{p-2q}\, i_{p+1} \dots i_{2m}}.
\end{aligned}$$

Evaluating this expression directly, we find that the associated integral is

$$(-)^{\frac{1}{2}m(m-1)} \frac{1}{(p-2q)!} \int Q_{m+i_1 \dots m+i_{p-2q}} dx^{i_1} \dots dx^{i_{p-2q}}\, \omega_{m-p+q}.$$

Using the expansion for a harmonic integral of multiplicity p in the form

$$\int \sum_0^q \beta_k\, \omega_{q-k}, \quad \text{where} \quad q = \left[\frac{p}{2}\right], \qquad \beta_k = \frac{1}{(2k+\epsilon)!} P^{(k)}_{i_1 \dots i_{2k+\epsilon}} dx^{i_1} \dots dx^{i_{2k+\epsilon}}.$$

it follows easily from the duality properties that the formula for the associated integrals is valid for all values of p, and can be written

$$(-1)^{\frac{1}{2}m(m-1)} \int \sum_0^q \beta_k'\, \omega_{m-q-k-\epsilon}, \quad \beta_k' = \frac{1}{(2k+\epsilon)!} P^{(k)}_{m+i_1 \dots m+i_{2k+\epsilon}}\, dx^{i_1} \dots dx^{2k+\epsilon}$$

where $\epsilon = 0$ or 1 according as p is even or odd.

11. We observe that if $p \leqslant m$ the number of harmonic integrals of the type

$$\frac{1}{p!} \int P_{i_1 \dots i_p} dx^{i_1} \dots dx^{i_p},$$

where $$g^{i\,m+j} P_{ij\,i_1 \dots i_{p-2}} = 0,$$

is $R_p - R_{p-2}$, or R_1 in the case $p = 1$. Now it is observed that the real and imaginary parts of the Abelian integrals of the first kind attached to V belong to this class, hence the number of Abelian integrals of the first kind of multiplicity p cannot exceed $\frac{1}{2}(R_p - R_{p-2})$.

In the case $p = 1$ it is easy to see that the number of integrals is exactly $\frac{1}{2}R_1$.

For if $\int P_i dx^i$ is a harmonic integral, so is $\int P_{m+i} dx^i$.

Hence $$\int (P_j - i P_{m+j}) dx^j \quad (\text{where } i^2 = -1)$$

is an Abelian integral of the first kind attached to V, and we can thus determine $\frac{1}{2}R_1$ such integrals. If there were more, a linear combination of them would give an integral whose real part had no periods, and this is known to be impossible.

In the case $p = m = 2$, I have given an exact formula for the number of double Abelian integrals of the first kind†.

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† W. V. D. Hodge, *Journal London Math. Soc.*, 8 (1933), 312. Other applications will be found in a paper in *Proc. Camb. Phil. Soc.*, 31 (1935), 18. The reference in that paper to a paper in the press refers to the present paper.

THE POTENTIAL OF A CIRCULAR CYLINDER BETWEEN TWO INFINITE PLANES

By R. C. KNIGHT.

[Received and read 14 December, 1933.]

Introduction.

The problem is to determine a solution of Laplace's equation which shall have given values on an infinite circular cylinder, lying symmetrically between two parallel infinite planes, and shall have zero or a constant value on those planes. These boundaries have been considered by Howland*, Bairstow†, and others in connection with the bi-harmonic equation; while Rigby‡ has considered a very similar potential problem in which the inner boundary is a sphere.

Taking a plane section perpendicular to the axis of the cylinder, the problem becomes a two-dimensional one with the boundaries a circle symmetrically placed between two parallel infinite straight lines. The method employed is to construct harmonic functions which vanish on the straight lines and to combine them, when arranged in suitable form, to give the required value on the circle.

Determination of the potential functions.

A solution of

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad (1)$$

* R. C. J. Howland, *Phil. Trans.* (A), 229 (1930), 49–86; R. C. J. Howland and A. C. Stevenson, *Phil. Trans.* (A), 232 (1933), 155–222; R. C. J. Howland and R. C. Knight, *Proc. Camb. Phil. Soc.*, 29 (1930), 277–287.

† L. Bairstow, B. M. Cave, and E. D. Lang, *Proc. Royal Soc.* (A), 100 (1922), 394–413.

‡ C. M. Rigby, *Proc. London Math. Soc.* (2), 33 (1932), 525–536.

is required satisfying the conditions:

(i) V is to have given values when $r = a$; the case of V constant and equal to unity is considered;

(ii) V is to be zero when $x = \pm c$, $c > a$.

The field due to a line charge, of line density σ, perpendicular to the plane at $z = a$, where $z = x + iy$, is

$$V = C - 2\sigma \log|z-a|,$$

where C is an arbitrary constant.

We take a unit σ at the point $z = \xi$, $|\xi| < c$, and form the image system of this in the two lines $z = \pm c$. This will be formed by units

$+\sigma$ at the points $\pm 4pc + \xi$ $\quad (p = 0, 1, 2, \ldots)$,

and $-\sigma$ at the points $\pm 2(2p-1)c - \xi$ $\quad (p = 0, 1, 2, \ldots)$.

The field due to these charges will be

$$V_0 = \Re W_0(\xi), \tag{2}$$

where

$$W_0(\xi) = A - 2\sigma \Big\{\log(z-\xi) + \sum_{p=1}^{\infty} [\log(z-4pc-\xi) + \log(z+4pc-\xi) - \log\{z-2(2p-1)c+\xi\} - \log\{z+2(2p-1)c+\xi\}]\Big\}. \tag{3}$$

$W_0(\xi)$ will be zero on $z = \pm c$, with suitable choice of A. The derivatives of $W_0(\xi)$ with respect to ξ will also be zero on the straight lines.

The first of the derivatives is

$$W_1(\xi) = 2\sigma \Big\{\frac{1}{z-\xi} + \sum_{p=1}^{\infty} \Big(\frac{1}{z-4pc-\xi} + \frac{1}{z+4pc-\xi} + \frac{1}{z-2(2p-1)c+\xi} + \frac{1}{z+2(2p-1)c+\xi}\Big)\Big\}. \tag{4}$$

$W_0(\xi)$ and $W_1(\xi)$ may be expanded in descending powers of c provided that

$$|z-\xi| < 4c \quad \text{and} \quad |z+\xi| < 4c, \tag{5}$$

and these conditions are satisfied if $|z| < c$. The first of these expansions is

$$W_1(\xi) = 2\sigma \Big\{\frac{1}{z-\xi} + \sum_{q=1}^{\infty} [A_{2q-1}(z-\xi)^{2q-1} + B_{2q-1}(z+\xi)^{2q-1}]\Big\}, \tag{6}$$

where $$A_{2q-1} = -\frac{2S_{2q}}{(4c)^{2q}}, \quad B_{2q-1} = -\frac{2(1-2^{-2q})\,S_{2q}}{(2c)^{2q}}, \tag{7}$$

and S_{2q} is the zeta-function for an integral argument, so that

$$S_{2q} = \sum_{n=1}^{\infty} n^{-2q}.$$

$W_1(\xi)$ is an odd function and, since we require only even functions on account of the symmetry of the problem, we differentiate $W_1(\xi)$ $2n-1$ times with respect to ξ and then put $\xi = 0$, giving

$$\begin{aligned} W_{2n}(0) &= 2\sigma\left\{\frac{(2n-1)!}{z^{2n}} + \sum_{q=n}^{\infty} \frac{(2q-1)!}{(2q-2n)!}\, z^{2q-2n}(B_{2q-1}-A_{2q-1})\right\} \\ &= 2\sigma(2n-1)!\left\{\frac{1}{z^{2n}} + \sum_{q=n}^{\infty} \binom{2q-1}{2n-1} z^{2q-2n}(B_{2q-1}-A_{2q-1})\right\} \\ &= 2\sigma(2n-1)!\left\{\frac{1}{z^{2n}} + \sum_{q=0}^{\infty} \binom{2q+2n-1}{2n-1} z^{2q}(B_{2q+2n-1}-A_{2q+2n-1})\right\}, \end{aligned} \tag{8}$$

so that*

$$\begin{aligned} V_{2n} &= \Re W_{2n}(0) \\ &= r^{-2n}\cos 2n\theta + \sum_{q=0}^{\infty} \binom{2q+2n-1}{2q} (B_{2q+2n-1}-A_{2q+2n-1})\, r^{2q}\cos 2q\theta. \end{aligned} \tag{9}$$

To determine V_0 in a similar form, we expand $W_0(0)$ in descending powers of c; then

$$\begin{aligned} W_0(0) &= A' - \left[\log z + \sum_{p=1}^{\infty}\left\{\log\left(1-\frac{z^2}{16p^2c^2}\right) - \log\left(1-\frac{z^2}{4(2p-1)^2c^2}\right)\right\}\right] \\ &= A' - \log z + \sum_{q=1}^{\infty} a_{2q} z^{2q}, \end{aligned} \tag{10}$$

where $$a_{2q} = \frac{1}{q(4c)^{2q}} S_{2q} - \frac{1-2^{-2q}}{q(2c)^{2q}} S_{2q} = \frac{1}{q}\,\frac{1}{(2c)^{2q}}\,\mu_{2q}, \tag{11}$$

and $$\mu_{2q} = \sum_{m=1}^{\infty} \frac{(-)^m}{m^{2q}}. \tag{12}$$

---

* σ has been taken to have the value $\frac{1}{2}$ and the $(2n-1)!$ has been omitted. This does not affect the solution, since we take a constant times each V_{2n} and determine the constants to satisfy the boundary conditions.

Hence $$V_0 = A'' - \log r + \sum_{m=1}^{\infty} \alpha_{2q} r^{2q} \cos 2q\theta.$$

A', A'' are constants which can be chosen to give any constant value of V on the straight boundaries. If the value is to be zero, we have

$$A'' = \log c - \sum_{q=1}^{\infty} \frac{1}{q} \frac{1}{(2c)^{2q}} c^{2q} \mu_{2q}$$

$$= \log c - \sum_{q=1}^{\infty} \frac{1}{q} \frac{1}{2^{2q}} \sum_{m=1}^{\infty} \frac{(-)^m}{m^{2q}}$$

$$= \log c - \sum_{q=1}^{\infty} \frac{1}{q} \frac{1}{2^{2q}} (1 - 2^{2q-1}) \frac{\pi^{2q}}{(2q)!} B_q,$$

where the B_q are Bernoulli's numbers.

After a little reduction A'' is found to have the value

$$A'' = \log \frac{4c}{\pi},$$

since $$\log \sin z = \log z - \sum_{n=1}^{\infty} \frac{2^{2n-1} z^{2n}}{n(2n)!} B_n,$$

and hence

$$V_0 = \log \frac{4c}{\pi} + \sum_{q=1}^{\infty} \alpha_{2q} r^{2q} \cos 2q\theta. \tag{13}$$

Returning to equation (9), we have

$$B_{2q+2n-1} - A_{2q+2n-1} = -\frac{2}{(2c)^{2q+2n}} (1 - 2^{-2q-2n}) S_{2q+2n} + \frac{2}{(4c)^{2q+2n}} S_{2q+2n}$$

$$= \frac{2}{(2c)^{2q+2n}} \mu_{2q+2n},$$

so that (9) may be written as

$$V_{2n} = r^{-2n} \cos 2n\theta + 2 \sum_{q=0}^{\infty} \binom{2q+2n-1}{2q} \frac{1}{(2c)^{2q+2n}} \mu_{2q+2n} r^{2q} \cos 2q\theta. \tag{14}$$

The equations (13) and (14) give the functions satisfying condition (ii) in

a form suitable to the circular boundary. To satisfy condition (i), we take

$$V = C_0 V_0 + \sum_{s=1}^{\infty} C_{2s} V_{2s}$$

$$= C_0 \left(\log \frac{4c}{\pi r} + \sum_{q=1}^{\infty} \frac{1}{q} \frac{1}{(2c)^{2q}} \mu_{2q} r^{2q} \cos 2q\theta \right)$$

$$+ \sum_{s=1}^{\infty} C_{2s} \left\{ r^{-2s} \cos 2s\theta + 2 \sum_{q=0}^{\infty} \binom{2q+2s-1}{2q} \frac{1}{(2c)^{2s+2q}} \mu_{2s+2q} r^{2q} \cos 2q\theta \right\}$$

$$= \mathrm{V}_0 + \sum_{n=1}^{\infty} (\mathrm{V}_{2n} r^{2n} + \mathrm{V}'_{2n} r^{-2n}) \cos 2n\theta. \qquad (15)$$

We can now solve the problem in which V is given on the circular boundary as a function of θ, if that function is expressible in a Fourier series.

Determination of the constants.

If V is to have the value unity when $r = a$, we must have

$$\mathrm{V}_0 = 1, \quad \mathrm{V}_n = \mathrm{V}_n' = 0 \quad (n \geqslant 1),$$

and the equations for the constants become

$$1 = C_0 \log \frac{4c}{\pi a} + 2 \sum_{s=1}^{\infty} \frac{1}{(2c)^{2s}} \mu_{2s} C_{2s}, \qquad (16.1)$$

and, when $q \geqslant 1$,

$$0 = \frac{a^{2q}}{(2c)^{2q}} \frac{1}{q} \mu_{2q} C_0 + \frac{1}{a^{2q}} C_{2q} + 2 \sum_{s=1}^{\infty} \binom{2q+2s-1}{2q} \frac{a^{2q}}{(2c)^{2q+2s}} \mu_{2q+2s} C_{2s}. \qquad (16.2)$$

These equations bear a marked similarity to those in Rigby's paper, but a different method of solution has been adopted here. This method is one of successive approximation.

In equations (16.1), (16.2), write

$$\lambda = a/2c \qquad (17)$$

and

$$D_{2n} = C_{2n}/a^{2n} \quad (n \geqslant 0); \qquad (18)$$

they then become

$$1 = D_0 \log(2/\pi\lambda) + 2 \sum_{s=1}^{\infty} \mu_{2s} \lambda^{2s} D_{2s},$$

$$0 = \frac{1}{q} \mu_{2q} \lambda^{2q} D_0 + D_{2q} + 2 \sum_{s=1}^{\infty} \binom{2q+2s-1}{2q} \mu_{2q+2s} \lambda^{2q+2s} D_{2s},$$

i.e.
$$1 = lD_0 + \sum_{s=1}^{\infty} {}^0a_{2s}\lambda^{2s} D_{2s}, \tag{19.1}$$

$$0 = {}^{2q}a_0\lambda^{2q} D_0 + D_{2q} + \lambda^{2q} \sum_{s=1}^{\infty} {}^{2q}a_{2s}\lambda^{2s} D_{2s}, \tag{19.2}$$

where
$${}^{2q}a_{2s} = 2\binom{2q+2s-1}{2q}\mu_{2q+2s} \quad (s \geqslant 1), \tag{20.1}$$

$${}^{2q}a_0 = \mu_{2q}/q, \tag{20.2}$$

$$l = \log(2/\pi\lambda). \tag{21}$$

Suppose that
$$D_{2s} = \sum_{r=0}^{\infty} D_{2s}^{(r)} \quad (s \geqslant 0), \tag{22}$$

where
$$D_0^{(0)} = 1/l, \quad D_{2s}^{(0)} = 0, \tag{23.1}$$

$$D_{2q}^{(1)} = -{}^{2q}a_0\lambda^{2q} D_0^{(0)} \quad (q \geqslant 1), \tag{23.2}$$

$$D_0^{(1)} = -\frac{1}{l} \sum_{s=1}^{\infty} {}^0a_{2s}\lambda^{2s} D_{2s}^{(1)}, \tag{23.3}$$

and generally, when $r \geqslant 2$,
$$D_0^{(r)} = \frac{1}{l} \sum_{s=1}^{\infty} {}^0a_{2s}\lambda^{2s} D_{2s}^{(r)} \tag{24.1}$$

and, for $q \geqslant 1$,
$$D_{2q}^{(r)} = -\lambda^{2q} \sum_{s=0}^{\infty} {}^{2q}a_{2s}\lambda^{2s} D_{2s}^{(r-1)}. \tag{24.2}$$

The convergence of the D's.

The inequalities used in investigating the convergence are

$$|\mu_{2n}| = \left| \sum_{m=1}^{\infty} \frac{(-)^m}{m^{2n}} \right| < 1,$$

$$|{}^{2q}a_{2s}| = 2\binom{2q+2s-1}{2q} |\mu_{2q+2s}| < 2^{2q+2s},$$

using the inequality $\binom{s+t}{s} < 2^{s+t}$ for the binomial coefficient; and for the special cases

$$|{}^{2q}a_0| < 1/q < 2^q; \quad |{}^0a_{2s}| < 2.$$

Suppose that
$$|D_{2q}^{(r)}| < A_r(2\lambda)^{2q}; \tag{25}$$

then
$$|D_0^{(r)}| < (2A_r/l) \sum_{s=1}^{\infty} \lambda^{2s}(2\lambda)^{2s}$$
$$= \frac{2A_r}{l} \frac{4\lambda^4}{1-4\lambda^4},$$

and
$$|D_{2q}^{(r+1)}| < A_r\lambda^{2q}\left\{\frac{2}{l}\frac{4\lambda^4}{1-4\lambda^4}2^{2q} + \sum_{s=1}^{\infty} 2^{2q+2s}\lambda^{2s}(2\lambda)^{2s}\right\}$$
$$= A_r(2\lambda)^{2q}\left\{\frac{2}{l}\frac{4\lambda^4}{1-4\lambda^4} + \frac{16\lambda^4}{1-16\lambda^4}\right\}$$
$$= A_{r+1}(2\lambda)^{2q},$$

where
$$A_{r+1} = \left(\frac{2}{l}\frac{4\lambda^4}{1-4\lambda^4} + \frac{16\lambda^4}{1-16\lambda^4}\right)A_r.$$

The inequality (25) is satisfied when $r = 1$, since
$$|D_{2q}^{(1)}| < (2\lambda^{2q}/l) < A_1(2\lambda)^{2q},$$

where
$$A_1 = 1/l,$$

so that $\sum_{r=0}^{\infty} D_{2q}^{(r)}$ converges if
$$\frac{2}{l}\frac{4\lambda^4}{1-4\lambda^4} + \frac{16\lambda^4}{1-16\lambda^4} < 1.$$

The best value of λ which satisfies this inequality is between 0·389 and 0·390, so that we have convergence if
$$\lambda < 0{\cdot}390.$$

From physical considerations it is to be expected that the series would converge up to $\lambda = 0{\cdot}5$; much has been lost in using the inequality $\binom{s+t}{s} < 2^{s+t}$, and in particular in taking $1/q < 2^q$, but no better inequality could be found which was not too complicated to be of any use.

The convergence of the potential function.

Returning to equation (15) for V, we substitute for the C's from equations (18) and (22) and obtain
$$V = \sum_{r=0}^{\infty}\left\{D_0^{(r)}V_0 + \sum_{s=1}^{\infty} D_{2s}^{(r)}a^{2s}V_{2s}\right\}.$$

TABLE I. Values of $-{}^{2q}a_{2s}$.

q	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
0	—	1·64493	1·89407	1·97110	1·99247	1·99803	1·99952	1·99988
1	·82247	5·68220	$1·97110\times10^{1}$	$4·18418\times10^{1}$	$7·19308\times10^{1}$	$1·09973\times10^{2}$	$1·55990\times10^{2}$	
2	·47352	9·85551	$6·97363\times10^{1}$	$2·51758\times10^{2}$	$6·59840\times10^{2}$	$1·42991\times10^{3}$	$2·72996\times10^{3}$	
3	·32852	$1·39473\times10^{1}$	$1·67839\times10^{2}$	$9·23776\times10^{2}$	$3·43179\times10^{3}$	$1·00098\times10^{4}$	$2·47519\times10^{4}$	
4	·24906	$1·79827\times10^{1}$	$3·29920\times10^{2}$	$2·57384\times10^{3}$	$1·28698\times10^{4}$	$4·86200\times10^{4}$	$1·51164\times10^{5}$	
5	·19981	$2·19947\times10^{1}$	$5·71965\times10^{2}$	$6·00591\times10^{3}$	$3·88959\times10^{4}$	$1·84756\times10^{5}$	$7·05432\times10^{5}$	
6	·16663	$2·59984\times10^{1}$	$9·09986\times10^{2}$	$1·23760\times10^{4}$	$1·00776\times10^{5}$	$5·87860\times10^{5}$	$2·70416\times10^{6}$	
7	·14285	$2·99995\times10^{1}$	$1·35999\times10^{3}$	$2·32560\times10^{4}$	$2·32560\times10^{5}$	$1·63438\times10^{6}$	$8·91480\times10^{6}$	
8	·12500	$3·39999\times10^{1}$	$1·93800\times10^{3}$	$4·06980\times10^{4}$	$4·90314\times10^{5}$	$4·08595\times10^{6}$	$2·60758\times10^{7}$	

Each series with (r) constant will be absolutely convergent if

$$|D_0^{(r)}|\left\{\log\left(\frac{4c}{\pi r}\right)+\sum_{q=1}^{\infty}\frac{1}{q}|\mu_{2q}|\frac{r^{2q}}{(2c)^{2q}}\right\}$$
$$+\sum_{s=1}^{\infty}a^{2s}|D_{2s}^{(r)}|\left\{r^{-2s}+2\sum_{q=0}^{\infty}\binom{2q+2s-1}{2q}|\mu_{2s+2q}|\frac{r^{2q}}{(2c)^{2q+2s}}\right\}$$

converges.

Using the inequalities given above, we find that this is less than

$$D_0^{(r)}\left[l+\log\left(\frac{a}{r}\right)\right]+\sum_{s=1}^{\infty}|D_{2s}^{(r)}|\left(\frac{a}{r}\right)^{2s}+\sum_{s=0}^{\infty}|D_{2s}^{(r)}|\sum_{q=0}^{\infty}\frac{2^{2s}a^{2s}}{(2c)^{2s}}\left(\frac{r}{c}\right)^{2q}$$
$$<D_0^{(r)}\left[l+\log\left(\frac{a}{r}\right)\right]+\sum_{s=1}^{\infty}A_r\left(\frac{2\lambda a}{r}\right)^{2s}+\sum_{q=1}^{\infty}A_r(2\lambda)^{4s}\frac{1}{1-(r^2/c^2)}.$$

It is necessary for V to converge at any rate for some $r>a$ and, provided that $a<r<c$, these series converge for $\lambda<\frac{1}{2}$, *i.e.* if the cylinder lies wholly between the planes. This is true for all values of (r) for each $D_{2s}^{(r)}$ and we have shown that the $D_{2s}^{(r)}$ converge to D_{2s} for values of λ up to 0·390. Hence the potential function converges under the same restriction.

Calculation of the constants.

The values of the coefficients ${}^{2q}a_{2s}$ were calculated by using the values for μ_{2p} given by Glaisher*, and checked by using the simple relations between those coefficients which have the same value of $q+s$. They are given in Table I and those not given can be calculated by using $\mu=-1$ without error to five figures.

The values of the $D^{(r)}$'s were calculated in a straightforward manner by using equations (21), (22), and then, for the final constants D's, (20). λ was taken to be 0·05, 0·10, ..., 0·40, and in most cases values to $D_{14}^{(r)}$ were found. The value of r was taken as far as was necessary to give the value of D_0 correct to five figures. Table II gives the calculated values of D_0 for the various values of λ.

TABLE II.

λ	·05	·10	·15	·20	·25	·30	·35	·40
D_0	0·3931	0·5403	0·6921	0·8653	1·0761	1·3498	1·7369	2·3656

* J. W. L. Glaisher, *Quart. J. of Math.* 45 (1914), 141.

The capacity of the cylinder.

The capacity of the cylinder, per unit length, since its potential is unity, is

$$C = -K \int_0^{2\pi} \left[\frac{\partial V}{\partial r} r\right]_{r=a} d\theta,$$

where K is the specific inductive capacity of the medium between the cylinder and the planes.

Hence, on using equations (15) and (18), we have

$$C = 2\pi K C_0 = 2\pi K D_0,$$

where D_0 is given in Table II.

In conclusion I should like to thank Prof. R. C. J. Howland for his advice and many helpful suggestions concerning the problem. I am also deeply grateful to the Research Committee of University College, Southampton, whose assistance has made possible my use of a calculating machine.

University College,
Southampton.

A THEOREM CONCERNING MEROMORPHIC FUNCTIONS OF FINITE ORDER*

By A. J. MACINTYRE.

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1. *Introduction.* In two recent papers J. M. Whittaker† has established the existence of what he calls "flat regions" for integral functions of finite order. Given an integral function $f(z)$ a region R is said to be a "flat region" if there is a constant $H>1$ independent of R such that the inequalities

$$\frac{1}{H}<\frac{\log|f(z_1)|}{\log|f(z_2)|}<H$$

are satisfied for any pair of points z_1 and z_2 belonging to R. Whittaker proved that, *if $f(z)$ is an integral function of finite order $\rho>0$ and σ is a given number less than $1-\frac{1}{2}\rho$, there is a positive constant h and a sequence of points $\xi_1, \xi_2, \ldots, \xi_\nu, \ldots$ such that $\lim_{\nu=\infty}|\xi_\nu|=\infty$ and*

$$\log|f(z)|>h\log M(|\xi_\nu|, f)$$

in the circle R_ν defined by

$$|z-\xi_\nu|<|\xi_\nu|^\sigma.$$

Here the sequence of circles $R_1, R_2, \ldots$ are all flat regions in which $\log|f(z)|$ tends to infinity, since $M(r,f)=\max|f(re^{i\theta})|$ tends to infinity with r.

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* The author is considerably indebted to Dr. E. F. Collingwood for assistance in the preparation of the manuscript and for improvements in the argument which enable (4) and (5) to be given their present forms without the appearance of undetermined constants.

† *Proc. Edinburgh Math. Soc.*, 2 (1930), 111–28; *Quart. J. of Math.* (Oxford), 2 (1931), 252–258. Theorem C of the latter is stated below.

In this note I prove an analogous theorem about meromorphic functions of finite order. My method differs from Whittaker's and leads to results of rather greater precision, as is seen on comparing his result, as just stated, with Theorem 1 of the present note. The extension to meromorphic functions leads also to results about functions with deficient values in the sense of R. Nevanlinna. An obvious example is Theorem 2, which is an immediate corollary of Theorem 1.

The notation used throughout is that of R. Nevanlinna*.

The aim of the note is the proof of the following theorem.

THEOREM 1. *Hypothesis.* $f(z)$ *is a meromorphic function of finite positive order* ρ; $\rho' \geqslant \rho$ *is any number such that*

$$T(r) = T(r, f) = o\left(\frac{r^{\rho'}}{\log r}\right), \tag{1}$$

and the poles of $f(z)$ *are of deficiency* $\delta > 0$, *so that*

$$\overline{\lim_{r=\infty}} \frac{N(r, \infty)}{T(r)} \leqslant 1-\delta. \tag{2}$$

Assertion. We can find a sequence of circles C_ν *with centres* ξ_ν *and radii* r_ν *satisfying the conditions*

$$\lim_{\nu=\infty} |\xi_\nu| = \infty; \quad \lim_{\nu-\infty} r_\nu |\xi_\nu|^{\frac{1}{2}\rho'-1} = \infty, \tag{3}$$

and a corresponding sequence of positive numbers ϵ_ν *satisfying*

$$\lim_{\nu=\infty} \epsilon_\nu = 0,$$

such that, for all z *in* C_ν, $f(z)$ *satisfies the inequalities*

$$1-\epsilon_\nu < \frac{\log|f(z)|}{\log|f(\xi_\nu)|} < 1+\epsilon_\nu \tag{4}$$

and

$$\frac{\delta}{e} < \frac{\log|f(\xi_\nu)|}{T(|\xi_\nu|, f)} < |\xi_\nu|^{\rho'}. \tag{5}$$

The proof of this theorem depends on three lemmas, which are discussed first.

* R. Nevanlinna, *Fonctions méromorphes* (Paris, 1929).

2. The mean value $I(\alpha r, r)$ is defined by

$$I(\alpha r, r) = \frac{1}{\pi r^2(1-\alpha^2)} \int_{\alpha r}^{r} \int_{-\pi}^{\pi} \log^{+} |f(te^{i\theta})| \, t \, dt \, d\theta$$

$$= \frac{2}{r^2(1-\alpha^2)} \int_{\alpha r}^{r} m(t, f) \, t \, dt,$$

where $0 \leqslant \alpha < 1$.

LEMMA 1. *If $f(z)$ is a meromorphic function of finite order $\rho > 0$, and if its poles are of deficiency $\delta > 0$, then*

$$\frac{\delta}{e} < \overline{\lim_{r=\infty}} \frac{I\{\rho r/(1+\rho), r\}}{T(r)} < 1. \tag{6}$$

Proof. The right-hand inequality is trivial. To prove the left-hand one we proceed as follows. Since

$$\overline{\lim_{r=\infty}} \frac{N(r, \infty)}{T(r)} \leqslant 1-\delta,$$

we have

$$\underline{\lim_{r=\infty}} \frac{m(r, f)}{T(r)} \geqslant \delta.$$

Let δ' be any positive number less than δ. Then, for all sufficiently large r,

$$I(\alpha r, r) = \frac{2}{r^2(1-\alpha^2)} \int_{\alpha r}^{r} m(t, f) \, t \, dt > \frac{2\delta'}{r^2(1-\alpha^2)} \int_{\alpha r}^{r} T(t, f) \, t \, dt > \delta' \, T(\alpha r).$$

Take $\rho'' > \rho$. Then, for some large r,

$$T(\alpha r) > \alpha^{\rho''} \, T(r).$$

For if not, then, for all r greater than a certain r_1,

$$T(\alpha r) \leqslant \alpha^{\rho''} \, T(r),$$

and, in particular, we have the sequence of inequalities

$$T(r_1) \leqslant \alpha^{\rho''} \, T\left(\frac{r_1}{\alpha}\right) \leqslant \alpha^{2\rho''} \, T\left(\frac{r_1}{\alpha^2}\right) \leqslant \ldots \leqslant \alpha^{n\rho''} \, T\left(\frac{r_1}{\alpha^n}\right) \leqslant \ldots.$$

Hence, writing $r_n = r_1/\alpha^n$, we have

$$\rho = \overline{\lim_{r=\infty}} \frac{\log T(r)}{\log r} \geqslant \overline{\lim_{n=\infty}} \frac{\log T(r_n)}{\log r_n} \geqslant \overline{\lim_{n=\infty}} \frac{\log T(r_1) - n\rho'' \log \alpha}{\log r_1 - n \log \alpha} \geqslant \rho'',$$

contrary to hypothesis. Hence, for some large r,

$$I(\alpha r, r) > \delta' \alpha^{\rho''} T(r).$$

Since $\delta - \delta'$ and $\rho'' - \rho$ may both be as small as we please, we conclude that

$$\overline{\lim_{r=\infty}} \frac{I(\alpha r, r)}{T(r)} \geqslant \delta \alpha^{\rho},$$

and, in particular, if we put $\alpha = \rho/(1+\rho) > 0$, we have, for all $\rho > 0$,

$$\alpha^{\rho} = \left(\frac{\rho}{1+\rho}\right)^{\rho} > \frac{1}{e},$$

and so

$$\overline{\lim_{r=\infty}} \frac{I(\alpha r, r)}{T(r)} > \frac{\delta}{e}.$$

3. We now make use of a dissection of the plane introduced into the subject by Valiron*. The sequences r_n and ν_n are defined by

$$\text{(i)} \quad r_{n+1} = r_n + K r_n^{1-\frac{1}{2}\rho'},$$

$$\text{(ii)} \quad \nu_n = \left[\frac{2\pi}{K} r_n^{\frac{1}{2}\rho'}\right],$$

where $K > 0$ and $r_1 > 0$ are arbitrarily chosen and the square brackets denote the integral part of the number written inside. The annulus $r_n < |z| < r_{n+1}$ is divided into ν_n "quadrilaterals" by radii inclined at equal angles, and in order to fix the dissection we lay down the rule that the positive real axis shall be one of these radii. In this way the whole plane outside the circle $|z| = r_1$ is divided into quadrilaterals. Consider a quadrilateral Q belonging to the annulus $r_n < |z| < r_{n+1}$. Q lies between and in contact with two other quadrilaterals Q' and Q'' belonging to the same annulus $r_n < |z| < r_{n+1}$. Q' and Q'' are "neighbours" of Q. For all $n > 1$ there are quadrilaterals belonging to the annulus $r_{n-1} < |z| < r_n$ whose frontiers have at least one point in common with the frontiers of Q', Q, or Q''. These quadrilaterals are also called neighbours of Q. It

* G. Valiron, *Acta Math.*, 52 (1929), 67–92 (74).

is clear, since ν_n is a non-decreasing function of n, that their number cannot exceed five. Similarly there are quadrilaterals belonging to the annulus $r_{n+1} < |z| < r_{n+2}$ whose frontiers have a point in common with the frontiers of Q', Q, or Q''. It is evident that we can choose r_1 so that their number also does not exceed five. The total number of neighbours thus defined cannot exceed twelve.

Having fixed the dissection of the plane into quadrilaterals, we define the function $\phi_K(z)$ in relation to this dissection as follows. Suppose first that z is an interior point of a quadrilateral Q. Then $\phi_K(z) = 0$ if neither Q nor any one of its neighbours and their frontiers as defined above contains a zero of any of the functions $f(z)$, $f(z)-1$, $1/f(z)$. Otherwise $\phi_K(z) = 1$. As thus defined $\phi_K(z)$ is constantly 0 or 1 in the interior of any quadrilateral Q. If z is a frontier point of Q, it is also a frontier point of other quadrilaterals not exceeding three in number. If $\phi_K = 0$ inside all those quadrilaterals of which z is a frontier point, then $\phi_K(z) = 0$. Otherwise $\phi_K(z) = 1$. For $|z| < r_1$ we take $\phi_K(z) = 1$.

The dissection being fixed, the function $\phi_K(z)$ thus defined for all values of z depends only on $f(z)$.

We now define the mean value $I_K(\alpha r, r)$ by the equation

$$I_K(\alpha r, r) = \frac{1}{\pi r^2(1-\alpha^2)} \int_{\alpha r}^{r} \int_{-\pi}^{\pi} \overset{+}{\log} |f(te^{i\theta})| \, \phi_K(te^{i\theta}) \, t \, dt \, d\theta,$$

and prove

LEMMA 2. *If $f(z)$ is a meromorphic function of finite order $\rho > 0$, and satisfies* (1), *then*

$$\lim_{r=\infty} \frac{I_K\{\rho r/(1+\rho), r\}}{T(r)} = 0. \tag{7}$$

Proof. For $0 < \alpha < 1$, writing $I_K(r)$ for $I_K(0, r)$, we have

$$I_K(\alpha r, r) \leqslant \frac{1}{1-\alpha^2} I_K(r). \tag{8}$$

Further, for any $0 < \mu < 1$,

$$\begin{aligned} I_K(r) &\leqslant I_K(\mu r) + \frac{1}{\pi r^2} \int_{\mu r}^{r} \int_{-\pi}^{\pi} \overset{+}{\log} |f(te^{i\theta})| \, t \, dt \, d\theta \\ &\leqslant I_K(\mu r) + \frac{2}{r^2} \int_{\mu r}^{r} T(t) \, t \, dt \\ &\leqslant I_K(\mu r) + (1-\mu^2) \, T(r). \end{aligned} \tag{9}$$

To obtain an upper bound for $I_K(\mu r)$ we introduce Nevanlinna's inequality*

$$\overset{+}{\log}|f(te^{i\theta})| \leqslant \frac{1+\mu}{1-\mu} m(r, f) + \sum_{|b_\nu|<r} \log\left|\frac{r^2-\bar{b}_\nu z}{r(z-b_\nu)}\right|, \tag{10}$$

where $z = te^{i\theta}$, $t \leqslant \mu r$, and the b_ν are the poles of $f(z)$. For $|b_\nu| < r$,

$$\log\left|\frac{r^2-\bar{b}_\nu z}{r(z-b_\nu)}\right| \leqslant \log 2r + \log\left|\frac{1}{z-b_\nu}\right|.$$

Hence, if A is a set of superficial measure mA contained in the circle $|z| < r$, the surface integral

$$S = \int_A \log\left|\frac{r^2-\bar{b}_\nu z}{r(z-b_\nu)}\right| dz^2 \leqslant mA \log 2r + \int_A \log\left|\frac{1}{z-b_\nu}\right| dz^2.$$

Here we may take A to consist wholly of a finite number of simply connected regions interior to the circle $|z| < r$. It is then well known that, for a given value of mA, the last integral on the right is a maximum when A is a circle with b_ν as centre. Therefore

$$S \leqslant mA\left(\log 2r + \log\frac{\pi e}{mA}\right). \tag{11}$$

Let $A(r)$ be the area in which the integrand of $I_K(r)$ does not vanish. Then, by (10) and (11),

$$I_K(\mu r) < \frac{A(\mu r)}{\pi\mu^2 r^2}\left\{\frac{1+\mu}{1-\mu} T(r, f) + n(r, \infty) \log\frac{2\pi r e}{A(\mu r)}\right\}.$$

Applying this inequality to $I_K(\mu^2 r)$, we find

$$I_K(\mu^2 r) < \frac{A(\mu^2 r)}{\pi\mu^4 r^2}\left\{\frac{1+\mu}{1-\mu} T(\mu r, f) + n(\mu r, \infty) \log\frac{2\pi\mu e r}{A(\mu^2 r)}\right\}. \tag{12}$$

Now

$$n(\mu r, \infty) \leqslant \frac{N(r, \infty)}{\log 1/\mu} \leqslant \frac{T(r)}{\log 1/\mu}. \tag{13}$$

$A(r)$ is certainly a non-decreasing function of r for which it is easy to find an upper bound. Consider a zero of one of the functions $f(z)$, $f(z)-1$, or $1/f(z)$, and suppose that it lies in the annulus $r_n < |z| < r_{n+1}$. It is then evident that $\phi_K(z) = 1$ in a group of not more than five quadrilaterals belonging to each of the annuli (r_{n-1}, r_n), (r_n, r_{n+1}), and (r_{n+1}, r_{n+2}), making fifteen quadrilaterals in all. If the zero under consideration lies either on

* R. Nevanlinna, *loc. cit.*, 25.

the circumference $|z|=r_n$ or on $|z|=r_{n+1}$, we have to add a further group of not more than five quadrilaterals either in the annulus (r_{n-2}, r_{n-1}) or in the annulus (r_{n+2}, r_{n+3}), making twenty in all. Now observe that the area of a quadrilateral belonging to $r_n<|z|<r_{n+1}$ is of the form $K^2 r_n^{2-\rho'}(1+\epsilon_n)$, where $\lim \epsilon_n=0$ $(\epsilon_n>0)$. Writing

$$n(r)=n(r,0)+n(r,1)+n(r,\infty),$$

we deduce from the foregoing that the area in which $\phi_K(z)=1$, arising from zeros of $f(z)$, $f(z)-1$, and $1/f(z)$ in $r_n<|z|<r_{n+1}$, does not exceed

$$20K^2\{n(r_{n+1})-n(r_n)\}r_n^{2-\rho'}(1+\epsilon_n).$$

Hence

$$A(r)\leqslant 20K^2\sum_{\nu=1}^{n+3} r_\nu^{2-\rho'}(1+\epsilon_\nu)\{n(r_{\nu+1})-n(r_\nu)\}+A(r_1),$$

where $r_n\leqslant r<r_{n+1}$. For all sufficiently large values of r this is less than

$$25K^2\int_{r_1}^{2r} t^{2-\rho'}\,dn(t)+A_1 = 25K^2\left\{(2r)^{2-\rho'}n(2r)+(\rho'-2)\int_{r_1}^{2r} t^{1-\rho'}n(t)\,dt\right\}+O(1)$$

$$[A_1=A(r_1)]$$

$$=o\left(\frac{r^2}{\log r}+\int_{r_1}^{2r}\frac{t\,dt}{\log t}\right)$$

by (1). Hence, since, for $r_1>1$,

$$\int_{r_1}^{2r}\frac{t\,dt}{\log t}=2\int_{r_1}^{2r}\left(\frac{1}{\log t}-\frac{1}{2\log t}\right)t\,dt<2\int_{r_1}^{2r}\left(\frac{1}{\log t}-\frac{1}{2(\log t)^2}\right)t\,dt$$

$$=\frac{4r^2}{\log 2r}-\frac{r_1^2}{\log r_1},$$

we have

$$A(r)=o\left(\frac{r^2}{\log r}\right). \tag{14}$$

μ is a fixed number still to be chosen. It follows from (12), (13), and (14) that

$$I_K(\mu^2 r)=o\{T(r)\}.$$

Hence, writing μ^2 in place of μ in (9),

$$I_K(r)<\{1-\mu^4+o(1)\}\,T(r)<5(1-\mu)\,T(r)$$

for all sufficiently large r. Hence, by (8),

$$\overline{\lim_{r=\infty}}\frac{I_K(\alpha r, r)}{T(r)}<\frac{5(1-\mu)}{1-\alpha^2}.$$

If we put $\alpha = \rho/(\rho+1)$ and $\mu = 1-\epsilon$, where ϵ is arbitrarily small, the lemma follows.

4. Consider the class of functions

$$p(z) = p_0 + p_1 z + \dots$$

which are regular and do not take the values 0 or 1 in the circle $|z| < R$. Schottky proved that under these conditions, given any positive number $\theta < 1$, there is a finite number $S^*(p_0, \theta)$, depending only on $|p_0|$ and θ, such that

$$|p(z)| \leqslant S^*(p_0, \theta)$$

for all $|z| < R$. Since the date of Schottky's paper much research has been devoted to the study of this upper bound $S^*(p_0, \theta)$. When we are dealing with a single function $p(z)$ we are generally interested in $S^*(p_0, \theta)$ as a function of θ and it is this aspect of the problem that has received most attention hitherto. When, however, as here, we are concerned with a family of functions $p_n(z)$ the properties of $S^*\{p_n(0), \theta\}$, regarded as a function of the two variables $p_n(0)$ and θ, become important. Quite recently, Ostrowski† has published an exhaustive investigation in which he has given very precise asymptotic values for $S^*(p_0, \theta)$, viz.

$$16 S^*(p_0, \theta) \cong (16|p_0|)^{(1+\theta)/(1-\theta)} \quad \text{as} \quad |p_0| \to \infty$$

and uniformly in θ, and

$$\frac{S^*(p_0, \theta)}{16} \simeq \left|\frac{p_0}{16}\right|^{(1+\theta)/(1-\theta)} \quad \text{as} \quad |p_0|^{1-\theta} \to 0,$$

and uniformly in $|p_0|$ and θ. He has also obtained equally precise *lower bounds* for $|p(z)|$ in terms of $|p_0|$ and θ for large and small values of $|p_0|$. The result which we actually require here can be stated as

LEMMA 3. *There exist absolute constants A and B such that, if $|p_0| > A$,*

$$\frac{1-\theta}{1+\theta} - \frac{B}{\log|p_0|} \leqslant \frac{\log|p(z)|}{\log|p_0|} \leqslant \frac{1+\theta}{1-\theta} + \frac{B}{\log|p_0|}, \tag{15}$$

for all $|z| < \theta R$ and $\theta \leqslant \frac{1}{2}$.

The lemma shows that a function $p(z)$ is "flat" in any circle $|z| \leqslant \theta R$ provided that $|p_0|$ is sufficiently large. We show in the next paragraph

† *Comm. Math. Helv.*, 5 (1933), 55–87. We have extracted only a small part of the content of his *V**, p. 85.

that there is a sequence of circles in which the meromorphic function $f(z)$ behaves like a $p(z)$, and we apply Lemma 3 to show that $f(z)$ is "flat" in these circles.

5. *Proof of Theorem* 1. Let $K_1, K_2, \ldots, K_\nu, \ldots$ be a sequence of positive numbers, for the time being quite arbitrary. Let us take any number K_ν of this sequence and form the function $\phi_{K_\nu}(z)$ depending only on K_ν and $f(z)$. Now, writing $\alpha = \rho/(\rho+1)$ and combining Lemmas 1 and 2, we have

$$\overline{\lim_{r=\infty}} \frac{\frac{1}{\pi r^2(1-\alpha^2)} \int_{\alpha r}^{r} \int_{-\pi}^{\pi} \overset{+}{\log} |f(te^{i\theta})| \{1-\phi_{K_\nu}(te^{i\theta})\} t\, dt\, d\theta}{T(r)}$$

$$= \overline{\lim_{r=\infty}} \frac{I(\alpha r, r) - I_{K_\nu}(\alpha r, r)}{T(r)} > \frac{\delta}{e}.$$

We can thus find a value $r(\nu)$ for which

$$\frac{1}{\pi r(\nu)^2(1-\alpha^2)} \int_{\alpha r(\nu)}^{r(\nu)} \int_{-\pi}^{\pi} \overset{+}{\log} |f(te^{i\theta})| \{1-\phi_{K_\nu}(te^{i\theta})\} t\, dt\, d\theta > \frac{\delta}{e} T\{r(\nu)\}.$$

On the left-hand side of this inequality we have the mean value of the non-negative function $\overset{+}{\log} |f(z)| \{1-\phi_{K_\nu}(z)\}$ taken over the ring

$$\alpha r(\nu) \leqslant |z| \leqslant r(\nu).$$

It follows that this ring contains a point ξ_ν at which

$$\{1-\phi_{K_\nu}(\xi_\nu)\} \overset{+}{\log} |f(\xi_\nu)| > \frac{\delta}{e} T\{r(\nu)\} \geqslant \frac{\delta}{e} T(|\xi_\nu|) > 0.$$

It follows that

$$\phi_{K_\nu}(\xi_\nu) = 0,$$

and so

$$\overset{+}{\log} |f(\xi_\nu)| > \frac{\delta}{e} T(|\xi_\nu|). \tag{16}$$

By the definition of $\phi_{K_\nu}(z)$, a point ξ_ν, at which $\phi_{K_\nu}(\xi_\nu) = 0$, lies in a quadrilateral which together with its neighbours forms a region $R(\xi_\nu)$ in which there are no zeros of $f(z)$, $f(z)-1$, or $1/f(z)$. Plainly the circle $C(\xi_\nu)$, defined by

$$|z-\xi_\nu| \leqslant \frac{K_\nu}{2} |\xi_\nu|^{1-\frac{1}{2}\rho'},$$

lies wholly in $R(\xi_\nu)$ if, as we may suppose, $|\xi_\nu|$ is sufficiently large. Choose

$\theta_\nu < \frac{1}{2}$ and apply Lemma 3 to $f(z)$ in $C(\xi_\nu, \theta_\nu)$ defined by

$$|z-\xi_\nu| \leqslant \frac{K\theta_\nu}{2} |\xi_\nu|^{1-\frac{1}{2}\rho'}.$$

This gives

$$\frac{1-\theta_\nu}{1+\theta_\nu} - \frac{B}{\log|f(\xi_\nu)|} < \frac{\log|f(z)|}{\log|f(\xi_\nu)|} < \frac{1+\theta_\nu}{1-\theta_\nu} + \frac{B}{\log|f(\xi_\nu)|}$$

for all z in $C(\xi_\nu, \theta_\nu)$.

Now take the number $K_{\nu+1}$ of our sequence and form the function $\phi_{K_{\nu+1}}(z)$ depending only on $K_{\nu+1}$ and $f(z)$. Arguing precisely as before, we can find a point $\xi_{\nu+1}$ such that

$$|\xi_{\nu+1}| > 2|\xi_\nu|; \quad \phi_{K_{\nu+1}}(\xi_{\nu+1}) = 0;$$

$$\overset{+}{\log}|f(\xi_{\nu+1})| > \frac{\delta}{e} T(|\xi_{\nu+1}|).$$

The circle $C(\xi_{\nu+1})$, defined by

$$|z-\xi_{\nu+1}| \leqslant \frac{K_{\nu+1}}{2} |\xi_{\nu+1}|^{1-\frac{1}{2}\rho'},$$

contains no zeros of $f(z)$, $f(z)-1$, or $1/f(z)$. Choosing $\theta_{\nu+1} < \frac{1}{2}$ and applying Lemma 3, we have

$$\frac{1-\theta_{\nu+1}}{1+\theta_{\nu+1}} - \frac{B}{\log|f(\xi_{\nu+1})|} < \frac{\log|f(z)|}{\log|f(\xi_{\nu+1})|} < \frac{1+\theta_{\nu+1}}{1-\theta_{\nu+1}} + \frac{B}{\log|f(\xi_{\nu+1})|}$$

for all z in the circle $C(\xi_{\nu+1}, \theta_{\nu+1})$ defined by

$$|z-\xi_{\nu+1}| \leqslant \frac{K_{\nu+1}\theta_{\nu+1}}{2} |\xi_{\nu+1}|^{1-\frac{1}{2}\rho'}.$$

In the same way we can find a point $\xi_{\nu+2}$ such that

$$|\xi_{\nu+2}| > 2|\xi_{\nu+2}|; \quad \phi_{K_{\nu+2}}(\xi_\nu) = 0;$$

$$\overset{+}{\log}|f(\xi_{\nu+2})| > \frac{\delta}{e} T(|\xi_{\nu+2}|),$$

and, given a positive $\theta_{\nu+2} < \frac{1}{2}$, $f(z)$ satisfies the inequalities

$$\frac{1-\theta_{\nu+2}}{1+\theta_{\nu+2}} - \frac{B}{\log|f(\xi_{\nu+2})|} < \frac{\log|f(z)|}{\log|f(\xi_{\nu+2})|} < \frac{1+\theta_{\nu+2}}{1-\theta_{\nu+2}} + \frac{B}{\log|f(\xi_{\nu+2})|}$$

in the circle $C(\xi_{\nu+2}, \theta_{\nu+2})$ defined by

$$|z-\xi_{\nu+2}| \leqslant \frac{K_{\nu+2}\theta_{\nu+2}}{2} |\xi_{\nu+2}|^{1-\frac{1}{2}\rho'}.$$

The sequences $K_1, K_2, \ldots, K_\nu, \ldots$ and $\theta_1, \theta_2, \ldots, \theta_\nu, \ldots$ are at our disposal and we can choose them so that

$$\theta_\nu < \tfrac{1}{2}; \quad \lim_{\nu=\infty} \theta_\nu = 0; \quad \lim_{\nu=\infty} K_\nu \theta_\nu = \infty.$$

For example, let us take

$$K_\nu = \nu, \quad \theta_\nu = (\nu+4)^{-\frac{1}{2}},$$

and repeat the above argument for all values of $\nu \geqslant 1$. In this way we obtain a sequence of points $\xi_1, \xi_2, \ldots, \xi_\nu, \ldots$ such that, for all values of ν,

$$2|\xi_\nu| < |\xi_{\nu+1}|,$$

and in the circle C_ν, defined by

$$|z-\xi_\nu| \leqslant \frac{\nu}{\sqrt{(\nu+4)}} |\xi_\nu|^{1-\frac{1}{2}\rho'} = r_\nu,$$

the function $f(z)$ satisfies the inequalities

$$1-\epsilon_\nu < \frac{\log|f(2)|}{\log|f(\xi_\nu)|} < 1+\epsilon_\nu, \tag{4}$$

where† $$\lim_{\nu=\infty} \epsilon_\nu = 0.$$

Moreover, we have

$$r_\nu |\xi_\nu|^{\frac{1}{2}\rho'-1} = \frac{\nu}{\sqrt{(\nu+4)}} \to \infty \quad \text{as} \quad \nu \to \infty,$$

and

$$\frac{\delta}{e} T(|\xi_\nu|, f) < \log|f(\xi_\nu)|.$$

† We have

$$\frac{1-\theta_\nu}{1+\theta_\nu} = 1 - \frac{2\theta_\nu}{1+\theta_\nu} > 1-4\theta_\nu = 1 - \frac{4}{\sqrt{(\nu+4)}} > 1 - \frac{4}{\sqrt{\nu}},$$

$$\frac{1+\theta_\nu}{1-\theta_\nu} = 1 + \frac{2\theta_\nu}{1-\theta_\nu} < 1+4\theta_\nu = 1 + \frac{4}{\sqrt{(\nu+4)}} < 1 + \frac{4}{\sqrt{\nu}},$$

since $\theta_\nu < \frac{1}{2}$. Further, since $|\xi_\nu| > 2^{\nu-1}|\xi_1|$,

$$\frac{B}{\log|f(\xi_\nu)|} < \frac{eB}{\delta T(|\xi_\nu|)} = o\left(\frac{1}{\log|\xi_\nu|}\right) = o\left(\frac{1}{\nu}\right).$$

Hence, putting $\epsilon_\nu = 5/\sqrt{\nu}$, we have, for all large ν,

$$1-\epsilon_\nu < \frac{1-\theta_\nu}{1+\theta_\nu} - \frac{B}{\log|f(\xi_\nu)|} < \frac{\log|f(z)|}{\log|f(\xi_\nu)|} < \frac{1+\theta_\nu}{1-\theta_\nu} + \frac{B}{\log|f(\xi_\nu)|} < 1+\epsilon_\nu.$$

To complete the proof of the theorem, we require to find an upper bound for $\log|f(\xi_\nu)|$. Let D_ν be the part of C_ν lying in the circle $|z| \leqslant |\xi_\nu|$. Plainly the area of D_ν is greater than $\pi r_\nu^2/4$. It follows from this and from (4) that the contribution to $I\{\rho|\xi_\nu|/(1+\rho), |\xi_\nu|\}$ made by D_ν is greater than

$$\frac{(1+\rho)^2}{4(1+2\rho)}\left(\frac{r_\nu}{|\xi_\nu|}\right)^2 \log|f(\xi_\nu)|(1-\epsilon_\nu);$$

and so this is less than

$$I\left(\frac{\rho|\xi_\nu|}{1+\rho},\ |\xi_\nu|\right) < T(|\xi_\nu|,\ f).$$

Hence, recalling that

$$r_\nu = \frac{\nu}{\sqrt{(\nu+4)}}\,|\xi_\nu|^{1-\frac{1}{2}\rho'},$$

we have

$$\log|f(\xi_\nu)| < \frac{4(1+2\rho)}{(1+\rho)^2}\,\frac{\nu+4}{\nu^2(1-\epsilon_\nu)}\,|\xi_\nu|^{\rho'}\,T(|\xi_\nu|,\ f)$$

$$< |\xi_\nu|^{\rho'}\,T(|\xi_\nu|,\ f),$$

for all sufficiently large values of ν. This completes the proof of Theorem 1.

We may observe that for an integral function Theorem 1 does not reduce to Whittaker's theorem. So far as flatness is concerned, our result is the more precise of the two, and there is a difference in the lower bounds of $\log|f(\xi_\nu)|$, though they are of the same order.

6. As a corollary of Theorem 1, we have a theorem about meromorphic functions with exceptional values, in the sense of R. Nevanlinna, which is not without interest. It is

THEOREM 2. *Hypothesis.* $f(z)$ *is a meromorphic function of finite order* $\rho > 0$; $\rho' \geqslant \rho$ *is any number such that*

$$T(r,\ f) = o\left(\frac{r^{\rho'}}{\log r}\right),$$

and the zeros of $f(z)-x$ *are of positive deficiency* δ, *so that*

$$\overline{\lim_{r=\infty}}\,\frac{N(r,\ x)}{T(r,\ f)} \leqslant 1-\delta.$$

Assertion. There is a sequence of circles C_ν, *satisfying the same conditions as in Theorem* 1, *such that, for all* z *in* C_ν,

$$\log|f(z)-x| < -\frac{\delta}{e}\,T(|\xi_\nu|,\ f). \tag{19}$$

Proof. This is immediate. The function

$$\frac{1}{f(z)-x}$$

satisfies the conditions of Theorem 1 and, as is well known,

$$T\left(r, \frac{1}{f-x}\right) = T(r, f)+O(1).$$

The assertion (19) then follows immediately from (4) and (5).

We may say that $|f(z)-x|\to 0$ in the sequence of circles C_ν like $\exp\{-(\delta/e)\,T(|\xi_\nu|, f)\}$. It is, of course, trivial that every large circle $|z|=r$ has at least one arc on which

$$\log|f(z)-x| < -\delta'\,T(|z|, f) \quad (\text{any } \delta' < \delta), \tag{20}$$

but nothing has hitherto been known *in general* about the order of magnitude of these individual arcs or of the regions containing them in which (20) is satisfied. It has been plausibly conjectured that x is an asymptotic value of $f(z)$, though this has only been proved in very special cases, as, for example, when x is exceptional in the sense of Picard. Theorem 2 may be regarded as related to this order of ideas, since it gives a sequence of regions of ascertained dimensions in which $f(z)$ tends to the deficient value x.

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THE RIEMANNIAN THEORY OF HANKEL TRANSFORMS

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1. This paper contains four theorems on the Riemannian theory of Hankel transforms, and is the extension of corresponding work by Pollard* on Fourier transforms. The idea of extending Pollard's work was first suggested to me by Prof. Hardy, to whom I am deeply indebted for many valuable suggestions and criticisms.

The Hankel repeated integral

$$f(x) = \int_0^\infty J_\nu(xy)\, y\, dy \int_0^\infty J_\nu(yt)\, t\, f(t)\, dt \tag{1.1}$$

gives rise to the reciprocity formulae

$$f(x) = \int_0^\infty \sqrt{(xy)}\, J_\nu(xy)\, g(y)\, dy, \tag{1.11}$$

$$g(x) = \int_0^\infty \sqrt{(xy)}\, J_\nu(xy)\, f(y)\, dy \tag{1.12}$$

connecting two functions f and g, and each of the two functions so connected is said to be the *Hankel transform* of the other.

Theorems 1 and 2 take the form "if (1.11) is true for all positive x, the integral being a Cauchy integral at infinity, and f and g satisfy certain conditions, then (1.12) is true for almost all y, the integral being a Cesàro first mean integral".

In Theorem 1, g is L in $(0, \infty)$, and in Theorem 2, g is L in $(0, 1)$ and g/x^2 is L in $(1, \infty)$, while $x^{\nu \mp \frac{1}{2}} f(x)$ has a finite integral in every finite interval.

* See Pollard, 2.

Theorem 3 is a deduction from Theorem 2, and Theorem 4 is an extension of Theorem 2 to the case when (1.11) ceases to be true for a finite number of values of x.

2. THEOREM 1. *If, for all positive values of x,*

$$f(x) = \int_0^\infty g(y) \sqrt{(xy)}\, J_\nu(xy)\, dy, \tag{2.1}$$

where $\nu \geqslant -\frac{1}{2}$, the integral being defined as a Cauchy integral at infinity, and

$$A = \int_0^\infty |g(y)|\, dy \tag{2.11}$$

is finite, then, for almost all positive values of y,

$$g(y) = \int_0^\infty f(x) \sqrt{(xy)}\, J_\nu(xy)\, dx, \tag{2.2}$$

the integral being a Cesàro integral.

The integral in (2.1), being a Cauchy* integral, is the limit of a Lebesgue integral. Thus (2.1) is equivalent to

$$f(x) = \lim_{\lambda \to \infty} \int_0^\lambda g(y) \sqrt{(xy)}\, J_\nu(xy)\, dy. \tag{2.3}$$

The integral in (2.2) is a Cesàro integral, so that (2.2) is equivalent to

$$g(y) = \lim_{\lambda \to \infty} \frac{1}{\lambda} \int_0^\lambda dt \int_0^t f(x) \sqrt{(xy)}\, J_\nu(xy)\, dx. \tag{2.4}$$

Since, for all values of x and y considered, $\sqrt{(xy)}\, J_\nu(xy)$ is uniformly bounded, we see that (2.11) ensures the absolute convergence of the integral in (2.3). $f(x)$ is therefore continuous, and the integral in (2.4) certainly exists. Let

$$\begin{aligned} J &= \frac{1}{\lambda} \int_0^\lambda dt \int_0^t f(x) \sqrt{(sx)}\, J_\nu(sx)\, dx \\ &= \frac{1}{\lambda} \int_0^\lambda dt \int_0^t \sqrt{(sx)}\, J_\nu(sx)\, dx \int_0^\infty g(y) \sqrt{(xy)}\, J_\nu(xy)\, dy. \end{aligned} \tag{2.5}$$

* The introduction of Cauchy integrals in this theorem is unnecessary since $f(x)$ is finite on account of (2.11), the integral being an L integral, but we shall want this form of the theorem in the proof of Theorem 2.

Since $|J| < K\lambda$, where K is an absolute constant, we may change the order of integration as we please. Thus

$$(2.6) \qquad J = \frac{\sqrt{s}}{\lambda} \int_0^\infty \sqrt{y}\, g(y)\, dy \int_0^\lambda dt \int_0^t x\, J_\nu(sx)\, J_\nu(yx)\, dx.$$

It is important to notice that the value of this integral depends only on that part of the y-range of integration which is in the immediate neighbourhood of s. We thus write

$$(2.61) \qquad J = \int_0^{s-\eta} \int_0^\lambda \int_0^t + \int_{s-\eta}^{s+\eta} \int_0^\lambda \int_0^t + \int_{s+\eta}^\infty \int_0^\lambda \int_0^t$$

$$= J_1 + J_2 + J_3.$$

I prove that, as λ tends to infinity, J_2 tends to $g(s)$ for almost all values of s concerned, and that J_1 and J_3 tend to zero. Again, J_2 is dominated entirely by the asymptotic behaviour of the Bessel functions, and we write

$$(2.7) \quad J_2 = \frac{\sqrt{s}}{\lambda} \int_{s-\eta}^{s+\eta} \sqrt{y}\, g(y)\, dy \int_0^\lambda \frac{t\{J_\nu(st)\,(d/dt)\,J_\nu(yt) - J_\nu(yt)\,(d/dt)\,J_\nu(st)\}}{s^2-y^2}\, dt$$

$$= \frac{\sqrt{s}}{\lambda} \int_{s-\eta}^{s+\eta} \sqrt{y}\, g(y)\, dy \int_0^\lambda \frac{\sqrt{t}\,J_\nu(st)\,(d/dt)\,\sqrt{t}\,J_\nu(yt) - \sqrt{t}\,J_\nu(yt)\,(d/dt)\,\sqrt{t}J_\nu(st)}{s^2-y^2}\, dt$$

$$= \frac{1}{\lambda} \int_{s-\eta}^{s+\eta} g(y)\, dy \times \int_0^\lambda \frac{y\sqrt{(st)}\,J_\nu(st)\,[\sqrt{(yt)}\,J_\nu(yt)]' - s\sqrt{(yt)}\,J_\nu(yt)\,[\sqrt{(st)}\,J_\nu(st)]'}{s^2-y^2}\, dt$$

$$= \frac{2}{\pi\lambda} \int_{s-\eta}^{s+\eta} g(y)\, dy \int_0^\lambda \frac{s\cos(yt-\omega)\,\sin(st-\omega) - y\cos(st-\omega)\,\sin(yt-\omega)}{s^2-y^2}\, dt$$

$$+ \frac{1}{\lambda} \int_{s-\eta}^{s+\eta} g(y)\, dy \times \int_0^\lambda \left\{ \frac{y\sqrt{(st)}\,J_\nu(st)\,[\sqrt{(yt)}\,J_\nu(yt)]' - s\sqrt{(yt)}\,J_\nu(yt)\,[\sqrt{(st)}\,J_\nu(st)]'}{s^2-y^2} \right.$$

$$\left. - \frac{2}{\pi} \frac{s\cos(yt-\omega)\sin(st-\omega) - y\cos(st-\omega)\sin(yt-\omega)}{s^2-y^2} \right\} dt$$

$$= J_4 + J_5,$$

where

$$\omega = \tfrac{1}{2}\nu\pi + \tfrac{1}{4}\pi. \tag{2.71}$$

Now

$$J_4 = \frac{1}{\pi\lambda}\int_{s-\eta}^{s+\eta} g(y)\,dy \int_0^\lambda \left\{\frac{\sin(s-y)t}{s-y} + \frac{\sin(st+yt-2\omega)}{s+y}\right\} dt \tag{2.72}$$

$$= \frac{2}{\pi\lambda}\int_{s-\eta}^{s+\eta} g(y)\,\frac{\sin^2\tfrac{1}{2}(s-y)\lambda}{(s-y)^2}\,dy$$

$$+ \frac{1}{\pi\lambda}\int_{s-\eta}^{s+\eta} g(y)\,\frac{\cos 2\omega - \cos(s\lambda+y\lambda-2\omega)}{(s+y)^2}\,dy = J_6 + J_7.$$

Suppose now that $s > 0$. Then

$$|J_7| \leqslant \frac{1}{\pi\lambda}\int_{s-\eta}^{s+\eta} \frac{2|g(y)|}{s^2}\,dy \leqslant \frac{2}{\pi s^2\lambda}\int_0^\infty |g(y)|\,dy = \frac{2A}{\pi\lambda s^2} = O\left(\frac{1}{\lambda}\right). \tag{2.73}$$

J_6 is similar to an integral which occurs in the theory of summation of Fourier series, and we know that, under certain conditions,

$$J_6 = g(s) + O_\eta\left(\frac{1}{\lambda}\right), \tag{2.74}$$

for almost all values of s, the constant in $O(1/\lambda)$ depending on η.

Now, by the first mean value theorem, the inner integrand in J_5 is equal to

$$\Big\{st\,[\sqrt{(Vt)}\,J_\nu(Vt)]'\,[\sqrt{(st)}\,J_\nu(st)]' - Vt\,[\sqrt{(Vt)}\,J_\nu(Vt)]''\,\sqrt{(st)}\,J_\nu(st)$$

$$-\sqrt{(st)}\,J_\nu(st)\,[\sqrt{(Vt)}\,J_\nu(Vt)]' - \frac{2}{\pi}\,st\sin(Vt-\omega)\sin(st-\omega)$$

$$-\frac{2}{\pi}\cos(st-\omega)\sin(Vt-\omega) - \frac{2}{\pi}\,Vt\cos(st-\omega)\cos(Vt-\omega)\Big\}\Big/(s+y),$$

where

$$0 < a \leqslant s-\eta \leqslant V \leqslant s+\eta \leqslant \beta < \infty, \tag{2.75}$$

and ω satisfies (2.71).

We write

$$J_5 = \frac{1}{\lambda}\int_{s-\eta}^{s+\eta}\int_0^k + \frac{1}{\lambda}\int_{s-\eta}^{s+\eta}\int_k^\lambda = J_8 + J_9, \tag{2.76}$$

k being finite.

In J_8 both Vt and st are finite on account of (2.75), so that the inner integrand is equal to

$$\left\{\frac{1}{2^\nu \Gamma(\nu+1)}\right\}^2 (st)^{\nu+\frac{1}{2}}(Vt)^{\nu-\frac{1}{2}}\{(\nu+\tfrac{1}{2})^2-(\nu^2-\tfrac{1}{4})-(\nu+\tfrac{1}{2})\}$$
$$+O(t^{2\nu+2})+O(t)+O(1) = O(1).$$

We see that

$$|J_8| \leqslant \frac{Kk}{\lambda}\int_{s-\eta}^{s+\eta}|g(y)|\,dy = O\left(\frac{1}{\lambda}\right), \tag{2.77}$$

K being a constant.

In J_9, since $Vt > ak$, $st > sk$, we can substitute asymptotic expansions and write

$$\sqrt{\left(\frac{\pi z}{2}\right)}\,J_\nu(z) = \cos(z-\omega) - \frac{4\nu^2-1}{8z}\sin(z-\omega) + O\left(\frac{1}{z^2}\right), \tag{2.78}$$

where z stands in turn for Vt and for st, and ω satisfies (2.71).

The inner integrand is then equal to

$$\frac{4\nu^2-1}{4\pi}\left(\frac{s}{V}-\frac{V}{s}\right)\cos(Vt-\omega)\sin(st-\omega)$$
$$+\frac{4\nu^2-1}{4\pi}\left[\frac{\cos(Vt-\omega)\cos(st-\omega)}{Vt} - \frac{\sin(Vt-\omega)\sin(st-\omega)}{st}\right]$$
$$+\frac{(4\nu^2-1)^2}{32\pi}\left[\frac{\cos(Vt-\omega)\cos(st-\omega)}{Vt} + \frac{\sin(Vt-\omega)\sin(st-\omega)}{st}\right] + O\left(\frac{1}{t}\right)$$
$$= \frac{4\nu^2-1}{4\pi}\,\frac{s^2-V^2}{sV}\cos(Vt-\omega)\cos(st-\omega) + O\left(\frac{1}{t}\right) = O(1).$$

Thus

$$|J_9| \leqslant K\,\frac{\lambda-k}{\lambda}\int_{s-\eta}^{s+\eta}|g(y)|\,dy = o(1), \tag{2.79}$$

as $\eta \to 0$, K being a constant.

Summing up, we have, on account of (2.74), (2.73), (2.77), and (2.79),

$$J_2 = g(s) + O_\eta\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{\lambda}\right) + o(1), \tag{2.8}$$

for almost all values of s, the last term tending to zero with η.

Having first fixed η, we see that

$$J_2 = g(s) + o(1), \tag{2.81}$$

the last term tending to zero as λ tends to infinity.

In order to prove that J_1 and J_3 tend to zero as λ tends to infinity, we appeal to Lebesgue's convergence theorem. Take J_1, which we can write in the form

$$J_1 = \frac{\sqrt{s}}{\lambda}\int_0^{s-\eta} \frac{\sqrt{y}\, g(y)\, dy}{s^2-y^2} \int_0^\lambda t\{s\, J_{\nu+1}(st)\, J_\nu(yt) - y\, J_\nu(st)\, J_{\nu+1}(yt)\}\, dt \tag{2.9}$$

$$= \int_0^{s-\eta} g(y)\, \phi(y, \lambda)\, dy,$$

where

$$\phi(y,\lambda) = \frac{\sqrt{(sy)}}{\lambda(s^2-y^2)} \int_0^\lambda t\{s\, J_{\nu+1}(st)\, J_\nu(yt) - y\, J_\nu(st)\, J_{\nu+1}(yt)\}\, dt \tag{2.91}$$

$$= \frac{1}{2\lambda}\int_0^\lambda dt \left\{\frac{\sqrt{(st)}\, J_{\nu+1}(st)\sqrt{(yt)}\, J_\nu(yt) - \sqrt{(st)}\, J_\nu(st)\sqrt{(yt)}\, J_{\nu+1}(yt)}{s-y}\right. \tag{2.92}$$

$$\left. + \frac{\sqrt{(st)}\, J_{\nu+1}(st)\sqrt{(yt)}\, J_\nu(yt) + \sqrt{(st)}\, J_\nu(st)\sqrt{(yt)}\, J_{\nu+1}(yt)}{s+y}\right\}.$$

Since $\sqrt{(st)}\, J_{\nu+1}(st)$, $\sqrt{(st)}\, J_\nu(st)$, $\sqrt{(yt)}\, J_{\nu+1}(yt)$, and $\sqrt{(yt)}\, J_\nu(yt)$ are uniformly bounded for all values of s, y, and t considered, we have

$$|\phi(y,\lambda)| \leqslant \frac{1}{2\lambda}\int_0^\lambda \left(\frac{K}{\eta} + \frac{K_1}{s}\right) dt < K_2 \tag{2.93}$$

(since η and s are fixed), K, K_1, and K_2 being constants independent of all the variables concerned. In order to prove that $\phi(y,\lambda)$ tends to zero, when λ tends to infinity, for almost all values of y, we write

$$\phi(y,\lambda) = \int_0^k + \int_k^\lambda = I_1 + I_2, \tag{2.94}$$

k being finite.

As before, we see that

$$|I_1| \leqslant \frac{1}{2\lambda}\int_0^k \left(\frac{K}{\eta}+\frac{K_1}{s}\right)dt = O\left(\frac{1}{\lambda}\right). \tag{2.95}$$

In I_2, since s and y are now fixed, we can substitute asymptotic expansions and, adopting the form for $\phi(y,\lambda)$ in (2.91), we see that the integrand is equal to

$$\begin{aligned}
&\frac{2}{\pi\sqrt{(sy)}}\Big[s\Big\{\sin(st-\omega)+\frac{4\nu^2-1}{8st}\cos(st-\omega)+O\Big(\frac{1}{t^2}\Big)\Big\}\\
&\qquad\times\Big\{\cos(yt-\omega)-\frac{4\nu^2-1}{8yt}\sin(yt-\omega)+O\Big(\frac{1}{t^2}\Big)\Big\}\\
&\quad -y\Big\{\cos(st-\omega)-\frac{4\nu^2-1}{8st}\sin(st-\omega)+O\Big(\frac{1}{t^2}\Big)\Big\}\\
&\qquad\times\Big\{\sin(yt-\omega)+\frac{4\nu^2-1}{8yt}\cos(yt-\omega)+O\Big(\frac{1}{t^2}\Big)\Big\}\Big]\\
&=\frac{1}{\pi\sqrt{(sy)}}\Big[(s+y)\sin(s-y)t+(s-y)\sin(st+yt-2\omega)\\
&\qquad+\frac{4\nu^2-1}{8t}\,\frac{y^2-s^2}{ys}\{\cos(s-y)t-\cos(st+yt-2\omega)\}+O\Big(\frac{1}{t^2}\Big)\Big].
\end{aligned}$$

Thus, since

$$\int_k^\lambda \sin at\,dt,\quad \int_k^\lambda \cos at\,dt,\quad \int_k^\lambda \frac{\sin at}{t}\,dt,\quad \text{and}\quad \int_k^\lambda \frac{\cos at}{t}\,dt$$

are finite, where $a\to 0$, we get

$$I_2 = O(1/\lambda). \tag{2.96}$$

We have, therefore, proved that J_1 tends to zero as λ tends to infinity. Exactly the same argument will prove that J_3 tends to zero as λ tends to infinity.

This establishes (2.2)

3. In (2.74) it was stated that $J_6\to g(s)$, as $\lambda\to\infty$, for almost all values of s. Lebesgue* has shown that this is the case provided that

$$\int_0^h |g(s+y)+g(s-y)-2g(s)|\,dy$$

* See, *e.g.*, Hobson, 1, 561.

has a differential coefficient equal to zero at $h = 0$. This condition will be satisfied whenever both

$$\int_0^h |g(s+y)-g(s)|\,dy \quad \text{and} \quad \int_0^h |g(s-y)-g(s)|\,dy$$

have differential coefficients equal to zero at $h = 0$.

In particular it will be satisfied at a point of continuity of $g(s)$ or, more generally, whenever

$$\{g(s+h)+g(s-h)\} \to 2g(s),$$

as $t \to 0$.

4. Before proceeding to Theorem 2, we require three lemmas—Lemma A, Lemma B, and Lemma C. Lemma A is intimately connected with Theorem 1; Lemmas B and C are lemmas of pure function theory and are extensions of de la Vallée Poussin's* theorems on generalized second derivatives. Theorem 2 depends in the main on Lemma C, which in turn depends on Lemma B.

In the course of our subsequent work, we have often to appeal to an extension of the Riemann-Lebesgue theorem to Bessel functions.

The extension is as follows.

(4.1) THEOREM A. *Let* $\int_a^b |\psi(\theta)|\,d\theta$ *be finite, where* b *may be infinite and* $a \geqslant 0$; *and let* $\nu \geqslant -\frac{1}{2}$. *Then, as* $\lambda \to \infty$,

$$\int_a^b \psi(\theta)\sqrt{\theta}\,J_\nu(\lambda\theta)\,d\theta = o\left(\frac{1}{\sqrt{\lambda}}\right). \tag{4.11}$$

Watson† uses a less general definition of the integral, but in other respects what he proves is rather more general. I give here a short independent proof.

Take first the case $a > 0$.

Then, in the range $a \leqslant \theta \leqslant b$, we have

$$J_\nu(\lambda\theta) = \sqrt{\left(\frac{2}{\pi\lambda\theta}\right)}\cos(\lambda\theta-\omega)+O\left\{\frac{1}{(\lambda\theta)^{\frac{3}{2}}}\right\}, \tag{4.2}$$

* See de la Vallée-Poussin, **3**, 285, Theorem 1; 290, Theorem 5.

† See Watson, **4**, 457.

where ω satisfies (2.71), so that

$$(4.3)\quad \int_a^b \psi(\theta)\sqrt{\theta}\,J_\nu(\lambda\theta)\,d\theta = \sqrt{\left(\frac{2}{\pi\lambda}\right)}\int_a^b \psi(\theta)\cos(\lambda\theta-\omega)\,d\theta + \int_a^b \psi(\theta)\sqrt{\theta}\,O\left\{\frac{1}{(\lambda\theta)^{\frac{3}{2}}}\right\}d\theta = J_1+J_2.$$

Now

$$(4.31)\qquad J_1 = o(1/\sqrt{\lambda}),$$

by the Riemann-Lebesgue theorem itself, and

$$(4.32)\qquad J_2 = O\,\frac{1}{\lambda^{\frac{3}{2}}}\int_a^b \frac{|\psi(\theta)|}{\theta}\,d\theta = O\,\frac{1}{\lambda^{\frac{3}{2}}}\int_a^b |\psi(\theta)|\,d\theta = O\left(\frac{1}{\lambda^{\frac{3}{2}}}\right) = o\left(\frac{1}{\sqrt{\lambda}}\right),$$

so that the theorem is established for $a > 0$.

Take now $a = 0$. Then

$$(4.4)\qquad \int_a^b \psi(\theta)\sqrt{\theta}\,J_\nu(\lambda\theta)\,d\theta = \int_a^{1/\sqrt{\lambda}} + \int_{1/\sqrt{\lambda}}^b = J_3+J_4.$$

Also, since, in J_3,

$$(4.41)\qquad J_\nu(\lambda\theta) = O\{1/\sqrt{(\lambda\theta)}\},$$

and in J_4 we may use (4.2), we have

$$(4.42)\qquad J_3 = O\,\frac{1}{\sqrt{\lambda}}\int_0^{1/\sqrt{\lambda}} |\psi(\theta)|\,d\theta = o\left(\frac{1}{\sqrt{\lambda}}\right),$$

and

$$(4.43)\quad J_4 = \sqrt{\left(\frac{2}{\pi\lambda}\right)}\int_{1/\sqrt{\lambda}}^b \psi(\theta)\cos(\lambda\theta-\omega)\,d\theta + O\,\frac{1}{\lambda^{\frac{3}{2}}}\int_{1/\sqrt{\lambda}}^b \frac{|\psi(\theta)|}{\theta}\,d\theta = J_5+J_6.$$

As before,

$$(4.44)\qquad J_5 = o(1/\sqrt{\lambda}).$$

Also

$$(4.45)\qquad J_6 = O\,\frac{1}{\lambda^{\frac{3}{2}}}\,\lambda^{\frac{1}{2}}\int_{1/\sqrt{\lambda}}^b \psi(\theta)\,d\theta = O\left(\frac{1}{\lambda}\right) = o\left(\frac{1}{\sqrt{\lambda}}\right).$$

Thus the theorem is completely established.

5. LEMMA A. *If, for all positive values of x,*

$$h(x) = \int_0^1 g(y) \sqrt{(xy)}\, J_\nu(xy)\, dy, \tag{5.1}$$

where $\nu \geqslant -\frac{1}{2}$, and

$$A_1 = \int_0^1 |g(y)|\, dy \tag{5.11}$$

is finite, then, for almost all values of y in $0 < y < 1$,

$$g(y) = \int_0^\infty h(x) \sqrt{(xy)}\, J_\nu(xy)\, dx, \tag{5.2}$$

the integral being a Cesàro integral, and, for all values of $y > 1$,

$$0 = \int_0^\infty h(x) \sqrt{(xy)}\, J_\nu(xy)\, dx, \tag{5.3}$$

the integral being a Cauchy integral at infinity.

To prove (5.2), we define $g(y)$ outside the range (0, 1) as being equal to zero. (5.2) then becomes a particular case of Theorem 1. To prove (5.3), which is equivalent to

$$0 = \lim_{\lambda \to \infty} \int_0^\lambda h(x) \sqrt{(xy)}\, J_\nu(xy)\, dx, \tag{5.4}$$

we write

$$\begin{aligned} \int_0^\lambda h(x) \sqrt{(sx)}\, J_\nu(sx)\, dx &= \int_0^\lambda \sqrt{(sx)}\, J_\nu(sx)\, dx \int_0^1 g(y) \sqrt{(xy)}\, J_\nu(xy)\, dy \\ &= \sqrt{s} \int_0^1 \sqrt{y}\, g(y)\, dy \int_0^\lambda x\, J_\nu(sx)\, J_\nu(yx)\, dx \\ &= \sqrt{s} \int_0^1 \frac{\sqrt{y}\, g(y)\, dy}{s^2 - y^2} \{s\lambda J_{\nu+1}(s\lambda) J_\nu(y\lambda) \\ &\qquad - y\lambda J_\nu(s\lambda) J_{\nu+1}(y\lambda)\} \\ &= J_1 - J_2, \end{aligned} \tag{5.5}$$

the change of order of integration being legitimate since A_1 is finite.

Since $s > 1$,

$$\int_0^1 \frac{|g(y)|}{s^2 - y^2}\, dy \leqslant \frac{A_1}{s^2 - 1}, \tag{5.61}$$

$$\int_0^1 \frac{y|g(y)|\, dy}{s^2 - y^2} \leqslant \frac{A_1}{s^2 - 1}, \tag{5.62}$$

so that, by Theorem A, (4.1),

$$J_1 = s^{\frac{3}{2}}\lambda J_{\nu+1}(s\lambda)\, o(1/\sqrt{\lambda}) = o(1), \tag{5.71}$$

$$J_2 = s^{\frac{1}{2}}\lambda J_{\nu}(s\lambda)\, o(1/\sqrt{\lambda}) = o(1), \tag{5.72}$$

as λ tends to infinity.

We have thus established (5.3).

6. Before we proceed to Lemmas B and C, we require a number of definitions and general assertions. In what follows we suppose that $h > 0$. Let*

$$G_\nu(x, h, F) = [(x+2\nu h+h)\, F(x+2h)+(x-2\nu h-h)\, F(x-2h)-2xF(x)]/4h^2. \tag{6.1}$$

When $h \to 0$, $G_\nu(x, h, F)$ has a greatest and least limit (finite or infinite). We call them the "upper and lower generalized second derivatives of order ν" of $F(x)$ at the point x. When they are equal, we call their common value the "generalized second derivative of order ν".

We write

$$\overline{\lim}\, G_\nu(x, h, F) = \overline{G}_2 F, \tag{6.11}$$

$$\underline{\lim}\, G_\nu(x, h, F) = \underline{G}_2 F. \tag{6.12}$$

We also write

$$\overline{\lim}\, \frac{F(x+2h)+F(x-2h)-2F(x)}{4h^2} = \overline{D}_2 F, \tag{6.21}$$

$$\underline{\lim}\, \frac{F(x+2h)+F(x-2h)-2F(x)}{4h^2} = \underline{D}_2 F, \tag{6.22}$$

$$\overline{\lim}\, \frac{F(x+2h)-F(x)}{2h} = \Lambda F, \tag{6.23}$$

$$\underline{\lim}\, \frac{F(x+2h)-F(x)}{2h} = \lambda F, \tag{6.24}$$

$$\overline{\lim}\, \frac{F(x)-F(x-2h)}{2h} = \Lambda' F, \tag{6.25}$$

$$\underline{\lim}\, \frac{F(x)-F(x-2h)}{2h} = \lambda' F. \tag{6.26}$$

* Watson introduced these derivatives in connection with Schlömilch series. See Watson, **4**, 647.

In (6.23)–(6.26) I follow de la Vallée Poussin's* notation.

We have generally

$$\overline{\lim}\,(l_1-l_2) \geqslant \overline{\lim}\,l_1-\overline{\lim}\,l_2, \tag{6.31}$$

$$\underline{\lim}\,(l_1-l_2) \leqslant \underline{\lim}\,l_1-\underline{\lim}\,l_2, \tag{6.32}$$

while

$$\overline{\lim}\,(l_1\pm l_2) = \overline{\lim}\,l_1\pm\lim\,l_2, \tag{6.33}$$

$$\underline{\lim}\,(l_1\pm l_2) = \underline{\lim}\,l_1\pm\lim\,l_2, \tag{6.34}$$

provided that $\lim\, l_2$ exists.

In particular, we see that

$$\begin{aligned}\overline{G_2}\,F = \overline{\lim}\Big[&\frac{x\{F(x+2h)+F(x-2h)-2F(x)\}}{4h^2}\\ &+(\nu+\tfrac{1}{2})\,\frac{F(x+2h)-F(x)}{2h}+(\nu+\tfrac{1}{2})\,\frac{F(x)-F(x-2h)}{2h}\Big]\end{aligned} \tag{6.35}$$

$$= x\,\overline{D_2}\,F+(2\nu+1)\,\frac{dF}{dx},$$

$$\underline{G_2}\,F = x\,\underline{D_2}\,F+(2\nu+1)\,\frac{dF}{dx}, \tag{6.36}$$

provided that dF/dx exists.

We shall take the "chord through A and B" to mean that part of the curve $y = px^{-2\nu}+q$ which goes through A and B, and which lies between A and B.

(6.4) LEMMA B. *If a continuous function $F(x)$ possesses a positive, non-zero, upper generalized second derivative of order ν in the interval (a, b), where $a > 0$, then, in this interval, every arc of the curve $y = F(x)$ lies below its chord.*

If the lower generalized second derivative of order ν is negative, every arc of the curve $y = F(x)$ lies above its chord.

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* See de la Vallée-Poussin, 3, 98.

Let P_1PP_2 be any arc of the curve $y = F(x)$, and let P_1P_2 be its chord. Let the abscissae of P_1, P_2 be x_1, x_2, where $0 < a \leqslant x_1 < x_2 \leqslant b$. Then the equation of the chord P_1P_2 is

$$y - F(x_1) = \frac{x^{-2\nu} - x_1^{-2\nu}}{x_2^{-2\nu} - x_1^{-2\nu}} \{F(x_2) - F(x_1)\}.$$

We shall prove the lemma on the assumption that the upper generalized second derivative of order ν is positive. The same proof will apply to the case where the lower generalized second derivative of order ν is negative.

Suppose, if possible, that the arc of the curve $y = F(x)$ lies, either wholly or partially, above its chord.

Consider the function

$$\phi(x) = F(x) - F(x_1) - \frac{x^{-2\nu} - x_1^{-2\nu}}{x_2^{-2\nu} - x_1^{-2\nu}} \{F(x_2) - F(x_1)\}.$$

In the interval (x_1, x_2), $\phi(x)$ is continuous and

$$\phi(x_1) = \phi(x_2) = 0.$$

Also, by hypothesis, it is possible to find c such that $x_1 < c < x_2$, and such that $\phi(c) > 0$. Now $\phi(x)$ attains its upper bound in (x_1, x_2), and, since $\phi(c) > 0$, this is positive. Let it assume its upper bound at c_1, where $x_1 < c < x_2$.

Since $$\overline{G}_2 k = \underline{G}_2 k = 0,$$

where k is a constant, and

$$\overline{G}_2 x^{-2\nu} = \underline{G}_2 x^{-2\nu} = \lim_{h \to 0} \frac{1}{4h^2} [(x + 2\nu h + h)(x + 2h)^{-2\nu} + (x - 2\nu h - h)(x - 2h)^{-2\nu} - 2x^{1-2\nu}]$$

$$= \lim_{h \to 0} \frac{1}{4h^2} \left[(x + 2\nu h + h) x^{-2\nu} \left\{ 1 - 4\nu \frac{h}{x} + 4\nu(2\nu + 1) \frac{h^2}{x^2} \right\} + (x - 2\nu h - h) x^{-2\nu} \left\{ 1 + 4\nu \frac{h}{x} + 4\nu(2\nu + 1) \frac{h^2}{x^2} \right\} - 2x^{1-2\nu} + O(h^3) \right]$$

$$= 0,$$

we have, by (6.33),

$$\overline{G}_2 \phi(c_1) = \overline{G}_2 F(c_1) > 0.$$

We can thus make $G_\nu(c_1, h, \phi) > 0$, by making h sufficiently small. But since $\phi(c_1+h) \leqslant \phi(c_1)$ and $\phi(c_1-h) \leqslant \phi(c_1)$, we see that, if $h < c_1/(2\nu+1)$, $G_\nu(c_1, h, \phi) \leqslant 0$.

This contradiction shows that no part of the curve $y = F(x)$ can lie above its chord.

7. LEMMA C. *Let $F(x)$ be continuous in (a, b), where $a > 0$. If a function $f(x)$ exists intermediate in value between the two generalized second derivatives of order ν of $F(x)$, or equal to one of them, and if, further, $x^{2\nu}f(x)$ is integrable in (a, b), then, in this interval,*

$$F(x) = \int_a^x u^{-2\nu-1}\,du \int_a^u t^{2\nu} f(t)\,dt + \frac{p}{x^{2\nu}} + q, \tag{7.1}$$

where p and q are constants.

Since $x^{2\nu}f(x)$ is integrable in (a, b), both the integrals

$$\int_a^u t^{2\nu} f(t)\,dt \quad \text{and} \quad \int_a^x u^{-2\nu-1}\,du \int_a^u t^{2\nu} f(t)\,dt$$

exist.

Throughout the proof of this lemma, we shall be using freely the properties of the majorant and minorant introduced by de la Vallée Poussin*. They are defined as follows.

(7.2) *Let $g(x)$ be integrable in (a, b). Then there exist two functions $\phi_1(x)$ and $\phi_2(x)$, both continuous in (a, b), $\phi_1(x)$ being greater, and $\phi_2(x)$ smaller, than $\int_a^x g(u)\,du$. Moreover, $\phi_1(x)$ and $\phi_2(x)$ can be made as near as we please to $\int_a^x g(u)\,du$. $\phi_1(x)$ is called the majorant, and $\phi_2(x)$ the minorant, of the integral $\int_a^x g(u)\,du$. Further, $\Lambda\phi_1$, $\lambda\phi_1$, $\Lambda'\phi_1$, and $\lambda'\phi_1$ are all greater than $g(x)$; and $\Lambda\phi_2$, $\lambda\phi_2$, $\Lambda'\phi_2$, and $\lambda'\phi_2$ are all less than $g(x)$.*

We adopt de la Vallée Poussin's† method of proof. Construct, relative to the integral $\int_a^u t^{2\nu} f(t)\,dt$, the majorant $\phi_1(u)$ and the minorant $\phi_2(u)$.

* See de la Vallée Poussin, 3.

† See de la Vallée Poussin, 3, 290–1.

Consider the two new functions

$$F(x)-\int_a^x u^{-2\nu-1}\phi_2(u)\,du, \quad F(x)-\int_a^x u^{-2\nu-1}\phi_1(u)\,du.$$

These two functions both exist because ϕ_1 and ϕ_2 are continuous. The function

$$F(x)-\int_a^x u^{-2\nu-1}\,du\int_a^u t^{2\nu}f(t)\,dt$$

is intermediate in value between them, being smaller than the first and greater than the second. Moreover, we can make either of the first two functions as near as we please to the third function, by proper choice of ϕ_1 and ϕ_2. Construct between the two points of abscissae a and b, the arcs AM_1B_1, AMB, and AM_2B_2 of the three curves

$$y_1 = F(x)-\int_a^x u^{-2\nu-1}\phi_2(u)\,du = F(x)-q_1,$$

$$y_2 = F(x)-\int_a^x u^{-2\nu-1}\,du\int_a^u t^{2\nu}f(t)\,dt = F(x)-q_2,$$

$$y_3 = F(x)-\int_a^x u^{-2\nu-1}\phi_1(u)\,du = F(x)-q_3.$$

Then $y_1 > y_2 > y_3$, and the three chords AB_1, AB, and AB_2 can be made as near to one another as we please, with AB below AB_1 and above AB_2.

We know, from the properties of the minorant enunciated in (7 . 2), that

(7 . 3) $\Lambda\phi_2$, $\lambda\phi_2$, $\Lambda'\phi_2$, and $\lambda'\phi_2$ are all less than $x^{2\nu}f(x)$.

Also, q_1 being continuous and possessing a continuous derivative $x^{-2\nu-1}\phi_2(x)$, we know that

(7 . 4) $\overline{D_2}q_1$ is intermediate among Λ, λ, Λ', and λ' of $x^{-2\nu-1}\phi_2(x)$.

This follows from a theorem of de la Vallée Poussin*. Also

$$(7\,.\,5)\qquad \Lambda\{x^{-2\nu-1}\phi_2(x)\} = \overline{\lim_{h\to 0}}\,\frac{(x+2h)^{-2\nu-1}\phi_2(x+2h)-x^{-2\nu-1}\phi_2(x)}{2h}$$

$$= x^{-2\nu-1}\Lambda\phi_2-(2\nu+1)x^{-2\nu-1}\phi_2,$$

* See de la Vallée Poussin, 3, 285.

by (6.33), since ϕ_2 is continuous, and similar formulae hold for λ, Λ', and λ' of $x^{-2\nu-1}\phi_2(x)$. From (7.4) we see that $\overline{D_2}q_1$ is intermediate among

$$x^{-2\nu-1}\Lambda\phi_2-(2\nu+1)x^{-2\nu-2}\phi_2,\quad x^{-2\nu-1}\lambda\phi_2-(2\nu+1)x^{-2\nu-2}\phi_2,$$

$$x^{-2\nu-1}\Lambda'\phi_2-(2\nu+1)x^{-2\nu-2}\phi_2,\quad \text{and}\quad x^{-2\nu-1}\lambda'\phi_2-(2\nu+1)x^{-2\nu-2}\phi_2,$$

and we have, by (7.3),

$$\overline{D_2}q_1\leqslant x^{-1}f(x)-(2\nu+1)x^{-2\nu-2}\phi_2(x). \tag{7.6}$$

Also, from (6.35), we have

$$\overline{G_2}q_1=x\overline{D_2}q_1+(2\nu+1)\frac{dq_1}{dx}, \tag{7.7}$$

because dq_1/dx exists and is equal to $x^{-2\nu-1}\phi_2(x)$ (ϕ_2 being continuous).

Hence

$$\overline{G_2}q_1\leqslant f(x)-(2\nu+1)x^{-2\nu-1}\phi_2(x)+(2\nu+1)x^{-2\nu-1}\phi_2(x)=f(x). \tag{7.81}$$

Finally, from (6.31), we have

$$\overline{G_2}y_1\geqslant\overline{G_2}F-\overline{G_2}q_1\geqslant f(x)-f(x)=0. \tag{7.82}$$

We can prove in the same way that

$$\underline{G_2}y_3\leqslant 0. \tag{7.9}$$

Now apply Lemma B. From (7.82) we see that the arc AM_1B_1 lies below its chord AB_1; and from (7.9) we see that the arc AM_2B_2 lies above its chord AB_2. The arc AMB will, *a fortiori*, lie between the two chords AB_1 and AB_2. Thus the invariable arc AMB lies between two variable chords AB_1 and AB_2 which can be made as near the chord AB as we please. The arc AMB must be, therefore, identical with its chord, and if $y=px^{-2\nu}+q$ is the equation of the chord, we get

$$F(x)-\int_a^x u^{-2\nu-1}\,du\int_a^u t^{2\nu}f(t)\,dt=\frac{p}{x^{2\nu}}+q,$$

which proves the lemma.

8. THEOREM 2. *If, for all positive values of x,*

$$f(x)=\int_0^\infty g(y)\sqrt{(xy)}\,J_\nu(xy)\,dy, \tag{8.1}$$

where $\nu \geqslant -\frac{1}{2}$, *the integral being defined as a Cauchy integral at infinity, and*

$$A_\infty = \int_1^\infty \frac{|g(y)|}{y^2}\, dy, \tag{8.11}$$

$$A_1 = \int_0^1 |g(y)|\, dy$$

are finite, and, further, $x^{\nu+\frac{1}{2}} f(x)$ *has a finite integral in every finite interval, then, for almost all positive values of* y,

$$g(y) = \int_0^\infty f(x) \sqrt{(xy)}\, J_\nu(xy)\, dx, \tag{8.2}$$

the integral being a Cesàro integral.

We begin by assuming that, for $0 \leqslant y < 1$,

$$g(y) = 0. \tag{8.21}$$

Then the integral

$$-\int_0^\infty \frac{g(y)}{y^2} \sqrt{(xy)}\, J_\nu(xy)\, dy$$

converges absolutely for all finite, positive values of x and represents a continuous function of x which we will call $x^{\nu+\frac{1}{2}} F(x)$. It is to be noted that $F(x)$ itself is continuous in any interval $0 < a \leqslant x \leqslant b$ of values of x.

By Theorem 1, for almost all positive values of y,

$$-\frac{g(y)}{y^2} = \lim_{\lambda\to\infty} \frac{1}{\lambda} \int_0^\lambda dt \int_0^t x^{\nu+\frac{1}{2}} F(x) \sqrt{(xy)}\, J_\nu(xy)\, dx,$$

or

$$-\frac{g(y)}{y^{\frac{5}{2}}} = \lim_{\lambda\to\infty} \frac{1}{\lambda} \int_0^\lambda dt \int_0^t x^{\nu+1} F(x)\, J_\nu(xy)\, dx. \tag{8.3}$$

Further, since, on account of (8.21), the integral

$$\int_0^\infty \frac{|g(y)|}{y^2}\, dy$$

is finite, we have, by Theorem A, (4.1),

$$x^{\nu+\frac{1}{2}} F(x) = o(1), \tag{8.4}$$

as $x \to \infty$.

We shall now prove the following Lemma.

(8.5) LEMMA D. *For every positive value of* x,

$$\lim_{h\to 0} G_\nu(x, h, F) = x^{\frac{1}{2}-\nu} f(x), \tag{8.51}$$

$G_\nu(x, h, F)$ being defined as in (6.1)

By observing that

$$[x\pm(2\nu+1)h - x^{\frac{1}{2}-\nu}(x\pm 2h)^{\nu+\frac{1}{2}}][x\pm(2\nu+1)h + x^{\frac{1}{2}-\nu}(x\pm 2h)^{\nu+\frac{1}{2}}]$$

$$= -(4\nu^2-1)h^2 - x^2 \sum_{m=3}^{\infty} \binom{2\nu+1}{m} \left(\pm\frac{2h}{x}\right)^m$$

$$= -(4\nu^2-1)h^2 - S(\pm h) \quad \text{(say)},$$

we see that

$$x\pm(2\nu+1)h = -\frac{(4\nu^2-1)h^2+S(\pm h)}{x\pm(2\nu+1)h+x^{\frac{1}{2}-\nu}(x\pm 2h)^{\nu+\frac{1}{2}}} + x^{\frac{1}{2}-\nu}(x\pm 2h)^{\nu+\frac{1}{2}},$$

so that

$$G_\nu(x, h, F) \tag{8.52}$$

$$= \int_0^\infty \frac{g(y)\,dy}{y\sqrt{y}} \left[\frac{(4\nu^2-1)h^2+S(h)}{4h^2} \cdot \frac{1}{x+(2\nu+1)h+x^{\frac{1}{2}-\nu}(x+2h)^{\nu+\frac{1}{2}}} \cdot \frac{J_\nu(xy+2yh)}{(x+2h)^\nu} \right.$$

$$\left. + \frac{(4\nu^2-1)h^2+S(-h)}{4h^2} \cdot \frac{1}{x-(2\nu+1)h+x^{\frac{1}{2}-\nu}(x-2h)^{\nu+\frac{1}{2}}} \cdot \frac{J_\nu(xy-2yh)}{(x-2h)^\nu} \right]$$

$$- \frac{x^{\frac{1}{2}-\nu}}{4h^2} \int_0^\infty \frac{g(y)\,dy}{y\sqrt{y}} \left[\sqrt{(x+2h)}\, J_\nu(xy+2yh) + \sqrt{(x-2h)}\, J_\nu(xy-2yh) \right.$$

$$\left. - 2\sqrt{x}\, J_\nu(xy) \right]$$

$$= J_1 + J_2.$$

We take h so small that it does not exceed cx, where $c < 1/(2\nu+3)$; this ensures that $x-2h$ and $x-(2\nu+1)h$ are both positive. We thus see that

$$\frac{|(4\nu^2-1)h^2+S(\pm h)|}{4h^2} = \left| \nu^2 - \tfrac{1}{4} + \frac{2h}{x} \sum_{m=3}^{\infty} \binom{2\nu+1}{m} \left(\pm\frac{2h}{x}\right)^{m-3} \right| < k_1,$$

$$|x\pm(2\nu+1)h + x^{\frac{1}{2}-\nu}(x\pm 2h)^{\nu+\frac{1}{2}}| > k_2 x,$$

and

$$\frac{|J_\nu(xy\pm 2yh)|}{(x\pm 2h)^\nu} = \frac{|\sqrt{(xy\pm 2yh)}\, J_\nu(xy\pm 2yh)|}{y^{\frac{1}{2}}(x\pm 2h)^{\nu+\frac{1}{2}}} < \frac{k_3}{y^{\frac{1}{2}} x^{\nu+\frac{1}{2}}},$$

k_1, k_2, and k_3 being constants independent of x and h, so that

$$|J_1| \leqslant \frac{k_4}{x^{\nu+\frac{3}{2}}} \int_0^\infty \frac{|g(y)|}{y^2}\, dy,$$

k_4 being a constant independent of x and h. Since x is positive, we see that J_1 converges uniformly in h, and we can take the limit under the sign of integration provided that the resulting integral converges.

We thus get

$$\lim_{h\to 0} J_1 = (\nu^2-\tfrac{1}{4})\, x^{-\nu-1} \int_0^\infty \frac{g(y)}{y\sqrt{y}}\, J_\nu(xy)\, dy, \tag{8.53}$$

since this integral converges on account of (8.21) and (8.11).

To deal with J_2 we write

$$\sqrt{\left(\frac{\pi z}{2}\right)}\, J_\nu(z) = \cos(z-\omega) - \frac{4\nu^2-1}{8z} \sin(z-\omega) + \theta(z), \tag{8.54}$$

where ω satisfies (2.71).

The integrand then becomes

$$-\sqrt{\left(\frac{2}{\pi}\right)} x^{\frac{1}{2}-\nu} \frac{g(y)}{4h^2y^2} [\cos(xy+2yh-\omega) + \cos(xy-2yh-\omega) - 2\cos(xy-\omega)]$$

$$+\sqrt{\left(\frac{2}{\pi}\right)} x^{\frac{1}{2}-\nu} \frac{(4\nu^2-1)\, g(y)}{32h^2y^3} \left[\frac{\sin(xy+2yh-\omega)}{x+2h} + \frac{\sin(xy-2yh-\omega)}{x-2h} - \frac{2\sin(xy-\omega)}{x}\right]$$

$$-\sqrt{\left(\frac{2}{\pi}\right)} x^{\frac{1}{2}-\nu} \frac{g(y)}{4h^2y^2} [\theta(xy+2yh) + \theta(xy-2yh) - 2\theta(xy)].$$

The terms in the first bracket are equal to

$$-4\cos(xy-\omega)\sin^2 yh,$$

and the terms in the second bracket are equal to

$$\frac{2x\sin(xy-\omega)\cos 2yh - 4h\cos(xy-\omega)\sin 2yh}{x^2-4h^2} - \frac{2\sin(xy-\omega)}{x}$$

$$= \frac{8h^2\sin(xy-\omega)\cos 2yh}{x(x^2-4h^2)} - \frac{4h\cos(xy-\omega)\sin 2yh}{x^2-4h^2} - \frac{4\sin(xy-\omega)\sin^2 yh}{x},$$

so that

$$(8.55)\quad J_2 = \sqrt{\left(\frac{2}{\pi}\right)}\, x^{\frac{1}{2}-\nu} \times \int_0^\infty g(y)\left[\cos(xy-\omega) - \frac{(4\nu^2-1)\sin(xy-\omega)}{8xy}\right]\left(\frac{\sin yh}{yh}\right)^2 dy$$
$$- \sqrt{\left(\frac{2}{\pi}\right)}\, x^{\frac{1}{2}-\nu}(4\nu^2-1)\int_0^\infty g(y)\,\frac{\cos(xy-\omega)}{8y(x^2-4h^2)}\,\frac{\sin 2yh}{y^2 h}\,dy$$
$$+ \sqrt{\left(\frac{2}{\pi}\right)}\, x^{\frac{1}{2}-\nu}(4\nu^2-1)\int_0^\infty g(y)\,\frac{\sin(xy-\omega)}{4xy(x^2-4h^2)}\,\frac{\cos 2yh}{y^2}\,dy$$
$$- \sqrt{\left(\frac{2}{\pi}\right)}\, x^{\frac{1}{2}-\nu}\int_0^\infty \frac{g(y)\,dy}{4h^2y^2}\,[\theta(xy+2yh)+\theta(xy-2yh)-2\theta(xy)]$$
$$= J_3+J_4+J_5+J_6.$$

Now J_3 is similar to an integral discussed by Pollard*, and it is obvious from his analysis that

$$(8.56)\quad \lim_{h\to 0} K_3 = \sqrt{\left(\frac{2}{\pi}\right)}\, x^{\frac{1}{2}-\nu} \times \int_0^\infty g(y)\left[\cos(xy-\omega) - \frac{(4\nu^2-1)\sin(xy-\omega)}{8xy}\right] dy,$$

provided that this integral converges. That this is so can be seen by writing the integral in the form

$$\sqrt{\left(\frac{2}{\pi}\right)}\, x^{\frac{1}{2}-\nu}\int_0^\infty g(y)\left[\sqrt{\left(\frac{\pi xy}{2}\right)}\, J_\nu(xy) - \theta(xy)\right] dy,$$

and observing that $\int_0^\infty g(y)\sqrt{(xy)}\,J_\nu(xy)\,dy$ converges to $f(x)$, and that

$$\theta(z) = O(1/z^2)$$

as $|z|\to\infty$.

J_4 and J_5 are obviously uniformly convergent from the test of Weierstrass for integrals, and we get

$$(8.57)\quad \lim_{h\to 0} J_4 = -\sqrt{\left(\frac{2}{\pi}\right)}\, x^{\frac{1}{2}-\nu}(4\nu^2-1)\int_0^\infty \frac{g(y)}{4x^2y^2}\cos(xy-\omega)\,dy,$$

$$(8.58)\quad \lim_{h\to 0} J_5 = \sqrt{\left(\frac{2}{\pi}\right)}\, x^{\frac{1}{2}-\nu}(4\nu^2-1)\int_0^\infty \frac{g(y)}{4x^3y^3}\sin(xy-\omega)\,dy,$$

since both of these integrals converge.

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* See Pollard, 2, 457–8.

In J_6 we write

$$\theta(xy+2yh)+\theta(xy-2yh)-2\theta(xy) = 4y^2h^2\theta''(\xi),$$

where $$xy-2yh < \xi < xy+2yh.$$

Since $\theta''(z) = O(1/z^2)$ when z is large, it is evident that J_6 converges uniformly with respect to h, and that

$$\lim_{h\to 0} J_6 = -\sqrt{\left(\frac{2}{\pi}\right)} x^{\frac{1}{2}-\nu} \int_0^\infty g(y)\,\theta''(xy)\,dy, \tag{8.59}$$

since the integral converges.

We thus get

$$\begin{aligned}
&\lim_{h\to 0} G_\nu(x, h, F) \\
&= (\nu^2-\tfrac{1}{4})\,x^{-\nu-1}\int_0^\infty \frac{g(y)}{y\sqrt{y}}\,J_\nu(xy)\,dy \\
&\qquad -x^{\frac{1}{2}-\nu}\int_0^\infty g(y)\,dy\left\{\sqrt{(xy)}\,J_\nu''(xy)+\frac{1}{\sqrt{(xy)}}\,J_\nu'(xy)-\frac{J_\nu(xy)}{4(xy)^{\frac{3}{2}}}\right\} \\
&= (\nu^2-\tfrac{1}{4})\,x^{-\nu-1}\int_0^\infty \frac{g(y)}{y\sqrt{y}}\,J_\nu(xy)\,dy \\
&\qquad -x^{\frac{1}{2}-\nu}\int_0^\infty g(y)\,dy\left\{\frac{\nu^2}{(xy)^{\frac{3}{2}}}\,J_\nu(xy)-\sqrt{(xy)}\,J_\nu(xy)-\frac{J_\nu(xy)}{4(xy)^{\frac{3}{2}}}\right\} \\
&= x^{\frac{1}{2}-\nu}\int_0^\infty g(y)\sqrt{(xy)}\,J_\nu(xy)\,dy = x^{\frac{1}{2}-\nu}f(x).
\end{aligned}$$

This establishes the lemma.

Since $F(x)$ is continuous in (a, b), where $a > 0$, and $x^{2\nu}x^{\frac{1}{2}-\nu}f(x) = x^{\nu+\frac{1}{2}}f(x)$ is integrable in (a, b), it follows from Lemma C, (7), that in (a, b)

$$\begin{aligned}
F(x) &= \int_a^x u^{-2\nu-1}\,du\int_a^u t^{\nu+\frac{1}{2}}f(t)\,dt+\frac{p}{x^{2\nu}}+q \\
&= \int_a^x u^{-2\nu-1}\,du\int_0^u t^{\nu+\frac{1}{2}}f(t)\,dt+\frac{p'}{x^{2\nu}}+q',
\end{aligned}$$

where p, q, p', and q' are constants independent of x, and it is obvious that

in this case $p' = 0$. Also

$$x^{2\nu+\frac{1}{2}} F'(x) = \int_0^x t^{\nu+\frac{1}{2}} f(t)\, dt \tag{8.6}$$

throughout the range $0 \leqslant x \leqslant b$ since the integral converges.

Substitute in (8.3) and integrate by parts. This gives

$$\begin{aligned}
&\int_0^t x^{\nu+1} F(x) J_\nu(xy)\, dx \\
&\quad = \frac{1}{y}\left[x^{\nu+1} J_{\nu+1}(xy) F(x)\right]_0^t - \frac{1}{y}\int_0^t x^{\nu+1} J_{\nu+1}(xy) F'(x)\, dx \\
&\quad = \frac{1}{y}\left[x^{\nu+1} J_{\nu+1}(xy) F(x)\right]_0^t + \frac{1}{y^2}\left[x^{-\nu} J_\nu(xy) x^{2\nu+1} F'(x)\right]_0^t \\
&\qquad\qquad - \frac{1}{y^2}\int_0^t \sqrt{x}\, J_\nu(xy) f(x)\, dx \\
&\quad = \frac{1}{y} t^{\nu+1} J_{\nu+1}(ty) F(t) + \frac{1}{y^2} t^{\nu+1} J_\nu(ty) F'(t) - \frac{1}{y^2}\int_0^t \sqrt{x}\, J_\nu(xy) f(x)\, dx,
\end{aligned}$$

since

$$\left[x^{\nu+1} J_{\nu+1}(xy) F(x)\right]_{x=0} = \left[\sqrt{x}\, J_{\nu+1}(xy) x^{\nu+\frac{1}{2}} F(x)\right]_{x=0} = 0,$$

and

$$\left[x^{-\nu} J_\nu(xy) x^{2\nu+1} F'(x)\right]_{x=0} = \left[x^{-\nu} J_\nu(xy) \int_0^x t^{\nu+\frac{1}{2}} f(t)\, dt\right]_{x=0} = 0.$$

Now

$$t^{\nu+1} J_{\nu+1}(ty) F(t) = \sqrt{t}\, J_{\nu+1}(ty)\, t^{\nu+\frac{1}{2}} F(t) = O(1)\, o(1) = o(1),$$

as $t \to \infty$, by (8.4), and it is obvious that

$$\frac{1}{\lambda}\int_0^\lambda t^{\nu+1} J_{\nu+1}(yt) F(t)\, dt \to 0$$

as $\lambda \to \infty$.

Again

$$\frac{1}{\lambda}\int_0^\lambda t^{\nu+1} J_\nu(ty) F'(t)\, dt = \frac{1}{\lambda}\left[t^{\nu+1} J_\nu(ty) F(t)\right]_0^\lambda - \frac{1}{\lambda}\int_0^\lambda F(t) \frac{d}{dt}\{t^{\nu+1} J_\nu(ty)\}\, dt.$$

The first term tends to zero as before, and the integrand in the second term is

$$F(t)\,t^{\nu}\,J_{\nu}(yt)+F(t)\,yt^{\nu+1}\,J_{\nu-1}(yt)=o\left(\frac{1}{t^{\nu+\frac{1}{2}}}\right)t^{\nu}\,O\left(\frac{1}{t^{\frac{1}{2}}}\right)+o\left(\frac{1}{t^{\nu+\frac{1}{2}}}\right)t^{\nu+1}\,O\left(\frac{1}{t^{\frac{1}{2}}}\right)$$

$$=o\left(\frac{1}{t}\right)+o(1)=o(1),$$

as $t\to\infty$, and the integrand tends to zero as above. Thus

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\int_0^{\lambda}dt\int_0^{t}x^{\nu+1}F(x)\,J_{\nu}(xy)\,dx=-\frac{1}{y^2}\lim_{\lambda\to\infty}\frac{1}{\lambda}\int_0^{\lambda}dt\int_0^{t}\sqrt{x}\,J_{\nu}(xy)\,f(x)\,dx,$$

that is, on account of (8.3), for almost all positive values of y,

$$g(y)=\lim_{\lambda\to\infty}\frac{1}{\lambda}\int_0^{\lambda}dt\int_0^{t}\sqrt{(xy)}\,J_{\nu}(xy)\,f(x)\,dx. \tag{8.7}$$

We have thus proved the theorem on the assumption made in (8.21). To obtain the result in the general case, we subtract from $f(x)$ the function $h(x)$ of Lemma A, (5). We thus obtain

$$f(x)-h(x)=\int_1^{\infty}g(y)\sqrt{(xy)}\,J_{\nu}(xy)\,dy=\int_0^{\infty}g_1(y)\sqrt{(xy)}\,J_{\nu}(xy)\,dy,$$

where $g_1(y)$ now satisfies (8.21).

An application of (8.7) then gives

$$g_1(y)=\lim_{\lambda\to\infty}\frac{1}{\lambda}\int_0^{\lambda}dt\int_0^{t}\sqrt{(xy)}\,J_{\nu}(xy)\{f(x)-h(x)\}\,dx,$$

and the theorem follows on account of (5.2) and (5.3).

9. THEOREM 3. *A finite function $f(x)$ cannot be represented for all positive values of x by two distinct integrals satisfying the conditions of Theorem 2.*

Let

$$f(x)=\int_0^{\infty}g_1(y)\sqrt{(xy)}\,J_{\nu}(xy)\,dy, \tag{9.1}$$

$$=\int_0^{\infty}g_2(y)\sqrt{(xy)}\,J_{\nu}(xy)\,dy. \tag{9.2}$$

Then $$\int_0^\infty (g_1-g_2)\sqrt{(xy)}\,J_\nu(xy)\,dy = 0.$$

Also this integral satisfies the conditions of Theorem 2, which gives, for almost all positive values of y,

$$g_1(y)-g_2(y) = 0,$$

and the integrals in (9.1) and (9.2) are essentially the same.

10. THEOREM 4. *The conclusion of Theorem* 3 *is still true if the integrals cease to represent* $f(x)$ *for a finite number of values of* x, *provided that, in addition,* $g(y)\to 0$ *when* $y\to\infty$.

Write $$g(y) = g_1(y)-g_2(y).$$

Then $$\int_0^\infty g(y)\sqrt{(xy)}\,J_\nu(xy)\,dy = 0,$$

save possibly at the exceptional points. Also $g(y)$ satisfies the conditions of Theorem 2 and tends to zero at infinity.

We now follow the argument of Theorem 2 and write

(10.1) $$-\int_0^\infty \frac{g(y)}{y^2}\sqrt{(xy)}\,J_\nu(xy)\,dy = x^{\nu+\frac{1}{2}}F(x).$$

Then, for positive values of x, $F(x)$ is continuous and takes the form $px^{-2\nu}+q$ in any interval not containing one of the exceptional points. In order to show that it assumes this form throughout, it is sufficient to show that

(10.2) $$\lim_{h\to 0} h\,G_\nu(x,\,h,\,F) = 0.$$

For suppose that ξ is an exceptional point and that $F(x)$ takes the form $px^{-2\nu}+q$ to the immediate left of ξ, and that it takes the form $p'x^{-2\nu}+q'$ to the immediate right of ξ.

From the continuity of $F(x)$ at ξ, we have

(10.3) $$\frac{p}{\xi^{2\nu}}+q = \frac{p'}{\xi^{2\nu}}+q'.$$

Also, from (10.2), we have

$$\lim_{h\to 0}\left[(\xi+2\nu h+h)\,\frac{F(\xi+2h)-F(\xi)}{2h}-(\xi-2\nu h-h)\,\frac{F(\xi)-F(\xi-2h)}{2h}\right]=0,$$

so that

$$\xi F'(\xi+0)=\xi F'(\xi-0),$$

whence $p=p'$.

From (10.3) we see that $q=q'$, so that there is no break in the form of $F(x)$ at ξ.

To establish (10.2), we proceed as in the proof of Lemma D, (8.5), and express $h\,G_\nu(x, h, F)$ as the sum of five integrals. It is easily seen that they are all uniformly convergent and tend to zero with h, except possibly the integral

$$J=\sqrt{\left(\frac{2}{\pi}\right)}x^{\frac{1}{2}-\nu}\int_0^\infty g(y)\left[\cos(xy-\omega)-\frac{(4\nu^2-1)\sin(xy-\omega)}{8xy}\right]\frac{\sin^2 yh}{y^2h}\,dy.$$

To show that J tends to zero with h, we write

$$J=\sqrt{\left(\frac{2}{\pi}\right)}x^{\frac{1}{2}-\nu}\left(\int_0^k+\int_k^\infty\right)=J_1+J_2,$$

k being finite.

Now

$$|J_1|\leqslant\sqrt{\left(\frac{2}{\pi}\right)}x^{\frac{1}{2}-\nu}\int_0^k\left|g(y)\left\{\cos(xy-\omega)-\frac{(4\nu^2-1)\sin(xy-\omega)}{8xy}\right\}\right|\left(\frac{\sin yh}{yh}\right)^2 h\,dy$$

$$\leqslant\sqrt{\left(\frac{2}{\pi}\right)}x^{\frac{1}{2}-\nu}h\int_1^k|g(y)|\,dy,$$

since, for $0\leqslant y<1$, $g(y)=0$.

Thus $J_1\to 0$.

We now choose k so that $|g(y)|<\epsilon$, when $y\geqslant k$. Then

$$|J_2|\leqslant\sqrt{\left(\frac{2}{\pi}\right)}x^{\frac{1}{2}-\nu}\epsilon\int_k^\infty\frac{\sin^2 yh}{y^2h}\,dy=\sqrt{\left(\frac{2}{\pi}\right)}x^{\frac{1}{2}-\nu}\epsilon\int_{kh}^\infty\left(\frac{\sin y}{y}\right)^2 dy$$

$$<\sqrt{\left(\frac{2}{\pi}\right)}x^{\frac{1}{2}-\nu}\epsilon\int_0^\infty\left(\frac{\sin y}{y}\right)^2 dy=\sqrt{\left(\frac{2}{\pi}\right)}x^{\frac{1}{2}-\nu}\,\frac{\pi}{2}\,\epsilon.$$

Thus J tends to zero with h and (10.2) is established. This gives by the

argument of § 8, for almost all positive values of y,

$$g(y) = 0,$$

and the theorem is proved.

References.

1. E. W. Hobson, *The theory of functions of a real variable*, 2 ed., 2 (Cambridge, 1926).
2. S. Pollard, " Identification of the coefficients in a trigonometrical integral ", *Proc. London Math. Soc.* (2), 25 (1926), 451–68.
3. C. de la Vallée-Poussin, *Cours d'analyse infinitésimale*, 3 ed., 1 (Louvain, Paris, 1914).
4. G. N. Watson, *Theory of Bessel functions* (Cambridge, 1922).

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ON GENERALIZED FOURIER SERIES WITH NON-NEGATIVE COEFFICIENTS

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1. Let $f(x)$ be a measurable real function of the real variable x, bounded in $-\infty < x < \infty$. Suppose further that the "mean value"

$$\lim_{\xi\to\infty} \frac{1}{2\xi}\int_{-\xi}^{\xi} f(t)\,e^{-i\lambda t}dt = c(\lambda) = M\{f(t)\,e^{-i\lambda t}\} \tag{1.1}$$

exists for $\lambda \gtrless 0$. Then it can readily be shown† that $c(\lambda)$ vanishes, except for at most an enumerable set of λ-values. Obviously we have always

$$c(-\lambda) = \bar{c}(\lambda); \tag{1.11}$$

so that $c(\lambda)$ and $c(-\lambda)$ are different from zero or equal to zero simultaneously.

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† In fact, $\mu_1, \ldots, \mu_N$ being unequal real numbers and $\gamma_1, \ldots, \gamma_N$ arbitrary complex numbers, we obviously have

$$\int_{-\xi}^{\xi}\left|f(t)-\sum_1^N \gamma_n e^{i\mu_n t}\right|^2 dt$$

$$= \int_{-\xi}^{\xi} f(t)^2\,dt - \sum_1^N \bar{\gamma}_n \int_{-\xi}^{\xi} f(t)\,e^{-i\mu_n t}\,dt - \sum_1^N \gamma_n \int_{-\xi}^{\xi} f(t)\,e^{i\mu_n t}\,dt + \sum_1^N\sum_1^N \gamma_n\gamma_m \int_{-\xi}^{\xi} e^{i(\mu_n-\mu_m)t}\,dt.$$

Hence, by (1.1) and (1.11), it follows that

$$\sum_1^N |c(\mu_n)|^2 \leqslant \sum_1^N |\gamma_n - c(\mu_n)|^2 + U^2, \quad \text{where} \quad U = \operatorname*{upper\ bound}_{-\infty<x<\infty} |f(x)|.$$

Thus, for $\gamma_n = c(\mu_n)$, we have $\sum_1^N |c(\mu_n)|^2 \leqslant U^2$; this being established, the truth of the above assertion follows by the arguments of Bohr in [3], 43.

Suppose that $c(\lambda)$ does not vanish identically and denote by $\lambda_1, \ldots, \lambda_n, \ldots$ those values of λ, in any order, which satisfy the inequalities

$$\lambda > 0, \quad |c(\lambda)| > 0.$$

We call the series

$$(1.2) \qquad c(0)+\Sigma\{c(\lambda_n)\, e^{i\lambda_n x}+c(-\lambda_n)\, e^{-i\lambda_n x}\} \sim f(x)$$

the *Fourier expansion* of $f(x)$.

Putting

$$c(0) = a_0, \quad c(\lambda_n) = \tfrac{1}{2}(a_n - ib_n) \quad (n = 1, 2, \ldots),$$

where $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ are real numbers, so that, by (1.11),

$$c(-\lambda_n) = \tfrac{1}{2}(a_n + ib_n) \quad (n = 1, 2, \ldots),$$

we can write the series (1.2) in the form

$$(1.21) \qquad a_0+\Sigma(a_n \cos\lambda_n x+b_n \sin\lambda_n x).$$

In this paper we are concerned with generalized Fourier series of the type (1.21) in which the cosine and sine coefficients are all non-negative or in which there is a well defined subset of those coefficients with that property.

2. Concerning the former case we enunciate the following

THEOREM A. *Let $f(x)$ be a real function of the real variable x, having all the properties mentioned in § 1 which ensure the existence of the generalized Fourier series* (1.21), *associated with $f(x)$. Suppose that*

$$(2.1) \qquad a_0 \geqslant 0, \quad a_n \geqslant 0, \quad b_n \geqslant 0, \quad n \geqslant 1.$$

Then the "partial sum"

$$(2.2) \qquad a_0+\sum_{0<\lambda_n\leqslant\omega}(a_n \cos\lambda_n x+b_n \sin\lambda_n x)$$

converges absolutely everywhere in $-\infty < x < \infty$ for every $\omega > 0$, and its value $s_\omega(x)$ satisfies the inequality

$$(2.3) \qquad \tfrac{1}{6}|s_\omega(x)| \leqslant U = \operatorname*{upper\,bound}_{-\infty<x<\infty} |f(x)|.$$

Finally, $s_\omega(x)$ is an almost periodic function† in the sense of Bohr, and (2.2) *is its Fourier expansion.*

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† Bohr [1], 30; [3], 27–28.

3. This theorem is included, as a special case, in

THEOREM B. *Let (ρ_κ) denote an arbitrary (finite or infinite) sequence of linearly independent† positive numbers and let $\{\mu_m\}$ be the subsequence of those terms λ_n of $\{\lambda_n\}$ [$\{\lambda_n\}$ belonging, in the sense of § 1, to a given function $f(x)$] which can be represented in the form*

$$\lambda_n = \sum_{\kappa=1}^{h} r_\kappa \rho_\kappa, \tag{3.1}$$

$h = h(n)$ being an integer and the $r_\kappa = r_\kappa(n)$ being rational.

With the notation of § 1, suppose that

$$c(0) \geqslant 0; \quad \mathfrak{R}c(\mu_m) \geqslant 0, \quad \mathfrak{R}ic(\mu_m) \geqslant 0 \text{ for } 0 < \mu_m < \Omega. \tag{3.2}$$

Then the "partial sum"

$$c(0) + \sum_{0<\mu_m\leqslant\omega} \{c(\mu_m)\, e^{i\mu_m x} + c(-\mu_m)\, e^{-i\mu_m x}\} \tag{3.3}$$

of the Fourier expansion (1.2) of $f(x)$ converges absolutely in $-\infty < x < \infty$ for $0 < \omega < \Omega$, and its sum function $\sigma_\omega(x)$, an almost periodic function whose Fourier expansion is identical with (3.3), can be written in the form

$$\sigma_\omega(x) = \phi_\omega(x) + \psi_\omega(x), \tag{3.4}$$

where $\phi_\omega(x)$ and $\psi_\omega(x)$ denote respectively the sum of the (absolutely convergent) series

$$c(0) + \sum_{0<\mu_m\leqslant\omega} \{c(\mu_m) + c(-\mu_m)\} \cos \mu_m x, \tag{3.31}$$

$$\sum_{0<\mu_m\leqslant\omega} \{c(\mu_m) - c(-\mu_m)\} \sin \mu_m x, \tag{3.32}$$

and satisfy the inequalities

$$|\phi_\omega(x)| \leqslant U \Big/ \left(1 - \frac{\omega}{\Omega}\right), \tag{3.41}$$

$$|\psi_\omega(x)| \leqslant U \left\{1 + \frac{\Omega}{\omega} \Big/ \left(1 - \frac{\omega}{\Omega}\right)\right\}; \quad U = \operatorname*{upper\ bound}_{-\infty<x<\infty} |f(x)|. \tag{3.42}$$

In fact, we can find a (finite or infinite) sequence $\{\beta_\kappa\}$, with positive linearly independent terms, even among the subsequences of $\{\lambda_n\}$, so that

† $\rho_1, \rho_2, \ldots$ are said to be linearly independent if no equation of the form $\sum_1^N r_\kappa \rho_\kappa = 0$ holds, where the r_κ are rational and not all zero.

each term of the last mentioned sequence can be represented in the form

$$\lambda_n = \sum_{\kappa=0}^{h(n)} r_\kappa^{(n)} \beta_\kappa,$$

the $r_\kappa^{(n)}$ being rational†.

If we choose such a *basic* sequence $\{\beta_\kappa\}$ as $\{\rho_\kappa\}$, (3.3) will be identical with (2.2). On the other hand, this choice ensures that, under the supposition (2.1), (3.2) is fulfilled, with $\{\mu_m\} \equiv \{\lambda_n\}$, for any positive Ω. Thus, from the truth of Theorem B, it follows that, for every $\omega > 0$, (2.2) is absolutely convergent in $-\infty < x < \infty$; further, that its sum is an almost periodic function $s_\omega(x)$ whose Fourier expansion is identical with (2.2); and, finally, that $s_\omega(x)$ is expressible as the sum of two functions, $\phi_\omega(x)$ and $\psi_\omega(x)$, satisfying the inequalities

$$|\phi_\omega(x)| \leqslant U,$$

$$|\psi_\omega(x)| \leqslant 5U,$$

obtained from (3.41) and (3.42) by making Ω equal to infinity and to 2ω respectively. Consequently, the assertions of Theorem A are all true, provided that Theorem B holds.

4. We now pass to the proof of Theorem B. For this purpose, following Bochner, we associate with (1.2) a sequence‡ of "Fejér-polynomials" $\Sigma_q(x)$, defined, for $q \geqslant 1$, by the relation

$$(4.1)\quad \Sigma_q(x) = \underset{(t)}{M}\left\{ f(x+t) \sum_{\nu_1=-P}^{P} \sum_{\nu_2=-P}^{P} \dots \sum_{\nu_q=-P}^{P} \left(1-\frac{|\nu_1|}{P}\right)\left(1-\frac{|\nu_2|}{P}\right) \dots \left(1-\frac{|\nu_q|}{P}\right) e^{-it\{(\nu_1/Q)\rho_1+(\nu_2/Q)\rho_2+\dots+(\nu_q/Q)\rho_q\}} \right\},$$

in which the mean value is taken with respect to t, while Q and P denote $q!$ and $qQ = q \cdot q!$ respectively; for $\alpha > q_0$, we put $\rho_\alpha = 0$ if the given sequence $\{\rho_\kappa\}$ consists of q_0 terms only.

First we examine these polynomials. (*Cp.* Bochner [1], 127-128; Bohr [3], 73-75.) In virtue of (1.1) and (1.2), and the linear independence of the ρ's (which belong to the given sequence) we can write (4.1) in the

† Bohr [2], 119–121; [3], 70–71.
‡ Bochner [1], 127–128; Bohr [3], 73–75.

form

$$\Sigma_q(x) = c(0) + \Sigma k_n^{(q)} \{c(\lambda_n) e^{i\lambda_n x} + c(-\lambda_n) e^{-i\lambda_n x}\}, \tag{4.11}$$

where

$$k_n^{(q)} = \prod_{\alpha=1}^{q} \left(1 - \frac{|\nu_\alpha|}{P}\right) \quad \text{for} \quad \lambda_n = \frac{1}{Q} \sum_{\alpha=1}^{q} \nu_\alpha \rho_\alpha, \tag{4.12}$$

the ν_α being integers with $|\nu_\alpha| < P$, while $k_n^{(q)} = 0$ otherwise.

With these definitions, $k_n^{(q)}$ is always equal to zero when λ_n does not belong to the sequence $\{\mu_m\}$ and, in the case (3.1), it satisfies the relation $0 \leqslant k_n^{(q)} < 1$, vanishing again, except for $h \leqslant q$, when

$$|r_\kappa Q| = \text{integer} < P \quad (\kappa = 1, \ldots, h).$$

Thus, if we put $p_m^{(q)} = k_n^{(q)}$ when $\mu_m = \lambda_n$, $\Sigma_q(x)$ admits the representation

$$\Sigma_q(x) = c(0) + \Sigma p_m^{(q)} \{c(\mu_m) e^{i\mu_m x} + c(-\mu_m) e^{-i\mu_m x}\}, \tag{4.2}$$

where, in virtue of the statements just detailed, we have

$$0 \leqslant p_m^{(q)} < 1, \quad \text{for} \quad q \geqslant 1, \quad m \geqslant 1; \tag{4.21}$$

$$p_m^{(q)} = 0, \quad \text{for} \quad m \geqslant m^* = m^*(q). \tag{4.22}$$

Now, from (4.12), we deduce a third property of $p_m^{(q)}$, namely

$$\lim_{q \to \infty} p_m^{(q)} = 1, \quad \text{for} \quad m \geqslant 1. \tag{4.23}$$

In fact, let us suppose that, for a fixed m,

$$\mu_m = \lambda_n = \sum_{\kappa=1}^{h} r_\kappa \rho_\kappa = \sum_{\kappa=1}^{h} \frac{u_\kappa}{v_\kappa} \rho_\kappa,$$

u_κ and $v_\kappa > 0$ being integers. Then, for $q > \max\{h, v_1, \ldots, v_h, |r_1|, \ldots, |r_h|\}$, we obtain

$$\lambda_n = \frac{\nu_1}{Q} \rho_1 + \ldots + \frac{\nu_h}{Q} \rho_h + \frac{0}{Q} \rho_{h+1} + \ldots + \frac{0}{Q} \rho_q,$$

with $\nu_\kappa = Q r_\kappa$ for $1 \leqslant \kappa \leqslant h$. Thus, by (4.12),

$$p_m^{(q)} = k_n^{(q)} = \prod_{\alpha=1}^{h} \left(1 - \frac{Q|r_\alpha|}{P}\right) = \prod_{\alpha=1}^{h} \left(1 - \frac{|r_\alpha|}{q}\right),$$

and, therefore, $p_m^{(q)}$ tends (increasing) to 1, as $q \to \infty$.

Finally, we derive, from (4.1), the important result that, for any $q \geqslant 1$ and any $x \gtrless 0$,

$$|\Sigma_q(x)| \leqslant U = \text{upper bound}_{-\infty<x<\infty} |f(x)|. \tag{4.3}$$

This follows from the property of the Bochner kernel

$$\kappa_q(t) = \prod_{\alpha=1}^{q} K_P\left(\frac{\rho_\alpha}{Q}\, t\right)$$

on the left of (4.1) that it is non-negative; this is obtainable from the same property of the Fejér kernel

$$K_n(x) = \sum_{\nu=-n}^{n} \left(1-\frac{|\nu|}{n}\right) e^{-ix},$$

in conjunction with $M\{\kappa_q(t)\} = 1$.

5. We now proceed to prove the absolute convergence of (3.3) for $0 < \omega < \Omega$. For this purpose, with the help of the results obtained in §4, we establish the absolute convergence of (3.31) and (3.32), which obviously yield a necessary and sufficient condition for the same behaviour of (3.3).

To attain this end, we next define, for any positive μ, the "Riesz-mean"

$$R_\mu(x, q) = c(0) + \sum_{0<\mu_m\leqslant\mu} \left(1-\frac{\mu_m}{\mu}\right) p_m^{(q)} \{c(\mu_m)\, e^{i\mu_m x} + c(-\mu_m)\, e^{-i\mu_m x}\}, \tag{5.1}$$

associated† with the Fejér polynomial $\Sigma_q(x)$.

Applying the formulae†

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{i2\kappa t}\frac{\sin^2\mu t}{\mu t^2}\, dt = \begin{cases} 1-(\kappa/\mu) & \text{for } 0 \leqslant \kappa < \mu, \\ 0 & \text{for } 0 < \mu \leqslant \kappa, \end{cases} \tag{5.2}$$

and using (4.2) and (5.1), we obtain the identity

$$R_\mu(x, q) = \frac{1}{\pi}\int_{-\infty}^{\infty} \Sigma_q(x+2t)\frac{\sin^2\mu t}{\mu t^2}\, dt;$$

hence, in virtue of (4.3) and (5.2), it follows at once that

$$|R_\mu(x, q)| \leqslant U. \tag{5.3}$$

† Bochner [2], 437; Szegö [1], 380.

Now from this result we derive

$$|R_\mu(x, q) \pm R_\mu(-x, q)| \leqslant 2U,$$

and hence we obtain the inequalities

$$(5.31) \qquad \left| c(0) + \sum_{0<\mu_m\leqslant\mu} \left(1-\frac{\mu_m}{\mu}\right) p_m^{(q)} \{c(\mu_m)+c(-\mu_m)\} \cos \mu_m x \right| \leqslant U,$$

$$(5.32) \qquad \left| \sum_{0<\mu_m\leqslant\mu} \left(1-\frac{\mu_m}{\mu}\right) p_m^{(q)} \{c(\mu_m)-c(-\mu_m)\} \sin \mu_m x \right| \leqslant U.$$

Taking $x = 0$ in (5.31) and using (1. 11), (3.2), and (4.21), we see that, for $0 < \omega < \mu < \Omega$, we have the inequalities

$$0 \leqslant \left(1-\frac{\omega}{\mu}\right) \left\{ c(0) + \sum_{0<\mu_m\leqslant\omega} p_m^{(q)} 2\Re c(\mu_m) \right\}$$

$$\leqslant c(0) + \sum_{0<\mu_m\leqslant\omega} \left(1-\frac{\mu_m}{\mu}\right) p_m^{(q)} 2\Re c(\mu_m)$$

$$\leqslant c(0) + \sum_{0<\mu_m\leqslant\mu} \left(1-\frac{\mu_m}{\mu}\right) p_m^{(q)} \{c(\mu_m)+c(-\mu_m)\} \leqslant U;$$

hence, *a fortiori*, we get the estimate

$$0 \leqslant c(0) + \sum_{0<\mu_m\leqslant\omega}^{m\leqslant m_0} p_m^{(q)} 2\Re c(\mu_m) \leqslant U \Big/ \left(1-\frac{\omega}{\Omega}\right) \quad (m_0 \geqslant 1),$$

which, in virtue of (4.23), implies that

$$0 \leqslant c(0) + \sum_{0<\mu_m\leqslant\omega}^{m\leqslant m_0} 2\Re c(\mu_m) \leqslant U \Big/ \left(1-\frac{\omega}{\Omega}\right).$$

Hence, by passing to the limit $m_0 \to \infty$, we deduce that

$$(5.4) \qquad 0 \leqslant c(0) + \sum_{0<\mu_m\leqslant\omega} \{c(\mu_m)+c(-\mu_m)\} \leqslant U \Big/ \left(1-\frac{\omega}{\Omega}\right),$$

and this yields the absolute convergence of (3.31) for $x \gtreqless 0$. For the absolute convergence of (3.32), we first observe that, differentiating term by term the sine polynomial on the left of (5.32), and using a theorem of S. Bernstein†, we deduce the inequality

$$(5.5) \qquad \left| \sum_{0<\mu_m\leqslant\mu} \left(1-\frac{\mu_m}{\mu}\right) p_m^{(q)} \mu_m \{c(\mu_m)-c(-\mu_m)\} \cos \mu_m x \right| \leqslant \mu U.$$

† Bernstein [1], 103.

Hence, putting $x=0$ and using (1.11), (3.2), (4.21), and (4.23), by following the lines of the proof of (5.4), we find that, for $0<\omega<\Omega$,

$$(5.6)\qquad 0\leqq \sum_{0<\mu_m\leqq\omega}\mu_m\{c(\mu_m)-c(-\mu_m)\}\leqq \Omega U\Big/\Big(1-\frac{\omega}{\Omega}\Big).$$

Therefore

$$\sum_{0<\mu_m\leqq\omega}\{c(\mu_m)-c(-\mu_m)\}|\sin\mu_m x|=\sum_{0<\mu_m\leqq\omega}\{c(\mu_m)-c(-\mu_m)\}\mu_m\left|\frac{\sin\mu_m x}{\mu_m x}\right||x|$$

$$\leqq|x|\sum_{0<\mu_m\leqq\omega}\mu_m\{c(\mu_m)-c(-\mu_m)\}\leqq|x|\,\Omega U\Big/\Big(1-\frac{\omega}{\Omega}\Big)\quad(x\gtrless 0);$$

this proves the absolute convergence of (3.32) for $0<\omega<\Omega$ and $-\infty<x<\infty$, without any exception.

6. It is obvious that the inequality (3.41) of Theorem B is an immediate consequence of (5.4). The counterpart of this inequality, embodied in (3.42), can be verified with the help of (5.6).

In fact, from (5.6), with the help of (4.21), we derive the inequality

$$\left|\sum_{0<\mu_m\leqq\omega}p_m^{(q)}\frac{\mu_m}{\omega}\{c(\mu_m)-c(-\mu_m)\}\sin\mu_m x\right|\leqq\frac{\Omega}{\omega}U\Big/\Big(1-\frac{\omega}{\Omega}\Big)\quad(0<\omega<\Omega).$$

Combining this with (5.32), which is valid for $\mu=\omega$, we find that, for any ω with $0<\omega<\Omega$,

$$\left|\sum_{0<\mu_m\leqq\omega}p_m^{(q)}\{c(\mu_m)-c(-\mu_m)\}\sin\mu_m x\right|\leqq U\left[1+\frac{\Omega}{\omega}\Big/\Big(1-\frac{\omega}{\Omega}\Big)\right].$$

But, for any fixed value of x, the sum of the sine series on the left of this inequality tends to $\psi_\omega(x)$, as $q\to\infty$, in consequence of (4.23) and the absolute convergence of (3.32). Thus (3.42) is true.

7. In order to establish the almost periodicity of $\sigma_\omega(x)$when $0<\omega<\Omega$, and to verify that its Fourier expansion is identical with (3.3), we shall prove the similar results concerning the functions $\phi_\omega(x)$ and $\psi_\omega(x)$ and the series (3.31) and (3.32) respectively. Herewith the remainder of Theorem B will be established.

Concerning the function $\phi_\omega(x)$ and the series (3.31) we observe that the property under consideration, by results due to Bohr†, is an immediate

† Bohr [1], 41–42, 51; [3], 33, 44–45.

consequence of the fact that (3.31) converges *uniformly* in $-\infty < x < \infty$ provided that $0 < \omega < \Omega$; this follows from (5.4). The almost periodicity of $\psi_\omega(x)$, together with

$$\psi_\omega(x) \sim \sum_{0<\mu_m\leqq\omega} \{c(\mu_m)-c(-\mu_m)\} \sin \mu_m x,$$

are consequences of the following remarkable result due to Bohr†:

Let $g(x)$ be an almost periodic function with $\gamma(\Lambda) = M\{g(t)\, e^{-i\Lambda t}\} = 0$ except for $\Lambda = \Lambda_n \gtrless 0$, $n \geqq 1$. Then $G(x) = \int_0^x g(\xi)\, d\xi$ will be an almost periodic function with the Fourier expansion

$$G(x) \sim C + \Sigma \gamma(\Lambda_n)\, e^{i\Lambda_n x}/i\Lambda_n \quad (C = \text{constant}),$$

if and only if $G(x)$ is bounded in $-\infty < x < \infty$.

This theorem can be applied, for $0 < \omega < \Omega$, to the sum-function $\tau_\omega(x)$ of the cosine series, *uniformly* convergent by (5.6),

$$(7.1) \qquad \sum_{0<\mu_m\leqq\omega} \mu_m \{c(\mu_m)-c(-\mu_m)\} \cos \mu_m x,$$

for, in virtue of (3.42),

$$\int_0^x \tau_\omega(\xi)\, d\xi = \sum_{0<\mu_m\leqq\omega} \{c(\mu_m)-c(-\mu_m)\} \sin \mu_m x = \psi_\omega(x)$$

is bounded in $-\infty < x < \infty$ when $0 < \omega < \Omega$. This completes the proof of Theorem B.

8. In the present section we formulate and verify a counterpart of the proposition just proved.

THEOREM C. *With the suppositions of Theorem B, the Riesz-sum*‡

$$(8.1) \qquad c(0) + \sum_{0<\mu_m\leqq\omega} \left(1-\frac{\mu_m}{\omega}\right) \{c(\mu_m)\, e^{i\mu_m x} + c(-\mu_m)\, e^{-i\mu_m x}\},$$

associated with the partial sum (3.3) of the Fourier expansion (1.2) of the function $f(x)$, converges absolutely for $x \gtrless 0$ when $0 < \omega < \Omega$, and its

† Bohr [1], 122–123; [3], 49–52.
‡ Bochner [2], 437; Szegö [1], 380.

sum-function $R_\omega(x)$, *an almost periodic function whose Fourier expansion is identical with* (8.1), *satisfies the inequality*

$$|R_\omega(x)| \leqslant U = \operatorname*{upper\ bound}_{-\infty<x<\infty} |f(x)|. \tag{8.2}$$

The absolute convergence of (8.1) is an immediate consequence of the same property of (3.3). Similarly, the almost periodicity of $R_\omega(x)$, together with

$$R_\omega(x) \sim c(0) + \sum_{0<\mu_m\leqslant\omega} \left(1-\frac{\mu_m}{\omega}\right)\{c(\mu_m)\,e^{i\mu_m x}+c(-\mu_m)\,e^{-i\mu_m x}\},$$

is an obvious consequence of the corresponding results concerning the function $\sigma_\omega(x)$ and the sum of the series

$$\sum_{0<\mu_m\leqslant\omega} \mu_m\{c(\mu_m)\,e^{i\mu_m x}+c(-\mu_m)\,e^{-i\mu_m x}\},$$

which, by (5.4) and (5.6), is uniformly convergent in $-\infty<x<\infty$, provided that $0<\omega<\Omega$.

Finally, the truth of (8.2) follows from (5.3), in conjunction with the relation

$$\lim_{q\to\infty} R_\omega(x, q) = R_\omega(x),$$

given, in virtue of (4.23), by the absolute convergence of (8.1), when $0<\omega<\Omega$.

9. Theorem B, which dealt with general real functions (bounded in $-\infty<x<\infty$) with a Fourier expansion (1.2), enables us to deduce, for the important subclass† consisting of the almost periodic functions, the following

THEOREM D. *Let* $f(x)$ *be a real almost periodic function* (*in the sense of Bohr*) *and suppose that, in its Fourier expansion* (1.2),

$$c(0)\geqslant 0, \quad \Re c(\mu_m)\geqslant 0, \quad \Re ic(\mu_m)\geqslant 0, \tag{9.1}$$

$\{\mu_m\}$ *denoting a subsequence of* $\{\lambda_n\}$ *as defined in* §3. *Let* $\sigma_\omega(x)$ *denote the sum-functions of the series* (3.3), *which, for* $\omega>0$, $x\gtrless 0$, *is absolutely convergent. Then, as* $\omega\to\infty$, *the sum-functions* $\sigma_\omega(x)$, *which, by Theorem B, are almost periodic, tend uniformly in* $-\infty<x<\infty$, *to an almost*

† *Cp.* Bohr [1], 35 (Theorem I), 42 (Theorem VII), 40 (Theorem IV), 45 (Theorem VIII), 46.

periodic limit-function $g(x)$ possessing the Fourier expansion

$$c(0)+\Sigma\{c(\mu_m)\,e^{i\mu_m x}+c(-\mu_m)\,e^{-i\mu_m x}\}. \tag{9.2}$$

In order to prove this theorem, we first remark that, in consequence of (4.1), (4.23), and the almost periodicity of $f(x)$, the Fejér polynomials (4.2) tend uniformly to a limit† $h(x)$ as $q\to\infty$ when $-\infty<x<\infty$. This $h(x)$ is a real almost periodic‡ function, and, in virtue of (4.23), its Fourier expansion is identical§ with the series (9.2). Now, putting

$$h(x)-\Sigma_q(x)=T(x,\,q),$$

we obtain a real almost periodic function whose Fourier expansion

$$T(x,\,q)\sim\Sigma\,(1-p_m^{(q)})\{c(\mu_m)\,e^{i\mu_m x}+c(-\mu_m)\,e^{-i\mu_m x}\}, \tag{9.3}$$

when written as a trigonometric series of the type (1.2), by (4.21) and (9.1), has only non-negative coefficients. Thus, applying Theorem A to $T(x,\,q)$, we note that the partial sum

$$\sum_{0<\mu_m\leqslant\omega}(1-p_m^{(q)})\{c(\mu_m)\,e^{i\mu_m x}+c(-\mu_m)\,e^{-i\mu_m x}\} \tag{9.4}$$

of (9.3) converges absolutely, for any $\omega>0$ and $x\gtreqless 0$, and its sum-function $t_\omega(x,\,q)$ satisfies the inequality

$$\tfrac{1}{6}\,|t_\omega(x,\,q)|\leqslant\operatorname*{upper\ bound}_{-\infty<x<\infty}|\,T(x,\,q)|=U_q. \tag{9.5}$$

This being established, we now pass to an upper estimate of the modulus of

$$\sigma_{\omega_2}(x)-\sigma_{\omega_1}(x)=\sum_{\omega_1<x\leqslant\omega_2}\{c(\mu_m)\,e^{i\mu_m x}+c(-\mu_m)\,e^{-i\mu_m x}\},$$

supposing that [*cp.* (4.22)]

$$\omega_2>\omega_1\geqslant\mu(q)=\max_{1\leqslant m\leqslant m^*(q)}\mu_m. \tag{9.6}$$

In consequence of this definition, we have, for any $x\gtreqless 0$,

$$\begin{aligned}t_{\omega_2}(x,\,q)-t_{\omega_1}(x,\,q)&=\sum_{\omega_1<\mu_m\leqslant\omega_2}(1-p_m^{(q)})\{c(\mu_m)\,e^{i\mu_m x}+c(-\mu_m)\,e^{-i\mu_m x}\}\\&=\sum_{\omega_1<x\leqslant\omega_2}\{c(\mu_m)\,e^{i\mu_m x}+c(-\mu_m)\,e^{i\mu_m x}\}\\&=\sigma_{\omega_2}(x)-\sigma_{\omega_1}(x);\end{aligned} \tag{9.7}$$

† Bochner [1], 128 (Theorem II, and the foregoing remarks).
‡ Bohr [1], 41 (Theorem VI) and 39 (Corr. of Theorem III).
§ Bohr [1], 62–63 (Theorem XXV).

for, by (9.6), $\mu_m > \omega_1$ implies that $m > m^*(q)$, and herewith, in virtue of (4.22), $p_m^{(q)} = 0$. Combining (9.7) with (9.5), with the supposition (9.6), we obtain the estimate

$$|\sigma_{\omega_2}(x) - \sigma_{\omega_1}(x)| \leqslant 12U_q.$$

Hence, since $$\lim_{q\to\infty} U_q = 0,$$

we establish the tendency (uniform in $-\infty < x < \infty$) of $\sigma_\omega(x)$ to an (almost periodic) function $g(x)$, as $\omega \to \infty$.

In conclusion, we remark that the Fourier expansion of $g(x)$ is obviously given by (9.2); and, consequently, in virtue of the "Uniqueness theorem", we have $g(x) = h(x)$ identically†.

10. Finally I mention the interesting special case of Theorems A and D in which $\lim_{n\to\infty} \lambda_n$ exists and is equal to ∞. Then, if a suitable rearrangement of the terms of (1.21) is made, we may suppose that the λ_n's increase with n. *With this assumption our first result is equivalent to*

$$|a_0 + a_1 \cos\lambda_1 x + b_1 \sin\lambda_1 x + \ldots + a_n \cos\lambda_n x + b_n \sin\lambda_n x| \leqslant 6U \quad (n \geqslant 0).$$

The corresponding special case of Theorem D can be enunciated as follows:

If in the Fourier series (1.21) *of a real almost periodic function* $f(x)$ *the* λ_n *tend monotonically (increasing) to* ∞, *as* $n \to \infty$, *the coefficients* a_0, a_n, b_n *being non-negative, then the series converges uniformly in* $-\infty < x < \infty$ *to the sum* $f(x)$.

The theorems just formulated can be regarded as generalisations of the two theorems‡ of Paley on Fourier series of pure periodic functions with positive coefficients.

In proving the second of the foregoing results, we have to apply Theorem D to the case where $\{\rho_n\}$ is a basic sequence for $\{\lambda_n\}$, while ω becomes infinite, with the values $\lambda_1, \lambda_2, \ldots$. Replacing the condition $\lim_{n\to\infty} \lambda_n = \infty$ by $\lim_{n\to\infty} \mu_n = \infty$, we arrive at an evident extension of the result before us to the series (9.2), rearranged so that μ_m increases with m.

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† Bohr [1], 54 [Theorem XVI); [3], 52.

‡ Paley [1], 205–206.

References.

S. Bernstein [1], *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle* (Paris, 1926), 1–204.

S. Bochner [1], "Beiträge zur Theorie der fastperiodischen Funktionen, I", *Math. Annalen*, 96 (1926), 119–147.

——— [2], "Properties of Fourier series of almost periodic functions", *Proc. London Math. Soc.* (2), 26 (1926), 433–452.

H. Bohr [1], [2], "Zur Theorie der fastperiodischen Funktionen, I, II", *Acta Math.*, 45 (1924), 29–127; 46 (1925), 101–214.

——— [3], "Fastperiodische Funktionen", *Ergebnisse der Math.*, 1 (Berlin, 1932), 417–516.

R. E. A. C. Paley [1], "On Fourier series with positive coefficients", *Journal London Math. Soc.*, 7 (1932), 205–208.

G. Szegö [1], "Zur Theorie der fastperiodischen Funktionen", *Math. Annalen*, 96 (1926), 378–382.

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SOME FORMULAE FOR PRIMALS IN FOUR DIMENSIONS (II)

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1. The present note is intended to supplement the work of a previous paper*, in which I gave the relations between the projective characters of a primal in S_4. The primal F in question is assumed to possess ordinary singularities, *i.e.* it has the generality of the projection of a non-singular V_3, situated in S_5, from an arbitrary point. Thus F is of order n, rank a, first class m, and class w. Its double surface B is of order μ_0, rank μ_1, class μ_2, with ν_2 apparent pinch-points; B has an apparent double curve of order $2k$, with T apparent triple points. On B there is a curve C_t, triple for B and F, which is of order t and rank r_t and contains ξ quadruple points which are quadruple for F and sextuple for B; it has classes of immersion t_1 in F and t_0 in B. There is, further, a pinch-curve C_j on B, of order j and rank r_j, having classes of immersion j_1 in F and j_0 in B; it meets C_t in τ stationary points, at which two of the tangent planes to B coincide. C_j is simple on B, and lies on all surfaces A of contact of tangents drawn to F from external points P; these surfaces touch everywhere along C_j.

To find the class z_j of immersion of C_j in A, we project A from P on S_3; then the projection of C_j meets the double curve of the projected surface in the points arising from the $[ja]$ tangents to F meeting C_j elsewhere, and the cuspidal curve in the projections of the j_1 tangents to F in points of C_j; so that

$$z_j = j(a-1) - [ja] - 3j_1.$$

Hence, from equations† (12), (13), and (34), we obtain

$$z_j = 5j_1 - 2j_0 - j. \qquad (46)$$

* Roth, *Proc. London Math. Soc.* (2), 35 (1933), 540. This is referred to as F.

† The numbering of the equations in this note follows that of F.

From this result we may deduce the rank r_j of C_j. For consider the pairs of tangent planes to A and B respectively at points of C_j. Since the developable of spaces containing the pairs of tangent planes, at points of C_j, is of class j_1, the correspondence formula gives*

$$z_j+j_0=j_1+r_j,$$

whence

$$r_j=4j_1-j_0-j,$$

as in equation (39)†.

2. The projection B' of B from an arbitrary point P contains a triple curve C_t', the projection of C_t, and a double curve C_k of order k, the projection of the apparent double curve on B. C_k contains i points‡ which are quadruple for B' and which lie on C_t'; it meets C_t' elsewhere in the ξ sextuple points and in t_1 points which are simple on both curves. Evidently we have

$$i=t(\mu_0-3)-12\xi-t_0. \tag{35}$$

The class z_k of immersion of C_k in B' is found by considering the intersections of C_k with the second polar of an arbitrary point. We thus obtain§

$$z_k=k(\mu_0-2)-3T-6i-12\xi-2t_1. \tag{47}$$

To find the rank r_k of C_k, we consider the correspondence set up, on an arbitrary line, by the pairs of tangent planes to B' in points of C_k. As in (11)′, we have

$$2(z_k-r_k)=\nu_2.$$

Then, from (35), (36), (37), (42), and (47), we deduce the equation

$$r_k=\mu_1(\mu_0-1)-\tfrac{5}{2}\nu_2-\tfrac{9}{2}\tau-\mu_2-5r_t. \tag{48}$$

The arithmetic genus of B is given by

$$p_a=\mu_0-1-\tfrac{2}{3}\mu_1+\frac{\mu_2}{6}+\frac{\nu_2+\tau}{12}.$$

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* We consider the trace of the tangent planes on an arbitrary plane and apply the general correspondence theorem (F, 540).

† There is a misprint in F, § 6; the plane-form $n=4$, $a=6$ has $j_0=6$. Also, in (31), the term $2[ja]$ should be $[ja]$, as stated in F, § 9.

‡ These arise from the apparent intersections of C_t and C_{2k}.

§ As verifications we may use the examples of rational surfaces B', the representation and character of which have been given by Roth, *Proc. Camb. Phil. Soc.*, 28 (1932), 300.

In terms of the characters of B' this becomes

$$p_a = \binom{\mu_0-1}{3} - (\mu_0-4)k + (\tfrac{1}{2}r_k - k) + 2T - (3\mu_0-16)t + 32\xi + 5(\tfrac{1}{2}r_t - t) + 6i + 2t_1, \quad (49)$$

showing the reduction due to the various singularities.

3. If F is plane-ruled, B contains a pencil of plane curves c_{n-2} of order $n-2$ and index 2, the genus of the curves and of the pencil being that of F. Each curve c_{n-2} has t' nodes on C_t, where*

$$t' = \binom{n}{2} - \tfrac{1}{2}a - 2n + 5.$$

A space through c_{n-2} meets B again in a curve c of order μ_0-n+2 which passes through these t' points. Hence, projecting B from an arbitrary point P, we observe that the number of apparent intersections of c_{n-2} and c is

$$\epsilon = (n-2)(\mu_0-n+2) - 2t' - (\mu_0-n+2-t').$$

Hence, from the projected figure, the class of immersion of c_{n-2} in B is

$$x = (n-2)(\mu_0-1) - \epsilon - 4t',$$

which on reduction gives

$$x = a+n-7.$$

As a verification, we note that this expression vanishes for the cubic plane-form ($n=3$, $a=4$) and gives the correct value 3 for the quartic form ($n=4$, $a=6$).

4. *The class of F.* We now suppose that F is not plane-ruled, so that it possesses ∞^3 tangent spaces. There are then ∞^1 spaces tangent to F in points of B and passing through a given point P; their curve of contact C_ρ is of order ρ, where ρ, the *first class of immersion* of B in F, is given by

$$\rho = a(n-1) - 2j - m. \quad (7)$$

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* Roth, *Proc. London Math. Soc.* (2), 33 (1931), 121.

The curve C_ρ contains the ν_2 points of contact of the tangents drawn to B from P; it passes through the τ stationary points on C_l and the t_1 points of contact of the spaces through P which touch F in points of C_l.

The second class X of immersion of B in F, *i.e.* the number of spaces tangent to F in points of B and passing through two given points P, P', is obtained by considering the intersections of C_ρ with the second polar of P'. Thus

$$\rho(n-2) = X+2\nu_2+2\tau+2t_1-j_1. \tag{50}$$

The ∞^1 tangent spaces passing through P, P' have a curve of contact C_m of order m which, since the surfaces A, A' each contain C_j, meets C_j in a number of points Q such that the tangent planes to A at Q meet PP'; and of these spaces, a finite number w pass through a third given point P''. To find w, consider the intersections of C_m with the first polar of P''; these arise from the tangent spaces through $PP'P''$, the X tangent spaces through PP', and the points Q, counted twice. Hence

$$m(n-1) = w+X+2z_j. \tag{51}$$

We may also find z_j from the fact that it is an additive character which can be expressed in terms of j, j_1, and j_0 only, since it must vanish when F is without singularities. We therefore assume that

$$z_j = \alpha j+\beta j_1+\gamma j_0,$$

and determine the coefficients by means of three special cases, for which the values of w and X are known, and hence, from (51), the value of z_j. We shall employ three ruled forms, the characters of which have been given elsewhere*.

The quartic primal whose section is a Del Pezzo surface has $m = 12$, $X = 8$, $w = 16$, $j = j_1 = j_0 = 4$, so that $z_j = 12$. The quartic primal containing a double plane has $m = 20$, $X = 4$, $w = 28$, $j = j_1 = 4$, $j_0 = 0$, so that $z_j = 28$. The quintic primal whose section is a Del Pezzo surface has characters $m = 12$, $X = 20$, $w = 10$, $j = 8$, $j_1 = 10$, $j_0 = 14$, whence $z_j = 18$. Inserting these values we have

$$z_j = 5j_1-2j_0-j, \quad \text{as before};$$

so that w is determined and, by elimination of the characters X, ν_2, τ, and t_1, may be reduced to the form given in equation (43). Another form of the result, which shows the effect of each singularity upon the class of F,

* Roth, *Proc. Lond. Math. Soc.* (2), 35 (1933), 380.

is obtained by eliminating j, j_1, and j_0 instead, using for this purpose (3), (11)′, (12), and (41). In this way we arrive at the equation

$$w = n(n-1)^3 - \mu_0(17n^2 - 63n + 64 - 4\mu_0) + \mu_1(21n - 46) - 8\mu_2 - 10\nu_2 + 12t(n-1) + 11r_\prime + 108\xi. \quad (52)$$

Thus, in particular, the reduction in class due to a double plane is found, from (52), to be

$$17n^2 - 63n + 60.$$

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ON THE DIFFERENTIAL STRUCTURE OF REAL FUNCTIONS

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This paper is concerned with the problem of the differential structure of arbitrary functions. If we limit ourselves to the consideration of the ordinary derivates (sometimes known as derived numbers) of a function of one variable, the following well-known theorem holds:

THEOREM A. *If $f(x)$ is any finite real function defined (at least) in a set E, and such that $D^+(f, x) < \infty$ at each point of E, then, p.p. in E, $D^+(f, x) = D_-(f, x)$; if, further, $f(x)$ is measurable, then its approximate derivative exists, and is equal to $D^+(f, x)$, p.p. in E.*

COROLLARY. *If, in a set E, $-\infty < D_+(f, x) \leqslant D^+(f, x) < \infty$, then $f'(x)$ exists p.p. in E.*

Rademacher and Stepanoff attacked the structure problem for functions of more than one variable, but did not obtain so complete a result. More recently, Haslam-Jones's introduction of *directed derivates*† enabled him to prove, for measurable functions, a theorem exactly analogous to that which holds for functions of one variable. Saks has used the work of Haslam-Jones to obtain a result for non-measurable functions‡.

For functions of one variable, Burkill and Haslam-Jones defined *approximate derivates*, having the same relation to the approximate derivative as the ordinary derivates have to the derivative, and proved for them a structure theorem analogous to Theorem A§. Their work was, however, limited in the first instance to measurable functions.

† U. S. Haslam-Jones, *Quart. J. of Math.* (Oxford), 3 (1932), 120–132.

‡ S. Saks, *Théorie de l'Intégrale* (1933), 234–239.

§ J. C. Burkill and U. S. Haslam-Jones, *Proc. London Math. Soc.* (2), 32 (1931), 346–355.

In the present paper I attempt to complete the work of Burkill and Haslam-Jones, by removing all restrictions of measurability, and also by extending the idea of approximate derivates to functions of two variables. The first section deals with the approximate derivates of non-measurable functions, giving the structure theorem. Part of this result was incorporated at the end of a recent paper by these two authors on Relative Measurability†. There follows a short discussion of sufficient conditions for a monotone function, in terms of approximate derivates. Next I give a new proof of Haslam-Jones's results for two dimensions, following the lines of a well-known method of proving Theorem A. The measurability of the function concerned is now seen to be irrelevant. Finally I define approximate directed derivates, and prove the corresponding structure theorem.

For the sake of completeness, the definitions of Burkill and Haslam-Jones are given in the course of the present paper.

1. *The approximate derivates of a function.*

1.1. If H is any set of points in one dimension, we denote by $d_e^+(x_0, H)$ the upper right-hand outer density of H at x_0, by $d_{i+}(x_0, H)$ the lower right-hand inner density, and so on. Let now $f(x)$ be any (real and finite) function, defined at the points of H and at no other points.

If $0 \leqslant \lambda < d_e^+(x, H)$, and x belongs to H, then we define the upper right λ-derivate, $AD^+(f, x, \lambda)$, as the lower bound of numbers α such that the set of points ξ for which ξ belongs to H and

$$f(\xi)-f(x) \geqslant \alpha(\xi-x) \quad (\xi > x)$$

has upper right-hand outer density at x less than or equal to λ. [If $\lambda \geqslant d_e^+(x, H)$, then $AD^+(f, x, \lambda)$ is not defined.] If $\lambda = 0$, we write simply $AD^+(f, x)$ for $AD^+(f, x, 0)$, and call it the upper right approximate derivate. The other λ-derivates and approximate derivates are similarly defined. If at the point x all the λ-derivates (upper and lower, left and right) exist and are equal, for $0 \leqslant \lambda < 1$, then the common value is known as the approximate derivative, $AD(f, x)$, of $f(x)$. It is readily seen that necessary and sufficient conditions for the existence of $AD(f, x)$ are:

(i) H has outer density 1 at x; and

(ii) $AD^+(f, x) = AD_+(f, x) = AD^-(f, x) = AD_-(f, x)$.

† *Quart. J. of Math.* (Oxford), 4 (1933), 233–239.

The main theorem of Burkill and Haslam-Jones is

THEOREM B. *If $f(x)$ is measurable and defined in a measurable set H, and if E is a set of points at each of which $AD^+[f, x, \lambda(x)] < \infty$, where $\lambda(x)$ depends on x but always satisfies $0 \leqslant \lambda < 1$, then $AD(f, x)$ exists p.p. on E.*

In order to obtain a corresponding theorem for non-measurable functions, it is possible to apply directly arguments similar to those employed for measurable functions; this course will be adopted in the case of functions of more than one variable. For the present, however, we assume the above result, and show how to apply it to non-measurable functions. This is effected by defining the approximate maximal and minimal functions, $A^*(f)$ and $A_*(f)$, associated with $f(x)$.

1.2. Let $f(x)$ be any function defined in a set H, not necessarily measurable. If $0 \leqslant \lambda < d_e^+(x, H)$, define $A^+(f, x, \lambda)$ as the lower bound of numbers κ such that the set of points ξ, for which ξ belongs to H and $f(\xi) \geqslant \kappa$, has upper right-hand outer density at x less than or equal to λ. If $\lambda \geqslant d_e^+(x, H)$, we do not define $A^+(f, x, \lambda)$. Substituting "left" for "right", we obtain $A^-(f, x, \lambda)$. Finally, we define the approximate maximal function of $f(x)$, $A^*(f, x)$, as $\max [A^+(f, x, 0), A^-(f, x, 0)]$. (If only one of these is defined, we take A^* as equal to that one.) We have clearly, if

$$0 \leqslant \lambda \leqslant \mu < d_e^+(x, H),$$

$$A^*(f, x) \geqslant A^+(f, x, \lambda) \geqslant A^+(f, x, \mu).$$

$A_*(f, x)$ is defined as $-A^*(-f, x)$.

THEOREM I. (*a*) *$A^+(f, x, \lambda)$, $A^*(f, x)$ are measurable functions of x.*

(*b*) *There exist a measurable set H_1, and a set N of measure zero, such that*

(i) *$A^*f(x) \geqslant f(x)$ if x lies in $H_1 H$,*

(ii) *$A^*(f, x) = A^+(f, x, \lambda) = A^-(f, x, \lambda)$*

whenever $0 \leqslant \lambda < 1$, if x lies in H_1,

(iii) *none of $A^*(f, x)$, $A^+(f, x, \lambda)$, $A^-(f, x, \lambda)$, $f(x)$ is defined at any point outside $H_1 + N$.*

(*c*) *For any κ, $mE\{A^*(f, x) > \kappa\} = m_e E\{f(x) \geqslant \kappa\}$.*

Denote by $E(\kappa)$ the set in which $f(x) \geqslant \kappa$, and by $G(\kappa)$ a measurable set which includes $E(\kappa)$ and is such that $mG(\kappa) = m_e E(\kappa)$. Let $M^*(\kappa)$, $M(\kappa, \lambda)$, $M_*(\kappa)$ be respectively the sets in which

$$\max [d_e^+\{x, E(\kappa)\}, d_e^-\{x, E(\kappa)\}] > 0; \quad d_e^+\{x, E(\kappa)\} > \lambda;$$

and

$$\min [d_{e+}\{x, E(\kappa)\}, d_{e-}\{x, E(\kappa)\}] = 1.$$

Since any outer density of $E(\kappa)$ is equal to the corresponding density of $G(\kappa)$, all these sets differ from $G(\kappa)$ only by sets of measure zero, and are therefore themselves measurable. Write $M(\kappa) = M^*(\kappa) + E(\kappa)$, then clearly

$$M(\kappa) \supset M^*(\kappa) \supset M(\kappa, \lambda) \supset M_*(\kappa)$$

and $N(\kappa) = M(\kappa) - M_*(\kappa)$ is a null set (that is, of measure zero).

We now have the following relations:

$$E\{A^*(f, x) > \kappa\} = \sum_{n=1}^{\infty} M^*(\kappa + 1/n),$$

$$E\{A^+(f, x, \lambda) > \kappa\} = \sum_{n=1}^{\infty} M(\kappa + 1/n, \lambda),$$

$$E\{f(x) > \kappa\} = \sum_{n=1}^{\infty} E(\kappa + 1/n) \subset \sum_{n=1}^{\infty} M(\kappa + 1/n).$$

Thus we see that $A^*(f, x)$ and $A^+(f, x, \lambda)$ are measurable; also we have

$$\begin{aligned} mE\{A^*(f, x) > \kappa\} &= \lim_{n \to \infty} mM^*(\kappa + 1/n) \\ &= \lim_{n \to \infty} m_e E(\kappa + 1/n) \\ &= m_e E\{f(x) > \kappa\} \end{aligned}$$

$\left[\text{since } \sum_{p=1}^{n} M^*(\kappa + 1/p) = M^*(\kappa + 1/n)\right]$.

Now let (r_n) describe all the rational numbers, and write

$$N = \Sigma N(r_n) + N(-\infty), \quad H_1 = M(-\infty) - N.$$

By the above H_1 is measurable and N is a null-set; it is clear that, if x lies outside $H_1 + N$, that is outside $M(-\infty)$, none of $A^*(f, x)$, $A^+(f, x, \lambda)$, $A^-(f, x, \lambda)$, $f(x)$, is defined. Suppose that $A^+(f, x, \lambda)$ is not defined; then x does not belong to $M(-\infty, \lambda)$ and therefore is not in H_1. If, however, $A^*(f, x)$ and $A^+(f, x, \lambda)$ are both defined, but $A^*(f, x) > A^+(f, x, \lambda)$, then there exists an r_n such that $A^*(f, x) > r_n > A^+(f, x, \lambda)$. Then x lies

in $M^*(r_n)$ but not in $M(r_n, \lambda)$, and therefore in $N(r_n)$ and in N. A similar argument applies to $A^-(f, x, \lambda)$. In the same way, if $f(x) > r_n > A^*(f, x)$, x lies in $E(r_n)$ and so in $M(r_n)$, but not in $M^*(r_n)$; thus x is in N and not in H_1. Therefore, if x is in H_1, every $A^+(f, x, \lambda)$, $A^-(f, x, \lambda)$ is defined and we have

$$A^+(f, x, \lambda) = A^-(f, x, \lambda) = A^*(f, x) \geqslant f(x) \quad [\text{if } f(x) \text{ is defined}].$$

This completes the theorem.

[It would be possible to give properties of the functions $A^*(f, x)$, $A_*(f, x)$ similar to those of the ordinary maximal and minimal functions; for example, $A^*(A^*f, x) = A^*(f, x)$. It does not, however, appear worth while to do so.]

1.3. In order to prove the theorem on approximate derivates we now need three lemmas.

LEMMA I. $A^*\{f(x)-\kappa x, x\} = A^*(f, x)-\kappa x$.

LEMMA II. *If* $AD^+(f, x, \lambda) < \infty$, *then* $f(x) \geqslant A^+(f, x, \lambda)$.

Let κ be greater than $AD^+(f, x, \lambda)$.
Then the set of points ξ, with

$$f(\xi) \geqslant f(x)+\kappa(\xi-x),$$

has upper right outer density at most λ at x. *A fortiori*, for any $\epsilon > 0$, the set at which $f(\xi) \geqslant f(x)+\epsilon$ has upper right outer density at most λ at x. Thus $A^+(f, x, \lambda) \leqslant f(x)+\epsilon$ or, finally, $A^+(f, x, \lambda) \leqslant f(x)$.

LEMMA III. *If* $AD^+(f, x_0, \lambda) = \kappa < \infty$, *and* $A^*(f, x_0) = f(x_0)$, *then* $AD^+(A^*f, x_0, \lambda) = \kappa$. (*f is still supposed defined in a set H.*)

Since $A^*(f, x) \geqslant f(x)$ p.p. where $f(x)$ is defined, we must have

$$AD^+(A^*f, x_0, \lambda) \geqslant AD^+(f, x_0, \lambda).$$

On the other hand, consider $\phi(x) = f(x)-(\kappa+\epsilon)x$, where ϵ is an arbitrary positive number. By Lemma I,

$$A^*(\phi, x_0) = A^*(f, x_0)-(\kappa+\epsilon)x_0 = \phi(x_0).$$

Since $AD^+(f, x_0, \lambda) < \kappa+\epsilon$, the set of points ξ at which $\phi(\xi) > \phi(x_0)$, that is $f(\xi) > (\kappa+\epsilon)(\xi-x_0)+f(x_0)$, has upper right outer density at x_0 less than or equal to λ. Applying Theorem I (*c*) to the function $\phi(x)$, we see that, in

any interval I, $mIE\{A^*(\phi, \xi) > \phi(x_0)\} = m_e IE\{\phi(\xi) > \phi(x_0)\}$†. Hence the set in which $A^*(\phi, \xi) > \phi(x_0)$ has upper right-hand density at x_0 less than or equal to λ. Since $A^*(\phi, x_0) = \phi(x_0)$, this gives $AD^+(A^*\phi, x_0, \lambda) \leqslant 0$, or, by Lemma I,

$$AD^+(A^*f, x_0, \lambda) \leqslant \kappa + \epsilon.$$

By combining this inequality, for arbitrary $\epsilon > 0$, with

$$AD^+(A^*f, x_0, \lambda) \geqslant AD^+(f, x_0, \lambda),$$

the lemma is proved.

We can now prove the main theorem. We denote by $f_E(x)$ the function which is equal to $f(x)$ at points of E and is undefined elsewhere.

THEOREM II. *If $f(x)$ is any function defined in a set H, and if E is a set of points at each of which $AD^+[f, x, \lambda(x)] < \infty$, $0 \leqslant \lambda(x) < 1$, then, p.p. on E, $AD^+(f, x, \lambda) = AD_-(f, x, \mu) = AD(A^*f, x) = AD(f_E, x)$ for all λ, μ $(0 \leqslant \lambda < 1, 0 \leqslant \mu < 1)$.*

By Lemma II and Theorem I (*b*), if x lies in $E - EN$, then

$$f(x) = A^+(f, x, \lambda) = A^*f(x) = A^-(f, x, \lambda),$$

all λ. Then, by Lemma III, $AD^+[A^*f, x, \lambda(x)] = AD^+(f, x, \lambda) < \infty$ on $E - EN$. By Theorem B, since A^*f is measurable, it follows that p.p. on $E - EN$, that is p.p. on E, $AD(A^*f, x)$ exists and is equal to $AD^+[f, x, \lambda(x)]$. Since $f(x) = A^*(f, x)$ on $E - EN$ or p.p. on E, it follows that $AD(f_E, x) = AD(A^*f, x)$ p.p. on E. (The condition that E has outer density 1 is certainly satisfied p.p. on E.)

Now let x be a point of E at which E has outer density 1 and $A^*(f, x) = f(x)$, also $AD(A^*f, x) = AD(f_E, x)$. Then, since

$$f(\xi) \leqslant A^*(f, \xi)$$

p.p. on H, we have, for any λ,

$$AD^+(f, x, \lambda) \leqslant AD^+(A^*f, x, \lambda) = AD(A^*f, x),$$

$$AD_-(f, x, \lambda) \geqslant AD_-(A^*f, x, \lambda) = AD(A^*f, x).$$

But, on the other hand,

$$AD^+(f, x, \lambda) \geqslant AD^+(f_E, x, \lambda) = AD(f_E, x),$$

$$AD_-(f, x, \lambda) \leqslant AD_-(f_E, x, \lambda) = AD(f_E, x).$$

† In applying Theorem I (*c*) to the function $\phi(x)$, we may suppose for the moment that $\phi(x)$ is defined only in IH.

Collecting these results, we have

$$AD^+(f, x, \lambda) = AD_-(f, x, \lambda) = AD(A^*f, x) = AD(f_E, x),$$

and, since almost every point of E satisfies the conditions for x, the theorem is proved.

1.4. It is well known that if $f(x)$ is continuous in the interval $a \leqslant x \leqslant b$, and $D^+(f, x) \geqslant 0$ except at a finite or enumerable set of points, then $f(b) \geqslant f(a)$. We now have the following analogous theorem.

THEOREM III. *If $f(x)$, defined for $a \leqslant x \leqslant b$, is such that*

(i) $AD^+(f, x, \lambda) \geqslant 0$ *except at an enumerable set of points x_n,*

(ii) $A^+(f, x_n, \lambda) \geqslant f(x_n)$ *for each n,*

(iii) $A^-(f, x, \mu) \leqslant f(x)$ *everywhere*

(where λ, μ are fixed numbers such that $0 \leqslant \mu < \lambda$), then $f(b) \geqslant f(a)$.

COROLLARY. *If $f(x)$ is approximately continuous and for fixed $\lambda > 0$, $AD^+(f, x, \lambda) \geqslant 0$ everywhere, then $f(b) \geqslant f(a)$.*

We prove first the following lemma.

If a set E is such that

(i) $d_e^+(x, E) > \lambda$ *whenever x is in E,*

(ii) *x is in E whenever $d_e^-(x, E) > \mu$, $0 \leqslant \mu < \lambda$,*

then, if a belongs to E and $b > a$, b also belongs to E.

We construct a Lebesgue chain thus:

Write $x_0 = a$.

If x_θ belongs to E, take $x_{\theta+1} > x_\theta$ such that $x_{\theta+1}$ belongs to E and $m_e[E(x_\theta, x_{\theta+1})] \geqslant \lambda(x_{\theta+1} - x_\theta)$; this is possible by condition (i).

If γ is of the second type, and x_θ is defined and belongs to E for all $\theta < \gamma$, we write

$$x_\gamma = \overline{\text{bound}}_{\theta<\gamma} x_\theta.$$

Then we can prove, by induction with respect to θ, that, if $\theta_1 < \theta \leqslant \gamma$,

$$m_e[E(x_{\theta_1}, x_\theta)] \geqslant \lambda(x_\theta - x_{\theta_1});$$

hence, by condition (ii), since x_{θ_1} may be taken arbitrarily near x_γ, x_γ must belong to E. Thus the construction of the chain according to the above

rules is always possible and we deduce that b can be reached; that is, b belongs to E†.

To prove the theorem, we take any $\epsilon > 0$ and consider the set E at which $f(x) \geqslant f(a) - \epsilon(x-a) - \sum_{x_n < x} \epsilon/2^n$.

Conditions (i) and (ii) of the theorem give condition (i) of the lemma, and condition (iii) gives condition (ii) of the lemma. Thus b belongs to E; that is,

$$f(b) \geqslant f(a) - \epsilon(b-a) - \epsilon.$$

Hence

$$f(b) \geqslant f(a).$$

If we consider, however, the assertion "if $f(x)$ is approximately continuous and $AD^+(f, x) \geqslant 0$ everywhere, then $f(b) \geqslant f(a)$", the case is quite different. I do not know if this theorem is true or false, but I have shown that it cannot be proved by the usual method of considering a set of points such as $E[f(x) \geqslant f(a)]$. In fact, I have constructed a set E such that

(i) If x belongs to E, $d^+(x, E) > 0$.

(ii) If x does not belong to E, then $d(x, E) = 0$.

The first property would, if E were the set $[f(x) \geqslant f(a)]$, be the result of $AD^+(f, x) > 0$, the second corresponds to approximate continuity. The construction is long and is not given here.

2. *The directed derivates of a function $f(x, y)$.*

2.1. For convenience we represent the point (x, y) by the complex variable $z = x+iy$. Let $f(x, y) = f(z)$ be a real finite function defined in some set H. Denote by $S_\alpha(\rho, \eta)$ the open region of points $z = re^{i\theta}$, where $0 < r < \rho$, $\alpha - \eta < \theta < \alpha + \eta$. Write

$$B^\alpha(f, \rho, \eta, z) = B^\alpha(\rho, \eta, z) = \overline{\text{bound}}\ \frac{f(z+h)-f(z)}{|h| \cos\phi}$$

for h lying in $S_\alpha(\rho, \eta)$, where $h = k+il = |h|e^{i(\alpha+\phi)}$;

$$\Delta^\alpha(f, \eta, z) = \Delta^\alpha(\eta, z) = \lim_{\rho \to 0} B^\alpha(\rho, \eta, z);$$

$$D^\alpha(f, z) = D^\alpha(z) = \lim_{n \to 0} \Delta^\alpha(\eta, z)‡.$$

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† This argument is derived with some modifications from J. C. Burkill, *Math. Zeitschrift*, 34 (1931), 272.

‡ These limits clearly exist. These definitions are reproduced from U. S. Haslam-Jones, *loc. cit.*

The lower directed derivate $D_\alpha(f, z)$ may be defined similarly, or simply as $-D^\alpha(-f, z)$.

2.2. It would clearly be possible to define directed derivates for functions of more than two variables, and in fact the arguments which will be used apply equally to such functions. In order to avoid confusion of quasi-geometrical detail, however, this extension has not been made.

We shall deal first with monotone functions. If $x_1 \geqslant x$ and $y_1 \geqslant y$ we write $z_1 \geqslant z$, or $z_1 > z$ except when $z_1 = z$. If $f(z_1) \geqslant f(z)$ whenever $z_1 \geqslant z$, we say that $f(z)$ is monotone increasing.

If there exist finite numbers a, b such that $D^\alpha(z) = a\cos\alpha + b\sin\alpha$ for all α, we say that there is an upper derivate plane at z. It is easy to show, by applying the Heine-Borel theorem with respect to α, that it is then possible, given any $\epsilon > 0$, to find $\delta > 0$ such that

$$f(z+re^{i\theta}) < f(z) + ar\cos\theta + br\sin\theta + \epsilon r$$

whenever $$0 < r \leqslant \delta.$$

We need five lemmas.

LEMMA I. *If $f(z)$ is measurable (in particular if it is defined throughout some rectangle and is monotone), then $D^\alpha(z)$, $D_\alpha(z)$ are measurable.*

It is sufficient to show that $B^\alpha(\rho, \eta, z)$ is measurable. Write

$$\phi(z) = \overline{\text{bound}}\,\{f(z+h) - A(k\cos\alpha + l\sin\alpha)\}$$

for h in $S_\alpha(\rho, \eta)$. The set in which $B^\alpha(\rho, \eta, z) > A$ is identical with the set in which $\phi(z) > f(z)$ and is, therefore, measurable since ϕ is lower semi-continuous†.

LEMMA II. *If $f(z)$ is monotone increasing, and $0 < \alpha < \frac{1}{2}\pi$, then $D^\alpha(z_0)$, $D_\alpha(z_0)$, $D^{\alpha+\pi}(z_0)$, $D_{\alpha+\pi}(z_0)$ are equal to the corresponding one-dimensional derivates along the line $z = z_0 + re^{i\alpha}$.*

For simplicity we take z_0 to be the origin. With each point $z = x+iy$ we associate the points

$$z' = x(1+i\tan\alpha), \quad z'' = y(\cot\alpha + i),$$

† This lemma is taken directly from Haslam-Jones's paper. From now on the methods diverge.

so that, if $y \geqslant x \tan\alpha$, $\quad f(z') \leqslant f(z) \leqslant f(z'')$;

if $y \leqslant x \tan\alpha$, $\quad f(z') \geqslant f(z) \geqslant f(z'')$.

Given $\epsilon > 0$, we can choose η_0 so small that $|z'|, |z''|$ lie between $(1-\epsilon)|z|$ and $(1+\epsilon)|z|$ whenever $|\arg z - \alpha| < \eta_0$. It follows that, if $\eta < \eta_0$,

$$B^\alpha(\rho, \eta)\cos\eta \leqslant (1+\epsilon) \overline{\text{bound}}_{0<r<\rho(1+\epsilon)} \frac{f(re^{i\alpha})-f(0)}{r}.$$

From this it is easy to deduce, ϵ being arbitrary, that

$$D^\alpha(0) \leqslant \overline{\lim_{r\to+0}} \frac{f(re^{i\alpha})-f(0)}{r}.$$

On the other hand, the reverse inequality is obvious. Similar considerations apply to the other derivates.

LEMMA III. *If $f(z)$ is monotone increasing and $0 < \alpha < \frac{1}{2}\pi$, then*

$$D^\alpha(z) = D_\alpha(z) = -D^{\alpha+\pi}(z) = -D_{\alpha+\pi}(z),$$

and all these are finite, almost everywhere.

By Lemma I, the set of points in which these conditions are not fulfilled is measurable. But applying Lemma II and the known properties of monotone functions of one variable, we see that this set is cut by each line of direction α in a set of linear measure zero. Hence it is of plane measure zero.

LEMMA IV. *If $f(z)$ is monotone increasing, then, for $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$,*

$$D^\alpha(z_0) = \lim_{\theta\to\alpha+0} \overline{\lim_{r\to+0}} \frac{f(z_0+re^{i\theta})-f(z_0)}{r};$$

for $\frac{1}{2}\pi < \alpha < \frac{3}{2}\pi$,

$$D^\alpha(z_0) = \lim_{\theta\to\alpha-0} \overline{\lim_{r\to+0}} \frac{f(z_0+re^{i\theta})-f(z_0)}{r}.$$

We give the proof for the first case, and again suppose, for simplicity, that z_0 is the origin.

Let $\theta = \alpha+\phi$, $0 < \phi < \frac{1}{2}\pi - \alpha$; then $re^{i\theta}$ belongs to $S_\alpha(2r, 2\phi)$, so that

$$\{f(re^{i\theta})-f(0)\}/r \leqslant \cos\phi\, B^\alpha(2r, 2\phi).$$

Letting r and then ϕ tend to zero, we have

$$\overline{\lim_{\theta\to\alpha+0}} \overline{\lim_{r\to+0}} \frac{f(re^{i\theta})-f(0)}{r} \leqslant D^\alpha(0).$$

On the other hand, given θ as above and any positive δ, ϵ, we can find $z = x+iy = \rho e^{i(\alpha+\psi)}$ such that

$$f(z)-f(0) \geqslant \rho \cos\psi\{D^1(0)-\epsilon\},$$

and $|\psi| < \phi$, $\rho < \delta\cos\theta$. Consider the point $z' = x(1+i\tan\theta)$; since $\alpha < \theta < \frac{1}{2}\pi$, $z' > z$, and so $f(z') \geqslant f(z)$; also $|z'| < \delta$. Then

$$\frac{f(z')-f(0)}{|z'|} \geqslant \frac{\rho}{|z'|}\cos\psi\{D^1(0)-\epsilon\}.$$

Hence

$$\frac{f(z')-f(0)}{|z'|} \geqslant \cos\theta\sec(\alpha+\psi)\cos\psi\{D^1(0)-\epsilon\}.$$

Let $\delta \to 0$; then, since $|z'| < \delta$ and $|\psi| < \phi$, we have

$$\overline{\lim_{r\to+0}}\ \frac{f(re^{i\theta})-f(0)}{r} \geqslant \cos\theta\,\chi(\alpha,\phi)\{D^1(0)-\epsilon\},$$

where

$$\chi(\alpha,\phi) = \underline{\text{bound}}_{|\psi|<\phi}\{\sec(\alpha+\psi)\cos\psi\}.$$

If now we let ϕ and then ϵ tend to zero, we easily obtain

$$\underline{\lim_{\theta\to\alpha+0}}\ \overline{\lim_{r\to+0}}\ \frac{f(re^{i\theta})-f(0)}{r} \geqslant D^1(0).$$

Thus the lemma is proved.

LEMMA V. *If $0 < \alpha < \beta < \frac{1}{2}\pi$, there is a linear transformation of (x, y) into (x', y') in which lines of direction α, β become parallel to the axes of x', y' respectively. Measurable sets remain measurable, and their measure is multiplied by some constant (in particular, null-sets become null-sets). If $f(x, y)$ is transformed to $\phi(x', y')$ the property of possessing a derivative plane, or an upper derivate plane, at a given point, is preserved. If $f(x, y)$ is monotone increasing, so is $\phi(x', y')$.*

These results are almost obvious.

THEOREM IV. *If $f(z)$ is monotone increasing (and defined throughout some open region), then it is totally differentiable almost everywhere.*

Take any α, β such that $0 < \alpha < \beta < \frac{1}{2}\pi$; by Lemma III,

$$D^\alpha(z) = D_\alpha(z) = -D^{\alpha+\pi}(z) = -D_{\alpha+\pi}(z) \text{ p.p.}$$

Hence, if we transform the variables (x, y) as in Lemma V, in the new system there is a measurable finite function $a(z')$ such that

$$D^0(\phi, z') = D_0(\phi, z') = -D^\pi(\phi, z') = -D_\pi(\phi, z') = a(z') \text{ p.p.}$$

Similarly $D^{\frac{1}{2}\pi}(z') = D^{\frac{3}{2}\pi}(z') = \ldots = b(z')$, say, p.p. Since $\phi(z')$ is monotone increasing, $a(z')$, $b(z')$ are positive or zero.

We say that, at almost all points, $D^\vartheta(\phi, z') \leqslant a(z') \cos\theta + b(z') \sin\theta$, all θ. (For convenience we now write z for z'.) If at the point z this statement is false for some θ_0 (other than $\frac{1}{2}\pi$, $\frac{3}{2}\pi$), then, by Lemma IV, we can choose a *rational* θ near θ_0 such that

$$\overline{\lim_{r\to+0}} \frac{\phi(z+re^{i\theta})-\phi(z)}{r} > a(z)\cos\theta + b(z)\sin\theta.$$

Thus, if the statement above is false, we can choose in succession θ (rational), ϵ (small and positive), m, n (positive or zero integers), ρ, η (small and positive), such that, in a set E of positive outer measure, we have

$$\overline{\lim_{r\to+0}} \frac{\phi(z+re^{i\theta})-\phi(z)}{r} > a(z)\cos\theta + b(z)\sin\theta + 5\epsilon,$$

$$m\epsilon \leqslant a(z) < (m+1)\,\epsilon,$$

$$n\epsilon \leqslant b(z) < (n+1)\,\epsilon,$$

$$B^0(\rho, \eta, z) < (m+1)\,\epsilon,$$

$$B^{\frac{1}{2}\pi}(\rho, \eta, z) < (n+1)\,\epsilon,$$

$$B^\pi(\rho, \eta, z) < (-m+1)\,\epsilon,$$

$$B^{\frac{3}{2}\pi}(\rho, \eta, z) < (-n+1)\,\epsilon.$$

(θ is clearly not equal to 0, $\pm\frac{1}{2}\pi$, or π.)

Let z_0 be a point of E at which E has outer density 1. Let δ be any positive number less than the least of

$$\tfrac{1}{4}\sin^2\theta, \quad \tfrac{1}{2}|\cos\theta|, \quad \tfrac{1}{2}|\cos\theta\tan\eta|, \quad \tfrac{1}{2}|\sin\theta\tan\eta|,$$

and take ρ_1 so small that in any circle, centre z_0 and radius $r < \rho_1$, the outer measure of E is greater than $(\pi-\delta^2)r^2$. Then, if $r < \min(\rho, \rho_1)$, we consider the square

$$r(\cos\theta-\delta) < x-x_0 < r\cos\theta, \quad 0 < y-y_0 < r\delta.$$

This is easily seen to be contained in the circle of centre z_0 and radius r, and hence it must contain at least one point, z_1 say, of E. Suppose now that $\pi < \theta < \frac{3}{2}\pi$.

From the inequalities for δ and r it is easy to see that $z_1 - z_0$ belongs to $S_\pi(\rho, \eta)$ and $z_0 + re^{i\theta} - z_1$ to $S_{\frac{1}{2}\pi}(\rho, \eta)$. Hence, since z_0 and z_1 belong to E,

$$\phi(z_1) - \phi(z_0) < (-m+1)\,\epsilon |x_1 - x_0| < (m-1)\,\epsilon r \cos\theta + \epsilon r\delta,$$

$$\phi(z_0 + re^{i\theta}) - \phi(z_1) < (-n+1)\,\epsilon |y_0 + r\sin\theta - y_1| < (n-1)\,\epsilon r \sin\theta + \epsilon r\delta,$$

since $m-1 \geqslant -1$, $n-1 \geqslant -1$.

From these we obtain

$$\frac{\phi(z_0 + re^{i\theta}) - \phi(z_0)}{r} < (m-1)\,\epsilon \cos\theta + (n-1)\,\epsilon \sin\theta + 2\epsilon\delta$$

$$< a(z_0)\cos\theta + b(z_0)\sin\theta + 2\epsilon\delta + 4\epsilon.$$

Taking δ sufficiently small, and letting $r \to 0$, we have a contradiction; for z_0 belongs to E and therefore

$$\overline{\lim_{r \to +0}} \frac{\phi(z_0 + re^{i\theta}) - \phi(z_0)}{r} > a(z_0)\cos\theta + b(z_0)\sin\theta + 5\epsilon.$$

In a similar manner we obtain a contradiction if θ lies in any other quadrant. Thus we have shown that

$$D^\theta(z) \leqslant a(z)\cos\theta + b(z)\sin\theta \text{ p.p. (all } \theta).$$

Similarly
$$D_\theta(z) \geqslant a(z)\cos\theta + b(z)\sin\theta \text{ p.p.}$$

If z is a point where these equations are true, we can, given any $\epsilon > 0$ and any θ_0, find a sector $|\theta - \theta_0| < \eta$, $0 < r < \rho$, in which

$$\left| \frac{\phi(z + re^{i\theta}) - \phi(z)}{r} - a(z)\cos\theta - b(z)\sin\theta \right| < \epsilon.$$

Applying the Heine-Borel theorem with respect to θ_0, we can cover the whole range $0 \leqslant \theta \leqslant 2\pi$ with a finite number of such sectors, and hence, for some positive ρ_0,

$$\left| \frac{\phi(z + re^{i\theta}) - \phi(z)}{r} - a(z)\cos\theta - b(z)\sin\theta \right| < \epsilon$$

whenever $0 < r < \rho_0$. That is, $\phi(z)$ is totally differentiable at z, and so

$\phi(z)$ is totally differentiable p.p. Transforming back to the original coordinates, we see that the same is true of $f(z)$.

2.3. It is convenient to give at this point the definition of the approximate derivative plane. We define first the points of density and dispersion, in the strong sense†, of a set E. Let z_0 be a fixed point, z_1, z_2 variable points with $z_1 \leqslant z_0 \leqslant z_2$; let I be the rectangle $z_1 \leqslant z \leqslant z_2$. If $\{m_e(EI)/mI\} \to 1$ as $|z_2 - z_1| \to 0$ (z_1, z_2 varying in any way), we say that z_0 is a point of outer density of E in the strong sense. If $\{m_e(EI)/mI\} \to 0$ as $|z_2 - z_1| \to 0$, we say that z_0 is a point of dispersion of E.

Let $f(z)$ be any function of z defined in a set H, and let z_0 be a point of H. We say that $f(z)$ has an approximate derivative plane at z_0 if there exist finite numbers a, b and a set $H_1 \subset H$ such that

(i) $f(z_0+k+il) - f(z_0) - ak - bl = o\{\sqrt{(k^2+l^2)}\}$ as k, $l \to 0$, taking only those values such that z_0+k+il belongs to H_1;

(ii) z_0 is a point of outer density of H in the strong sense;

(iii) z_0 is a point of dispersion of $H - H_1$ in the strong sense.

[For measurable functions (iii) is a consequence of (iii)′: z_0 is a point of density of H_1 in the strong sense.]

We can now prove the complete analogue of Theorem A.

THEOREM V. (*a*) *If $f(z)$ is any function defined in a set H, then, p.p. in H, either* (i) $D^\alpha(f, z) = +\infty$, *all* α, *or* (ii) *an upper derivate plane exists.*

(*b*) *If $f(z)$ is measurable, then p.p. where the upper derivate plane exists it is also the approximate derivative plane.*

If z, α are such that $D^\alpha(f, z) < \infty$, we can find ρ, η such that $B^\alpha(\rho, \eta, z) < \infty$. Then, for any rational α_1, α_2 such that $\alpha - \eta < \alpha_1 < \alpha_2 < \alpha + \eta$ and some finite k, we have $f(z+re^{i\theta}) - f(z) < kr$ whenever $0 < r < \rho$, $\alpha_1 \leqslant \theta \leqslant \alpha_2$. If the theorem (*a*) is false we can choose fixed rational α_1, α_2, and positive ρ_0 (small), k_0 (large), such that, in a set E_0 of positive outer measure, we have

$$f(z+re^{i\theta}) - f(z) < k_0 r \quad (\alpha_1 \leqslant \theta \leqslant \alpha_2,\ 0 < r < \rho_0),$$

while no upper derivate plane exists. We then make a transformation as in

† S. Saks, *Théorie de l'Intégrale*, 223. It is there shown that, if E is any set, almost every point of E is a point of outer density in the strong sense.

Lemma V. (The condition $0 < \alpha_1 < \alpha_2 < \frac{1}{2}\pi$ is clearly unnecessary now that we are no longer dealing with monotone functions.) In the transformed system there exist numbers δ, M (depending in a simple manner on ρ_0, k_0) and a set E (the transform of E_0) of positive outer measure, such that, if z_1 belongs to E,

$$\phi(z)-\phi(z_1) \leqslant M|z-z_1|$$

whenever $$|z-z_1| < \delta, \quad 0 \leqslant \arg(z-z_1) \leqslant \tfrac{1}{2}\pi,$$

while there is no upper derivate plane at z_1.

Let z_0 be a point of outer density of E. Consider the square A;

$$z_0-\tfrac{1}{2}\delta(1+i) \leqslant z \leqslant z_0.$$

Clearly $m_e(EA) > 0$. Put $\phi_1(z) = \phi(z)-M(x+y)$, and define $\psi(z)$ as the upper bound of $\phi_1(z')$ for $z \leqslant z' \leqslant z_0$. Then $\psi(z)$ is clearly defined and monotone decreasing throughout the interior of A, but it may take the value $+\infty$. [Since z_0 is a density-point of E, there always exist points z' for which $\phi(z')$ is defined and $z \leqslant z' \leqslant z_0$.]

If, however, z belongs to EA we have, for all values of z' occurring in the definition of $\psi(z)$,

$$\phi_1(z')-\phi_1(z) \leqslant M|z-z'|-M(x'-x+y'-y) \leqslant 0,$$

so that $\psi(z) = \phi_1(z)$. Hence, taking z_1 as any point of EA, $\psi(z)$ is defined, monotone decreasing, and finite throughout the interior of the rectangle B, namely $z_1 \leqslant z \leqslant z_0$, and $m_e(EB) > 0$. Let z_2 be a point of outer density of E, belonging to E, lying in the interior of B, at which $\psi(z)$ is totally differentiable; such points must exist. Let $\partial\psi/\partial x = a$, $\partial\psi/\partial y = b$, at z_2. Let η be an arbitrary positive number less than $\frac{1}{2}$. Choose ρ_0 such that, if $r < 2\rho_0$, then

(i) The circle, centre z_2, radius $2r$, contains a part of E of outer measure greater than $(4\pi-\eta^2)r^2$.

(ii) This circle lies within B.

(iii) $|\psi(z_2+re^{i\theta})-\psi(z_2)-r(a\cos\theta+b\sin\theta)| \leqslant \eta r$, all θ. Consider the point $z' = z_2+re^{i\theta}$, where $r < \rho_0$. The square $z'-\eta r(1+i) \leqslant z \leqslant z'$ lies within the circle of centre z_2 and radius $2r$, and has area $\eta^2 r^2$; it therefore contains some point z_3 of E. Since $z' \geqslant z_3$ and $|z_3-z'| < \delta$,

$$\phi(z')-\phi(z_3) \leqslant M|z_3-z'| < 2|M|\eta r.$$

Since z_2, z_3 both belong to EB and $|z_3-z_2|<2r<2\rho_0$,

$$\begin{aligned}\phi(z_3)-\phi(z_2) &= \phi_1(z_3)-\phi_1(z_2)+M(x_3-x_2+y_3-y_2)\\ &= \psi(z_3)-\psi(z_2)+M(x_3-x_2+y_3-y_2)\\ &\leqslant (M+a)(x_3-x_2)+(M+b)(y_3-y_2)+\eta|z_3-z_2|\\ &\leqslant (M+a)\,r\cos\theta+(M+b)\,r\sin\theta+\eta 2r\\ &\qquad +|M+a|\,\eta r+|M+b|\eta r.\end{aligned}$$

Hence $\phi(z')-\phi(z_2)\leqslant (M+a)\,r\cos\theta+(M+b)\,r\sin\theta+N\eta r$, where

$$N=4|M|+|a|+|b|+2 \text{ (independent of } \eta).$$

From this, since it is true for all $r<\rho_0$ and all θ, we easily deduce

$$D^{\alpha}(\phi, z_2)\leqslant (M+a)\cos\alpha+(M+b)\sin\alpha+N\eta,$$

and so

$$D^{\alpha}(\phi, z_2)\leqslant (M+a)\cos\alpha+(M+b)\sin\alpha \text{ (all } \alpha).$$

On the other hand, since z_2 is a point of outer density of E, any sector, of arbitrarily small angle and radius, drawn from z_2, contains points of EB for which $\phi(z)=\psi(z)+M(x+y)$. Hence we easily see that

$$\begin{aligned}D^{\alpha}(\phi, z_2) &\geqslant D^{\alpha}(\phi_{EB}, z_2)\\ &\geqslant D^{\alpha}(\psi_{EB}, z_2)+M(\cos\alpha+\sin\alpha)\\ &\geqslant (M+a)\cos\alpha+(M+b)\sin\alpha.\end{aligned}$$

Thus $D^{\alpha}(\phi, z_2)=(M+a)\cos\alpha+(M+b)\sin\alpha$ (all α), that is ϕ has an upper derivate plane at z_2, contrary to supposition. This proves the first part of the theorem.

To prove the second part, we remark that, at any point where an upper derivate plane exists, we can find M, δ such that

$$f(z+re^{i\theta})<f(z)+Mr \text{ for all } \theta \text{ and all } r<\delta.$$

Arguing just as in the first part, but without the transformation, we obtain a set EB of positive outer measure at which the theorem is not true, and a monotone function $\psi(z)$ such that $f(z)=\psi(z)+M(x+y)$ at each point of EB. At any point which belongs to EB and is a point of density of the measurable set $\{f(z)=\psi(z)+M(x+y)\}$, and at which $\psi(z)$ is totally differentiable, it is clear that the plane given by

$$\left(M+\frac{\partial\psi}{\partial x}\right)\cos\theta+\left(M+\frac{\partial\psi}{\partial y}\right)\sin\theta$$

is the approximate derivative plane for $f(z)$. This plane cannot lie above the upper derivate plane for any value of θ; hence the two planes must coincide, which is contrary to the assumption that the theorem was not true at any point of EB.

This completes the proof of Theorem V.

3. *The approximate directed derivates of* $f(x, y)$.

3.1. Let E be any set in two dimensions, α any direction, and z_0 a given point. Write $S_\alpha(z_0, \rho, \eta)$ for the sector $0 < |z-z_0| < \rho$, $|\arg(z-z_0)-\alpha| < \eta$. Define also the semi-sectors

$$S_{\alpha+}(z_0, \rho, \eta) = \{0 < |z-z_0| < \rho,\ \alpha < \arg(z-z_0) < \alpha+\eta\},$$

$$S_{\alpha-}(z_0, \rho, \eta) = \{0 < |z-z_0| < \rho,\ \alpha-\eta < \arg(z-z_0) < \alpha\}.$$

We define the directed densities as follows:

$$d_e{}^\alpha(z_0, E) = \overline{\lim_{\eta\to 0}}\ \overline{\lim_{r\to 0}} \frac{m_e[ES_\alpha(z_0, \rho, \eta)]}{m[S_\alpha(z_0, \rho, \eta)]},$$

$$d_{e,\alpha}(z_0, E) = \underline{\lim_{\eta\to 0}}\ \underline{\lim_{r\to 0}} \frac{m_e[ES_\alpha(z_0, \rho, \eta)]}{m[S_\alpha(z_0, \rho, \eta)]}.$$

Substituting inner for outer measure, we obtain $d_i{}^\alpha(z_0, E)$, $d_{i,\alpha}(z_0, E)$. Finally, by replacing $S_\alpha(z_0, \rho, \eta)$ by $S_{\alpha+}(z_0, \rho, \eta)$ or $S_{\alpha-}(z_0, \rho, \eta)$ we obtain the "asymmetric" directed densities which we denote by $d_e^{\alpha+}(z_0, E)$, $d_i^{\alpha-}(z_0, E)$, and other similar notations. At each point where the outer (inner) density of E, measured over concentric squares, exists and is equal to 0, or 1, it is easy to see that each of the outer (inner) densities, symmetric or asymmetric, is equal to 0, or 1. In particular, we see that

$$d_{e,\alpha}(z, E) = d_{e,\alpha+}(z, E) = d_{e,\alpha-}(z, E) = 1 \ \text{(all } \alpha\text{) p.p. in } E.$$

Among other relations we may mention:

$$d_e{}^\alpha(z, E) \leqslant \max[d_e^{\alpha+}(z, E);\ d_e^{\alpha-}(z, E)]$$

$$\geqslant \tfrac{1}{2} d_e^{\alpha+}(z, E).$$

3.2. Let $f(z)$ be any function defined in a set H and nowhere else. For any z of H and any number κ, write $E(\kappa, z)$ for the set of points ζ such that ζ belongs to H and $f(\zeta)-f(z) \geqslant \kappa|\zeta-z|$. If z is a point of H at which $d_e{}^\alpha(z, H) > 0$, and λ is such that $0 \leqslant \lambda < d_e{}^\alpha(z, H)$, we define $AD^\alpha(f, z, \lambda)$ as the lower bound of numbers κ such that $d_e{}^\alpha[z, E(\kappa, z)] \leqslant \lambda$. Substituting $d_e^{\alpha+}$, $d_e^{\alpha-}$ for $d_e{}^\alpha$, we obtain $AD^{\alpha+}(f, z, \lambda)$, $AD^{\alpha-}(f, z, \lambda)$.

We say that $f(z)$ has an upper approximate derivate plane (UAD-plane) at z if there exist finite numbers a, b, and a set $H_1 \subset H$ such that

(i) z is a point of outer density of H_1 (in the strong sense)†.

(ii) The plane given by $a\cos\theta + b\sin\theta$ is the derivative plane (D-plane) of $f_{H_1}(z)$ at the point z.

(iii) Given any $\epsilon > 0$, the set of points $\zeta = z+k+il$, such that

$$f(\zeta) > f(z) + ak + bl + \epsilon\sqrt{(k^2+l^2)},$$

has z as a point of dispersion.

We see that this involves, for all λ such that $0 \leqslant \lambda < 1$,

$$AD^{\alpha}(f, z, \lambda) = AD^{\alpha+}(f, z, \lambda) = AD^{\alpha-}(f, z, \lambda) = a\cos\alpha + b\sin\alpha,$$

all α, also that, if $f(z)$ is measurable and has a UAD-plane at z, it must be also the AD-plane (approximate derivative plane).

3.3. In order to prove the "structure theorem" we use an argument similar to that employed in one dimension‡, but with various complications and additions due to the facts that we are working in two dimensions and with non-measurable functions. We need two lemmas.

3.31. LEMMA I. *If (E_n) is an increasing sequence of sets, and if E is a set such that $m_e E > 0$ and no point of E is a point of dispersion of the set $\sum_{n=1}^{\infty} E_n$, then there is a point of E which is a point of outer density of some set E_n.*

We can find a rectangle R (open) such that $m_e(ER) > \frac{1}{2}mR$. Let G be a measurable set such that $G \supset \Sigma E_n R$ and $mG = m_e \Sigma E_n R$. No point of ER is a point of dispersion of G, but since G is measurable almost all points outside G are points of dispersion. Hence $mG \geqslant m_e ER > \frac{1}{2}mR$ and so $m_e \Sigma E_n R > \frac{1}{2}mR$. Then we can find E_n such that $m_e E_n R > \frac{1}{2}mR$. We now take G_n measurable, including $E_n R$ and with $mG = m_e E_n R > \frac{1}{2}mR$; it follows that $m_e(EG_n R) > 0$. Almost every point of G_n is a point of density of G_n, and therefore a point of outer density of E_n. The lemma follows at once.

† The phrases "point of density" and "point of dispersion" are to be understood in the strong sense unless the contrary is stated.

‡ J. C. Burkill and U. S. Haslam-Jones, *loc. cit.*, *Proc. London Math. Soc.* (2), 32 (1931), 346–355.

LEMMA II. *If E_1, E_2 are such that $m_e E_1 > 0$ and no point of E_1 is a point of dispersion of E_2, then there is a point of E_2 which is a point of outer density of E_1.*

Find R as before such that $m_e E_1 R > \frac{1}{2} mR$. Take measurable sets G_1, G_2 including $E_1 R$, $E_2 R$ with $mG_1 = m_e E_1 R$, $mG_2 = m_e E_2 R$. As before, $m_e E_2 R = mG_2 \geqslant m_e E_1 R > \frac{1}{2} mR$, and so $m_e(G_1 E_2 R) > 0$. It follows that some point of E_2 is a point of density of G_1, and therefore a point of outer density of E_1.

3.32. THEOREM VI. *If $f(z)$ is any function defined in a set H, then almost all points of H fall into one of two sets:*

(i) *A set E_∞, where $AD^{\alpha+}(f, z, \lambda) = AD^{\alpha-}(f, z, \lambda) = +\infty$, all α and λ $(0 \leqslant \lambda < 1)$; hence $AD^{\alpha}(f, z, \lambda) = \infty$, all α, if $0 \leqslant \lambda < \frac{1}{2}$.*

(ii) *A set E at each point of which the UAD-plane exists and is also the AD-plane of $f_E(z)$.*

Suppose that, for some z, α, λ, κ, we have

$$AD^{\alpha+}(f, z, \lambda) < \kappa < \infty.$$

Take λ_0, λ_1 such that $\lambda < \lambda_0 < \lambda_1 < 1$. Denote by $E(\kappa, z)$ as before the set of points ζ belonging to H such that $f(\zeta) - f(z) \geqslant \kappa |\zeta - z)$, and by $S_{\alpha,\beta}(z, \rho)$ the set of points $\zeta = z + re^{i\theta}$, where $0 < r < \rho$, $\alpha < \theta < \beta$. Then we can find $\eta < \frac{1}{4}\pi$ such that

$$\overline{\lim_{\rho \to 0}} \frac{m_e[S_{\alpha, \alpha+\eta}(z, \rho) E(\kappa, z)]}{\frac{1}{2}\rho^2 \eta} < \lambda_0.$$

We can then find $\gamma < \eta(\lambda_1 - \lambda_0)/\lambda_1$ such that

$$\overline{\lim_{\rho \to 0}} \frac{m_e[S_{\alpha, \alpha+\gamma}(z, \rho) E(\kappa, z)]}{\frac{1}{2}\rho^2 \gamma} < \lambda_0 < \lambda_1.$$

Then we have at once

$$\overline{\lim_{\rho \to 0}} \frac{m_e[S_{\alpha+\gamma, \alpha+\eta}(z, \rho) E(\kappa, z)]}{\frac{1}{2}\rho^2 (\eta - \gamma)} < \frac{\lambda_0 \eta}{\eta - \gamma} < \lambda_1.$$

We then find rational α_1, α_2, α_3 so near α, $\alpha+\gamma$, $\alpha+\eta$ respectively that these inequalities are still true if we replace α by α_1, etc. (this is clearly trivial). A similar argument holds if $AD^{\alpha-}(f, z, \lambda) < \infty$. Thus, if the theorem is false we can find fixed rational $\alpha_1 < \alpha_2 < \alpha_3$ and fixed κ_0 (large),

λ_1 (near 1), ρ_0 (small) such that, if z is any point of a certain set E_1, of positive outer measure, we have

$$m_e[S_{\alpha_1,\,\alpha_2}(z,\,\rho)\,E(\kappa_0,\,z)] < \tfrac{1}{2}\lambda_1\rho^2(\alpha_2-\alpha_1), \tag{1}$$

$$m_e[S_{\alpha_2,\,\alpha_3}(z,\,\rho)\,E(\kappa_0,\,z)] < \tfrac{1}{2}\lambda_1\rho^2(\alpha_3-\alpha_2), \tag{2}$$

whenever $0<\rho<\rho_0$. (Note also that $\alpha_3-\alpha_1<\frac{1}{4}\pi$.) On the other hand, the theorem is to be false at every point of E_1; that is, either the UAD-plane does not exist, or it is not the AD-plane of $f_E(z)$.

We can find a linear function $L(z)$ such that if $\alpha_1 \leqslant \arg(z-z_1) \leqslant \alpha_3$ then $L(z)-L(z_1) \geqslant \kappa_0|z-z_1|$. Write $f_1(z)=f(z)-L(z)$, and $E(z)$ for the set of points ζ, where $f_1(\zeta) \geqslant f_1(z)$. Clearly we have from (1), for $0<\rho<\rho_0$, z in E_1,

$$m_e[S_{\alpha_1,\,\alpha_2}(z,\,\rho)\,E(z)] < \tfrac{1}{2}\lambda_1\rho^2(\alpha_2-\alpha_1), \tag{3}$$

(and similarly for $S_{\alpha_2,\,\alpha_3}$).

3.33. Choose λ_2, λ_3 such that $\lambda_1<\lambda_2<\lambda_3<1$.

Write

$$\alpha_4 = \alpha_2+(\alpha_3-\alpha_2)(\lambda_2-\lambda_1)/\lambda_2, \tag{4}$$

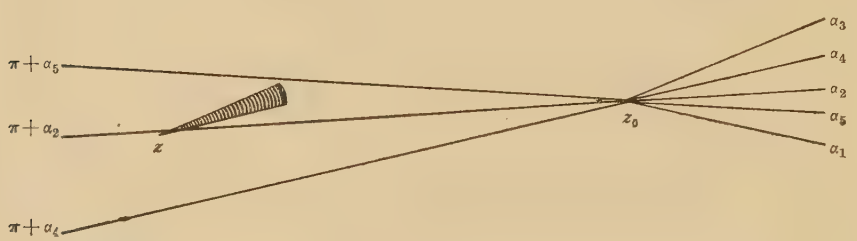
$$\alpha_5 = \alpha_2-(\alpha_2-\alpha_1)(\lambda_2-\lambda_1)/\lambda_2. \tag{5}$$

We can choose δ so small that if $\alpha_2 \leqslant \arg(z_0-z) \leqslant \alpha_4$ and $|z-z_0|=r$, then

$$S_{\alpha_4,\,\alpha_3}(z,\,\delta r) \subset S_{\pi+\alpha_5,\,\pi+\alpha_4}(z_0,\,r); \tag{6}$$

similarly, if $\alpha_5 \leqslant \arg(z_0-z) \leqslant \alpha_2$ and $|z-z_0|=r$,

$$S_{\alpha_1,\,\alpha_5}(z,\,\delta r) \subset S_{\pi+\alpha_5,\,\pi+\alpha_4}(z_0,\,r). \quad \text{(See figure.)} \tag{7}$$



The figure is drawn for the first case; the shaded sector is $S_{\alpha_4,\,\alpha_3}(z,\,\delta r)$.

Let

$$\beta = \min\left[\frac{\delta^2(\alpha_3-\alpha_4)}{\alpha_4-\alpha_5},\quad \frac{\delta^2(\alpha_5-\alpha_1)}{\alpha_4-\alpha_5}\right]. \tag{8}$$

We can find a positive $\rho_1 < \rho_0$ and a sub-set E_2 of E_1, $m_e E_2 > 0$, such that, if z belongs to E_2,

(i) $$m_e[E_1 S_{\pi+\alpha_4, \pi+\alpha_5}(z, \rho)] > \tfrac{1}{2}\rho^2(\alpha_4-\alpha_5)\{1-\beta(\lambda_3-\lambda_2)\} \tag{9}$$

for $0 < \rho \leqslant \rho_1$;

(ii) Given any $\epsilon > 0$, z is a point of outer density of the set of ρ such that $|f_1(\zeta)-f_1(z)| < \epsilon$†. (10)

3.34. We now say that, if z_0, z_1 are points of E_2 such that

$$\alpha_5 \leqslant \arg(z_0-z_1) \leqslant \alpha_4, \quad |z_0-z_1| < \rho_1, \quad \text{then} \quad f_1(z_0) \leqslant f_1(z_1).$$

To prove this we construct a sequence of points of E_1, $z_2, z_3, \ldots, z_n, \ldots$, all lying in $S_{\pi+\alpha_5, \pi+\alpha_4}(z_0, \rho_1)$ and tending to z_0 as $n \to \infty$, such that

$$f_1(z_n) < f_1(z_{n-1}) \quad (n = 2, 3, \ldots).$$

Suppose that z_{n-1} is already defined, and lies in $S_{\pi+\alpha_5, \pi+\alpha_4}(z_0, \rho_1)$. (If $n-1 = 1$, z_{n-1} may lie on the boundary of this sector, but that does not matter.) Write $|z_0-z_{n-1}| = r_{n-1}$ and consider the sector

$$S = S_{\alpha_4, \alpha_3}(z_{n-1}, \delta r_{n-1}).$$

Then $S \subset S_{\pi+\alpha_5, \pi+\alpha_4}(z_0, r_{n-1})$, by (6). We have, using (9), (8),

$$\begin{aligned} m_e(SE_1) &\geqslant mS - [mS_{\pi+\alpha_5, \pi+\alpha_4}(z_0, r_{n-1}) - m_e E_1 S_{\pi+\alpha_5, \pi+\alpha_4}(z_0, r_{n-1})] \\ &> mS - \beta(\lambda_3-\lambda_2)\tfrac{1}{2}r_{n-1}^2(\alpha_4-\alpha_5) \\ &> mS - \delta^2(\lambda_3-\lambda_2)\tfrac{1}{2}r_{n-1}^2(\alpha_3-\alpha_4) \\ &> mS(1-\lambda_3+\lambda_2). \end{aligned} \tag{11}$$

Since z_{n-1} belongs to E_1, we have, by (3) and (4),

$$\begin{aligned} m_e[SE(z_{n-1})] &\leqslant m_e[S_{\alpha_2, \alpha_3}(z_{n-1}, \delta r_{n-1}) E(z_{n-1})] \\ &< \tfrac{1}{2}\lambda_1(\alpha_3-\alpha_2)\delta^2 r_{n-1}^2 \\ &< \lambda_2 mS, \end{aligned} \tag{12}$$

since $\alpha_3-\alpha_4 = \lambda_1(\alpha_3-\alpha_2)/\lambda_2$.

† Almost all points of H satisfy condition (ii), *cf.* Sierpinski, *Fundamenta Math.*, 4 (1923), 124.

Hence $m_e[S\{E_1-E(z_{n-1})\}] > (1-\lambda_3)mS$. Thus we can choose z_n lying in S, belonging to E_1 but not to $E(z_{n-1})$ [so that $f_1(z_n) < f_1(z_{n-1})$], and such that

$$|z_n-z_{n-1}| > \sqrt{(1-\lambda_3)}\,\delta r_{n-1}.$$

Since $\alpha_3-\alpha_1 < \frac{1}{4}\pi$, we see that

$$r_n = |z_0-z_n| < r_{n-1}-|z_n-z_{n-1}|/\sqrt{2}$$

$$< \vartheta_0 r_{n-1}$$

say, where $\vartheta_0 < 1$.

Since $f_1(z_n) < f_1(z_{n-1}) < \ldots < f_1(z_1)$, and z_{n-1} lies in E_1, we obtain also from (12)

$$m_e[E(z_1)\,S_{\pi+\alpha_5,\pi+\alpha_4}(z_0, r_{n-1})] \leqslant m_e[E(z_{n-1})\,S_{\pi+\alpha_5,\pi+\alpha_1}(z_0, r_{n-1})]$$

$$\leqslant mS_{\pi+\alpha_5,\pi+\alpha_1}(z_0, r_{n-1}) - [mS - m_e S\,E(z_{n-1})]$$

$$< mS_{\pi+\alpha_5,\pi+\alpha_4}(z_0, r_{n-1})[1-(1-\lambda_2)\beta], \qquad (13)$$

since $mS \geqslant \beta m S_{\pi+\alpha_5,\pi+\alpha_4}(z_0, r_{n-1})$.

If, on the other hand, z_{n-1} lies in $S_{\pi+\alpha_5,\pi+\alpha_2}(z_0, \rho_1)$, we consider similarly the sector $S = S_{\alpha_1,\alpha_5}(z_{n-1}, \delta r_{n-1})$. Thus, in any case, we obtain z_n as above, and we have the sequence desired. Since $r_n < \vartheta_0 r_{n-1}$, $z_n \to z_0$ as $n \to \infty$. Then, applying (13) and (10), we see that $f_1(z_0) \leqslant f_1(z_1)$. [Suppose that $f_1(z_0) > f_1(z_1)$, and take $\epsilon = f_1(z_0)-f_1(z_1)$.]

3.35. Let now z_0 be any point of outer density of E_2, and z_1 a point of E_2 lying in $S_{\pi+\alpha_5,\pi+\alpha_4}(z_0, \rho_1)$. Consider the parallelogram R

$$\alpha_5 \leqslant \arg(z-z_1) \leqslant \alpha_4;$$

$$\pi+\alpha_5 < \arg(z-z_0) < \pi+\alpha_4.$$

(Since $\alpha_4-\alpha_5 < \frac{1}{4}\pi$, the maximum diameter of R is $|z_0-z_1| < \rho_1$.) Define, in R, $\phi(z)$ as the upper bound of $f_1(z')$ for points z' which lie in $E_2 R$ and satisfy $\alpha_5 \leqslant \arg(z'-z) \leqslant \alpha_4$. From the last paragraph, $\phi(z) = f_1(z)$ if z is in $E_2 R$, also $\phi(z)$ is defined and finite in R, and monotone decreasing in the angle $\alpha_5 \leqslant \theta \leqslant \alpha_4$ (in an obvious sense). Hence (by a simple transformation) $\phi(z)$ is totally differentiable p.p. in R. We now say that p.p. in $E_2 R$ the D-plane of $\phi(z)$ is the UAD-plane of $f_1(z)$ and the AD-plane of $f_{1E}(z)$. Since $f_1(z)$ differs from $f(z)$ only by a linear function, this obviously contradicts the supposition that the original theorem was false at points of

E, for $E_2 \subset E$ and $m_e E_2 R > 0$, since z_0 is a density-point of E_2. Thus the main theorem will be proved.

3.36. Suppose that the first statement is false, so that there exists a sub-set E_3 of $E_2 R$, $m_e E_3 > 0$, at which

(i) z is a point of outer density of $E_2 R$.

(ii) The D-plane of $\phi(z)$ exists and is not the UAD-plane of $f_1(z)$.

Then it is clear that z cannot be a point of dispersion of the set of points ζ at which $f_1(\zeta) > \phi(\zeta)$ (if it were, all the conditions for a UAD-plane would be satisfied). By Lemma I, there is a point z of E_3 at which the set $\{f_1(\zeta) > \phi(\zeta) + 1/n\}$ has outer density 1, in the strong sense. Since $f_1(z) = \phi(z)$ and $\phi(z)$ is differentiable at z, this clearly gives

$$AD^{\alpha+}(f_1, z, \lambda) = AD^{\alpha-}(f_1, z, \lambda) = +\infty,$$

all λ, and the same for $f(z)$, contrary to supposition.

There remains only this possibility: that in a sub-set E_4 of $E_2 R$, $m_e E_4 > 0$, we have

(i) z is a point of outer density of $E_2 R$.

(ii) The D-plane of $\phi(z)$ exists and is the UAD-plane of $f_1(z)$, but not the AD-plane of $f_{1E}(z)$. This means that the D-plane of $\phi(z)$ is not the LAD-plane of $f_{1E}(z)$; that is, z is not a point of dispersion of the set at which

$$f_{1E}(\zeta) < \phi(\zeta).$$

Apply Lemma II, taking E_4 as E_1, and for E_2 the set of points of ER where $f_1(\zeta) < \phi(\zeta)$ and $\phi(\zeta)$ is differentiable. Then there exists a point ζ_0 of E such that

(i) $f_1(\zeta_0) < \phi(\zeta_0)$.

(ii) $\phi(\zeta)$ is differentiable at ζ_0.

(iii) ζ_0 is a point of outer density of E_4 and *a fortiori* of the set $E_2 R$ at which $\phi(z) = f_1(z)$. Clearly

$$AD^{\alpha+}(f_1, \zeta_0, \lambda) = AD^{\alpha-}(f_1, \zeta_0, \lambda) = +\infty, \quad \text{all} \quad \alpha, \lambda;$$

the same is then true for $f(z)$, contrary to the supposition that ζ_0 is a point of E.

This completes the proof of the theorem.

3.4. *Remarks.*

(i) We have the following corollary:

If $f(z)$ is measurable and defined in H, then p.p. in H, either

(1) $AD^{\alpha+}(f, z, \lambda) = AD^{\alpha-}(f, z, \lambda) = +\infty$, *all* α, λ;

or (2) *the AD-plane exists.*

(ii) It appears strange at first sight that the "symmetric" directed approximate derivate should only appear incidentally, the asymmetric derivates playing a more fundamental rôle. A situation in some ways similar, however, arises in one dimension; by saying that $AD^{+}(f, x, \lambda) < \infty$ we are, in effect, saying that $D^{+}(f, x, \lambda) < \infty$ on some set whose lower right inner density at x is at least $1-\lambda$. If we replace this by the symmetric condition

$$\overline{\lim_{\delta \to 0}} \frac{1}{2\delta} m_e[(x-\delta, x+\delta) E\{f(\xi) \geqslant \kappa |\xi - x|\}] \leqslant \lambda,$$

we can say nothing unless $\lambda < \frac{1}{2}$, in which case we merely use this relation to establish the one-sided condition.

(iii) An examination of the proof of the theorem shows that it would still be true if in the definition of $AD^{\alpha+}(f, z, \lambda)$ we replaced the condition

$$\overline{\lim_{\eta \to 0}} \; \overline{\lim_{r \to 0}} \frac{m_e[E(\kappa, z) S_{\alpha+}(z, \rho, \eta)]}{\frac{1}{2}\rho^2 \eta} \leqslant \lambda$$

by the weaker condition

$$\underline{\lim_{\eta \to 0}} \; \overline{\lim_{r \to 0}} \frac{m_e[E(\kappa, z) S_{\alpha+}(z, \rho, \eta)]}{\frac{1}{2}\rho^2 \eta} \leqslant \lambda.$$

The definition here given, however, appears more natural.

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AN EXTENSION OF THE HADAMARD-PÓLYA DETERMINANTAL CRITERIA FOR UNIFORM FUNCTIONS

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1. *Introduction.*

The idea that the essential properties of a function represented by a Taylor series may be deduced from the coefficients of that series is exemplified in the results obtained by Hadamard in a famous thesis*. Here, if the uniform function

$$f(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots \tag{1}$$

has a radius of convergence $r > 0$, the ratios of the expressions

$$\overline{\lim_{n\to\infty}} \left| \sqrt[n]{D_n^{(p)}} \right|,$$

where

$$D_n^{(p)} = \begin{vmatrix} c_n & c_{n+1} & \cdots & c_{n+p} \\ c_{n+1} & c_{n+2} & \cdots & c_{n+p+1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ c_{n+p} & c_{n+p+1} & \cdots & c_{n+2p} \end{vmatrix},$$

are used to obtain the distances of the polar singularities of $f(z)$ from the origin. The associated set of determinants $D_p^{(n)}{}_n$ is used similarly to obtain the moduli of the regular zeros of the function.

The expressions $\overline{\lim_{n\to\infty}} \left| \sqrt[n]{D_k^{(n)}} \right|$ combine the formal properties of both types of determinant, but just as Hadamard found the *ratios* to be the effective criteria, so it will be seen that it is $\overline{\lim_{n\to\infty}} \left| \sqrt[n^2]{D_k^{(n)}} \right|$ which plays the

* 1, § 85; 2; 4, § 6.

important part in characterising the function. Pólya, in a comprehensive paper*, has obtained interesting results regarding the $D_k^{(n)}$ for an extensive class of functions. We know that, if $f(z)$ is regular at the origin, as in (1), then the set of the reciprocals of the index-points of its singularities is bounded. Pólya has shown that, if $f(z)$ is uniform and regular in the complementary set, and if this set is connex and of bounded conform-radius, then the $|D_k^{(n-1)}|^{1/\{(n+k-1)n\}}$ are bounded for $k < n$, however large n may be.

We are concerned with the four basic types of uniform function: (*a*) integral functions, (*b*) meromorphic functions, (*c*) functions with an isolated essential singularity and no others, (*d*) functions with an essential singularity which is a limit point of poles. Since any function of type (*c*) or (*d*) can be obtained from the corresponding function of type (*a*) or (*b*) by a suitable bilinear transformation on z, and since integral functions form a sub-class of meromorphic functions, the four types will be regarded as belonging to one class. Hadamard uses his criteria to *separate* the four types of function†, while it is possible to prove from Pólya's result that these four types possess the *common* property stated below:

THEOREM. *For the class of functions considered,*

$$\lim_{n\to\infty} |\sqrt[n^2]{D_k^{(n)}}| = 0, \tag{2}$$

where k is finite‡.

Owing to the extensive range of functions considered, the proof of Pólya's theorem requires an elaborate technique. We shall prove (2) by elementary methods and at the same time extend the result in a more powerful theorem.

THEOREM. *For the class of functions considered,*

$$\overline{\lim_{n\to\infty}} \frac{n^2 \log n}{\log 1/|D_k^{(n)}|} \leqslant \rho, \tag{3}$$

where k is finite and ρ is the order of the meromorphic or integral function involved.

If the function is of order zero, the limit is regular and relation (3) becomes an equality. Equality is attained also in the case of an integral function

* **4**, § 18; **6**.

† **2**, 131–3.

‡ This theorem includes as a special case the classical result, due to Kronecker, that, for a rational function, $D_0^{(n)} = 0$ for every $n \geqslant m$. See **1**, 321.

for which the sequence of dominant terms has Hadamard gaps and in the case of many other integral and meromorphic functions*.

The proofs of results (2) and (3) are given for integral functions in § 2, for meromorphic functions in § 3, and for functions of types (*c*) and (*d*) in § 4. Theorems 1–3 contain the result (2) implicitly and it is, therefore, not stated independently.

2. *Integral functions.*

If $f(z)$, defined by (1), is an integral function of order ρ, then

$$\overline{\lim_{n\to\infty}} \frac{n \log n}{\log 1/|c_n|} \leqslant \rho$$

and
$$|\sqrt[n]{c_n}| < (1/n)^{1/\rho'}, \quad |c_n| < (1/n)^{n/\rho'}$$

for any $n \geqslant m$ associated with a given $\rho' > \rho$.

In order to obtain a dominant for $|D_0^{(n)}|$ we consider the determinant obtained from $D_0^{(n)}$ by replacing every c_n by $c_n' = K(1/n)^{n/\rho'}$ and take every term positive. If the constant K is suitably chosen, every $c_n' > |c_n|$. It is easy to show that the maximum term is $c_n'^{n+1}$ and that all terms are *effectively*† of the same order. Hence

$$|D_0^{(n)}| < (n+1)!\, K^{n+1}(1/n)^{n(n+1)/\rho'}, \tag{4}$$

from which it follows that

$$\lim_{n\to\infty} |\sqrt[n]{D_0^{(n)}}| = 0, \quad \lim_{n\to\infty} |\sqrt[n^2]{D_0^{(n)}}| = 0,$$

and that
$$\overline{\lim_{n\to\infty}} \frac{n^2 \log n}{\log 1/|D_0^{(n)}|} \leqslant \rho'$$

for every $\rho' > \rho$.

For $D_k^{(n)}$ $(k > 0)$ we consider the integral function

$$c_k + c_{k+1}z + \ldots + c_{k+n}z^n + \ldots \equiv z^{-k}\{f(z) - c_0 - c_1 z - \ldots - c_{k-1}z^{k-1}\}, \tag{5}$$

and for $D_k^{(n)}$ $(k < 0)$, writing $k' = -k$,

$$c_0 z^{k'} + c_1 z^{k'+1} + \ldots + c_n z^{k'+n} + \ldots \equiv z^{k'} f(z), \tag{6}$$

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* For example, e^z. Also the Γ-function and its reciprocal.

† Every term u of the dominant satisfies the relation $\lim_{n\to\infty}\left(\frac{n^2 \log n}{\log 1/u}\right) = \rho'$.

both of which have the same order ρ as $f(z)$. Hence

THEOREM 1. *For every integral function of order ρ,*

$$\overline{\lim_{n\to\infty}} \frac{n^2 \log n}{\log 1/|D_k^{(n)}|} \leqslant \rho,$$

where k is finite.

3. *Meromorphic functions.*

Let $f(z)$, defined by (1), be a meromorphic function of order ρ having its sequence of poles at $1/a_n$, where $|a_n|$ tends *steadily* to zero. If λ is the exponent of convergence of the poles, then $\lambda \leqslant \rho$*, and

$$|a_n|^{\lambda'} < \frac{1}{n}, \tag{7}$$

for every $\lambda' > \lambda$ and $n > n_0$.

Consider the polynomials

$$Q_m(z) \equiv (1-a_1 z)(1-a_2 z)\dots(1-a_m z) \equiv 1+a_{m1}z+a_{m2}z^2+\dots+a_{mm}z^m,$$

and the associated operators† L_m, where

$$L_m c_s \equiv c_s + a_{m1}c_{s-1} + a_{m2}c_{s-2} + \dots + a_{mm}c_{s-m}$$

and

$$L_0 c_s \equiv c_s, \quad c_{-s} = 0 \quad (s > 0).$$

We use these to write $D_0^{(n)}$ in the form

$$D_0^{(n)} \equiv \begin{vmatrix} c_0 & c_1 & \dots & c_n \\ c_1 & c_2 & \dots & c_{n+1} \\ \dots & \dots & & \dots \\ \dots & \dots & & \dots \\ c_n & c_{n+1} & \dots & c_{2n} \end{vmatrix} = \begin{vmatrix} c_0 & L_1 c_1 & \dots & L_n c_n \\ c_1 & L_1 c_2 & \dots & L_n c_{n+1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ c_n & L_1 c_{n+1} & \dots & L_n c_{2n} \end{vmatrix}. \tag{8}$$

$L_m c_s$ is the coefficient of z^s in $Q_m(z)f(z)$, which has its poles at the same points as $f(z)$ except that the first m have been removed.

As in the case of integral functions (§2), we shall replace every element in the second determinant by a majorant of positive sign so chosen that all the terms in the dominant so formed are effectively of the same order. It will be sufficient, therefore, to find a suitable majorant for

$$|c_n L_1 c_n L_2 c_n \dots L_n c_n|.$$

* 5, ch. III.

† 1, 323.

To evaluate $L_m c_n$ we consider a majorant for the function $Q_m(z)f(z)$ on the circle $|z| = r$, where

$$|a_{m+1} r| < 1, \tag{9}$$

which contains no poles of the function, and use the following

LEMMA* 1. *If about each pole at $1/a_m$ a circle of radius $|a_m|^h$ is drawn, where h $(> \rho)$ is a positive constant, then, for $|z| = r$ sufficiently large and z outside all the circles,*

$$\log|f(z)| < r^{\rho'}$$

for every $\rho' > \rho$.

We assume that the small circles do not overlap and that $|z| = r$ does not intersect any of them, a restriction to be removed later.

Choosing $\rho' > \lambda'$ and taking $r = m^{1/\rho'}$, we observe from (7) that (9) is satisfied for $m > n_0$. Hence, on $|z| = r$, $|f(z)| < e^m$ for m sufficiently large, while, if M is a suitable constant,

$$|Q_m(z)| < M \prod_{s=1}^{m} \{1+(r/s^{1/\lambda'})\} < M \prod_{s=1}^{m} \{1+(r/s^{1/\rho'})\} < 2^m M m^{m/\rho'}/(m!)^{1/\rho'}.$$

From Binet's formula† it follows that

$$n \log n > \log(n!) > n(\log n - 1), \tag{10}$$

giving

$$m^{m/\rho'}/(m!)^{1/\rho'} < \exp\{(m \log m/\rho') - (m \log m - m)/\rho'\} = e^{m/\rho'}.$$

Hence, for m sufficiently large,

$$|Q_m(z)f(z)| < 2^m M e^{m(1+1/\rho')}.$$

Using Cauchy's inequality for the function $Q_m(z)f(z)$ on $|z| = r$,

$$|L_m c_n| < |Q_m(z)f(z)|/r^n < 2^m M e^{m(1+1/\rho')}/m^{n/\rho'}$$

for m sufficiently large, so that, if K is a suitable constant,

$$|L_m c_n| < K\, 2^m e^{m(1+1/\rho')}/m^{n/\rho'}$$

for every m. Hence

$$|c_n L_1 c_n L_2 c_n \dots L_n c_n| < K^n \{2e^{(1+1/\rho')}\}^{n(n+1)/2}/(n!)^{n/\rho'}. \tag{11}$$

* 3, 214.

† 7, 243.

From (8), (10), and (11) it follows that

$$D_0^{(n)} < (n+1)! \times \exp\left\{n \log K + \frac{n(n+1)}{2}\left(\log 2 + 1 + \frac{1}{\rho'}\right) - \frac{n^2}{\rho'}(\log n - 1)\right\},$$

whence

$$\frac{n^2 \log n}{\log|1/D_0^{(n)}|} \leqslant \frac{n^2 \log n}{(n^2 \log n/\rho') + O(n^2)} \leqslant \rho' + O(1/\log n) \tag{12}$$

for every $\rho' > \rho$.

A slight modification suffices to cover the cases of multiple poles or groups of poles in the cases in which the sets of circles in Lemma 1 overlap. For if $a_m = a_{m+1} = \ldots = a_{m+p}$, although the circle must be taken to exclude the point $1/a_{m+p}$ in p successive cases, yet the radius satisfies the better inequality for a_{m+p} in (9). The close grouping of a set of poles produces a similar effect.

As in § 2, for $D_k^{(n)}$ we consider the functions (5) and (6), both of which are meromorphic functions of the same order and with the same poles as $f(z)$, when k is finite. Hence, from (12), we have

THEOREM 2. *For every meromorphic function of order* ρ,

$$\overline{\lim_{n\to\infty}} \frac{n^2 \log n}{\log 1/|D_k^{(n)}|} \leqslant \rho,$$

where k *is finite.*

We note that, if the meromorphic function has only a finite number of poles, the proof reduces to that of § 2 after the initial stages.

4. *One essential singularity.*

If $f(z)$ in (1) represents a function with a unique essential singularity at the point $z = c$, $(c \neq 0)$, the substitution $z = cz'$ converts $f(z)$ into a function of the same type but with the essential singularity at $z' = 1$. This results in $D_k^{(n)}$ being multiplied by $c^{(n+1)^2}$ but has no effect on either of (2) or (3). Consequently, we shall consider only the case of functions in which the essential singularity is at the point 1.

Consider the function

$$f(z) = \sum_0^\infty \frac{a_s}{(1-z)^s} \equiv c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots. \tag{13}$$

According as the function $F(t)$ of order ρ, with the Taylor expansion

$$F(t) \equiv a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n + \ldots,$$

is either (a) an integral function or (b) a meromorphic function, having no pole at $t = 1$, so $f(z)$ is a function with an essential singularity at $z = 1$ which is (c) isolated, or (d) a limit point of poles. Evaluating the coefficients c_n by means of Cauchy's integral over a closed contour, including $z = 0$ but not including $z = 1$, we have

$$c_0 = F(1),$$

$$c_n = \frac{1}{n!}\left[\frac{d^n f(z)}{dz^n}\right]_{z=0} = \frac{1}{n!}\left[\frac{d^n t^{n-1}}{dt^n} F(t)\right]_{t=0} \quad (n > 0)$$

$$= F'(1) + {}^{n-1}C_1 \frac{F''(1)}{2!} + {}^{n-1}C_2 \frac{F'''(1)}{3!} + \ldots + {}^{n-1}C_{n-1} \frac{F^{n'}(1)}{n!},$$

where $t = 1/(1-z)$ and the contour in the t-plane includes the point $t = 1$ but no pole of $F(t)$ in case (b). Hence

$$c_n = b_1 + {}^{n-1}C_1 b_2 + {}^{n-1}C_2 b_3 + \ldots + b_n,$$

where

$$F(t+1) \equiv b_0 + b_1 t + b_2 t^2 + \ldots + b_n t^n + \ldots$$

is an integral or meromorphic function of order ρ. Thus, if $D_0^{(n)}$ refers to $f(z)$ of (13),

$$D_0^{(n)} = \begin{vmatrix} b_0 & b_1 & \ldots \\ b_1 & b_1 + b_2 & \ldots \\ b_1 + b_2 & b_1 + {}^2C_1 b_2 + b_3 & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ b_1 + {}^{n-1}C_1 b_2 + \ldots + b_n & b_1 + {}^nC_1 b_2 + \ldots + b_{n+1} & \ldots \end{vmatrix}$$

$$\begin{vmatrix} \ldots & b_1 + {}^{n-1}C_1 b_2 + \ldots + b_n \\ \ldots & b_1 + {}^nC_1 b_2 + \ldots + b_{n+1} \\ \ldots & b_1 + {}^{n+1}C_1 b_2 + \ldots + b_{n+2} \\ \ldots & \ldots \\ \ldots & \ldots \\ \ldots & b_1 + {}^{2n-1}C_1 b_2 + \ldots + b_{2n} \end{vmatrix}$$

Writing $b_0 = b_0 + 0$ and applying difference processes,

$$(14) \qquad D_0^{(n)} = \begin{vmatrix} 0 & b_1 & \dots & b_n \\ b_1 & b_2 & \dots & b_{n+1} \\ \dots & \dots & \dots & \\ \dots & \dots & \dots & \\ b_n & b_{n+1} & \dots & b_m \end{vmatrix} + b_0 \begin{vmatrix} b_1+b_2 & \dots & b_n+b_{n+1} \\ b_2+b_3 & \dots & b_{n+1}+b_{n+2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ b_n+b_{n+1} & \dots & b_{2n-1}+b_{2n} \end{vmatrix}.$$

These are the $D_0^{(n)}$ and $D_0^{(n-1)}$ determinants of the respective functions

$$F(t+1)-b_0, \quad b_0\left\{\frac{F(t+1)-b_0}{t} + \frac{F(t+1)-b_0-b_1 t}{t^2}\right\},$$

which are functions of order ρ. Writing these determinants as $D_0'^{(n)}$, $D_0''^{(n-1)}$, we deduce from (14) that

$$|D_0^{(n)}| \leqslant |D_0'^{(n)}| + |D_0''^{(n-1)}|,$$

and by use of the inequalities (4) or (12), as the case may be, it is readily shown that

$$\overline{\lim_{n\to\infty}} \frac{n^2 \log n}{\log 1/|D_0^{(n)}|} \leqslant \rho.$$

The case of $D_k^{(n)}$ is dealt with similarly, except that (14) may involve a sum of $k+1$ determinants. Hence we have

THEOREM 3. *For all functions of types* (c) *and* (d),

$$\overline{\lim_{n\to\infty}} \frac{n^2 \log n}{\log 1/|D_k^{(n)}|} \leqslant \rho,$$

where k is finite and ρ is the order of the integral or meromorphic function involved.

5. *An extension.*

It is well known that the order of the sum-function of a finite number of meromorphic (including integral) functions is that of the strongest component. Using the transformation $t = 1/(1-z)$, as in § 4, we obtain the following addition theorem.

THEOREM 4. *For the sum-function of a finite number of functions with the same essential singularity* (*which may be at infinity*),

$$\overline{\lim_{n\to\infty}} \frac{n^2 \log n}{\log 1/|D_k^{(n)}|} \leqslant \rho,$$

where ρ is the order of the strongest integral or meromorphic function involved.

It is possible to obtain a corresponding theorem, with suitable restrictions, for the sum of a countable number of such functions, but the main interest in the paper lies in the possibility of replacing the inequality in (3) by an equality. The attainment of this result will ensure a precise formulation of the "order" of a uniform function.

References.

1. P. Dienes, *The Taylor series* (1931).
2. J. Hadamard, "Essai sur l'études des fonctions données par leur développement de Taylor", *Journal de Math.* (4), 8 (1892), 101–186.
3. A. J. Macintyre and R. Wilson, "On the order of interpolated integral functions and of meromorphic functions with given poles", *Quart J. of Math.* (Oxford), 5 (1934), 211–20.
4. S. Mandelbrojt, *Les singularités des fonctions analytiques representées par une série de Taylor* (1932).
5. R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes* (1929).
6. G. Pólya, "Über gewisse notwendige Determinantenkriterien für die Fortsetzbarkeit einer Potenzreihe", *Math. Annalen*, 99 (1928), 687–706.
7. E. T. Whittaker and G. N. Watson, *Modern analysis* (1915).

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THE ZEROS OF EPSTEIN'S ZETA-FUNCTIONS

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1. In this paper we consider the distribution of the zeros of the functions

$$Z(s) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{1}{(am^2+bmn+cn^2)^s} = \Sigma\Sigma' \frac{1}{\{\phi(m, n)\}^s},$$

a particular case of Epstein's zeta-functions. Here a, b, and c are real numbers, $a > 0$, $c > 0$, and $\Delta = 4ac-b^2 > 0$, so that $\phi(m, n)$ is a positive definite quadratic form. The dash indicates that the summation is taken over all values of m and n except $m = n = 0$. We are primarily interested in the case where a, b, and c are integers, though some of our analysis applies to other cases as well.

These functions are closely connected with Dedekind's zeta-function $\zeta(s, \sqrt{-k})$ of the corpus $P(\sqrt{-k})$, where k is a positive integer with no squared factor. For*

$$\zeta(s, \sqrt{-k}) = \frac{1}{l} \sum_r \left\{ \Sigma\Sigma' \frac{1}{\{\phi_r(m, n)\}^s} \right\},$$

where ϕ_r runs through all classes of forms of discriminant $-\Delta$, $-\Delta$ being the discriminant of $P(\sqrt{-k})$, and l being the number of units of the corpus. Thus

$$\zeta(s, \sqrt{-1}) = \tfrac{1}{4}\Sigma\Sigma' \frac{1}{(m^2+n^2)^s}, \quad \zeta(s, \sqrt{-3}) = \tfrac{1}{2}\Sigma\Sigma' \frac{1}{(m^2+mn+n^2)^s},$$

$$\zeta(s, \sqrt{-5}) = \tfrac{1}{2}\Sigma\Sigma' \frac{1}{(m^2+5n^2)^s} + \tfrac{1}{2}\Sigma\Sigma' \frac{1}{(3m^2+4mn+3n^2)^s},$$

and so on.

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* E. Landau, "Über die Klassenzahl imaginär-quadratischer Zahlkörper", *Göttinger Nachrichten* (1918), 287.

This suggests that the Epstein zeta-functions arising in this way fall into two classes. If the number of classes of quadratic forms of discriminant $-\Delta$ is unity, the corresponding Epstein function $Z(s)$ is a multiple of Dedekind's zeta-function. It has all the characteristic properties of the Riemann zeta-function $\zeta(s)$, and it may be expected that any known theorem on $\zeta(s)$ may be extended to $Z(s)$. It is also reasonable to conjecture that there will be a "Riemann hypothesis" for such a $Z(s)$, viz. that all its complex zeros lie on $\sigma = \frac{1}{2}$. The simplest of these functions is

$$\Sigma\Sigma'(m^2+n^2)^{-s}.$$

A typical function of the other class is $Z(s) = \Sigma\Sigma'(m^2+5n^2)^{-s}$. This might be described as a "partial zeta-function", since it is only one term of a sum which makes up the Dedekind zeta-function of $P(\sqrt{-5})$. It has many of the properties of Riemann's $\zeta(s)$, it has a functional equation, an approximate functional equation, mean-value formulae, and so on. But it has no Euler product in terms of primes such as that on which the deeper properties of $\zeta(s)$ appear to depend. It seems to us to be of interest for precisely this reason, since it serves as a test by which we may distinguish between properties of $\zeta(s)$ which depend on the Euler product and those which do not.

Our main theorem is that every $Z(s)$ for which a, b, and c are integers and $\sqrt{\Delta}$ is irrational has an infinity of zeros on the critical line $\sigma = \frac{1}{2}$. The corresponding theorem for $\zeta(s)$ is usually considered as a step in the direction of the Riemann hypothesis. It is therefore interesting to have an example of a function for which the same step can be taken, but for which the "Riemann hypothesis" is probably false.

Whether the analogue of the Riemann hypothesis is actually false for these "partial zeta-functions" is not easy to prove. We have carried out some calculations with the function $\Sigma\Sigma'(m^2+5n^2)^{-s}$ with this object in view. They appear to indicate that the first six zeros of this function above the real axis lie on $\sigma = \frac{1}{2}$, and that the next two do not lie on this line. If our work is correct the question is settled, but the calculations are very cumbrous, and can hardly be considered conclusive. If the result suggested is true, the contrast between this $Z(s)$ and the Riemann $\zeta(s)$, for which the first 138 zeros* lie on $\sigma = \frac{1}{2}$, is striking.

We also prove that almost all the zeros of any $Z(s)$ for which the coefficients a, b, c are integers lie in the immediate neighbourhood of $\sigma = \frac{1}{2}$; but the deeper results in this direction for $\zeta(s)$ depend on the Euler product, and so cannot be extended to $Z(s)$.

* J. I. Hutchinson, "On the roots of the Riemann zeta-function", *Trans. American Math. Soc.*, 27 (1925), 49–60.

2. In the general case, Epstein's function $Z(s)$ is regular everywhere except for a simple pole at $s = 1$, and it satisfies the functional equation*

$$\left(\frac{2\pi}{\sqrt{\Delta}}\right)^{-s}\Gamma(s)\,Z(s) = \left(\frac{2\pi}{\sqrt{\Delta}}\right)^{s-1}\Gamma(1-s)\,Z(1-s). \tag{1}$$

Let

$$f(s) = e^{\frac{1}{2}\pi i(\frac{1}{2}-s)}(2\pi/\sqrt{\Delta})^{-s}\,\Gamma(s)\,Z(s), \tag{2}$$

and

$$W(t) = f(\tfrac{1}{2}+it). \tag{3}$$

Then the functional equation is $f(s) = f(1-s)$, and hence $W(t)$ is real for real t. The zeros of $Z(s)$ on $\sigma = \frac{1}{2}$ correspond to real zeros of $W(t)$, so that we have to prove that $W(t)$ has an infinity of changes of sign.

Our method is similar to one of the Hardy-Littlewood proofs† for $\zeta(s)$. This proof depends on the theorem of Landau that

$$J = \int_T^{T'} t^{\alpha}\left(\frac{t}{e\xi}\right)^{it} dt = O(T^{\alpha+\frac{1}{2}}) \quad (T \leqslant T' \leqslant 2T).$$

In our analysis we require the following more precise result.

LEMMA 1. *If J denotes the above integral, and $\alpha > 0$, we have*

$$J = O\{T^{\alpha}/\log(T/\xi)\} \quad (\xi < T), \tag{4}$$

$$J = (2\pi)^{\frac{1}{2}}\,\xi^{\alpha+\frac{1}{2}}\,e^{i(\frac{1}{4}\pi-\xi)} + O(T^{\alpha+\frac{1}{2}}) + O\{T^{\alpha}/\log(\xi/T)\} + O\{T^{\alpha}/\log(T'/\xi)\} \quad (T < \xi < T'), \tag{5}$$

$$J = O\{T^{\alpha}/\log(\xi/T')\} \quad (T' < \xi), \tag{6}$$

and

$$J = O(T^{\alpha+\frac{1}{2}}) \tag{7}$$

for all $\xi > A$.

Of these, (7) is equivalent to Landau's Satz 429, with α instead of $\frac{1}{8}$. (4) and (6) are immediate consequences of the second mean-value theorem;

* See H. S. A. Potter, *Proc. London Math. Soc.* (2), 36 (1934), 501–515, for a proof and reference to previous authors.

† See Landau's *Vorlesungen über Zahlentheorie*, VII Teil, Kap. 7.

e.g., for $\xi < T$,

$$\int_T^{T'} t^\alpha \cos\left(t \log \frac{t}{e\xi}\right) dt = T'^\alpha \int_{T''}^{T'} \cos\left(t \log \frac{t}{e\xi}\right) dt = T'^\alpha \int_{T''}^{T'} \frac{d \sin\{t \log(t/e\xi)\}}{\log(t/\xi)}$$

$$= \frac{T'^\alpha}{\log(T''/\xi)} \int_{T''}^{T'''} d \sin\left(t \log \frac{t}{e\xi}\right) = O\left\{\frac{T^\alpha}{\log(T/\xi)}\right\},$$

and similarly for the imaginary part of J.

If $T < \xi < T'$, we write

$$J = \int_T^{\xi-\rho} + \int_{\xi-\rho}^{\xi+\rho} + \int_{\xi+\rho}^{T'} = J_1 + J_2 + J_3,$$

where $\rho \leqslant \xi - T$, $\rho \leqslant T' - \xi$. The above argument shows that

$$J_1 = O\left\{\frac{T^\alpha}{\log\{(\xi+\rho)/\xi\}}\right\} = O\left(\frac{T^\alpha \xi}{\rho}\right) = O\left(\frac{T^{\alpha+1}}{\rho}\right),$$

and a similar result holds for J_3. Also, if $\rho^3 < A\xi^2$,

$$J_2 = \int_{-\rho}^{\rho} (\xi+u)^\alpha \exp\left[i(\xi+u) \log\{(\xi+u)/e\xi\}\right] du$$

$$= \int_{-\rho}^{\rho} \xi^\alpha \{1 + O(\rho/\xi)\} \exp\{-i\xi + \tfrac{1}{2}iu^2/\xi + O(\rho^3/\xi^2)\} du$$

$$= \xi^\alpha e^{-i\xi} \int_{-\rho}^{\rho} e^{\frac{1}{2}iu^2/\xi} \{1 + O(\rho/\xi) + O(\rho^3/\xi^2)\} du$$

$$= \xi^\alpha e^{-i\xi} \int_{-\rho}^{\rho} e^{\frac{1}{2}iu^2/\xi} du + O(\rho^2 \xi^{\alpha-1}) + O(\rho^4 \xi^{\alpha-2}),$$

and, if $\rho^2 > A\xi$,

$$\int_{-\rho}^{\rho} e^{\frac{1}{2}iu^2/\xi} du = (2\xi)^{\frac{1}{2}} \int_0^{\frac{1}{2}\rho^2/\xi} e^{iv} v^{-\frac{1}{2}} dv = (2\pi\xi)^{\frac{1}{2}} e^{\frac{1}{4}i\pi} - (2\xi)^{\frac{1}{2}} \int_{\frac{1}{2}\rho^2/\xi}^{\infty} e^{iv} v^{-\frac{1}{2}} dv$$

$$= (2\pi\xi)^{\frac{1}{2}} e^{\frac{1}{4}i\pi} + O(\xi/\rho).$$

Hence

$$J_2 = (2\pi)^{\frac{1}{2}} \xi^{\alpha+\frac{1}{2}} e^{i(\frac{1}{4}\pi - \xi)} + O(T^{1+\alpha}\rho^{-1}) + O(\rho^2 T^{\alpha-1}) + O(\rho^4 T^{\alpha-2}).$$

Taking $\rho = T^{\frac{3}{5}}$, we obtain (5), provided that $T + T^{\frac{3}{5}} \leqslant \xi \leqslant T' - T^{\frac{3}{5}}$. If $T < \xi < T + T^{\frac{3}{5}}$,

$$\int_{\xi - T^{\frac{3}{5}}}^{T} t^\alpha \left(\frac{t}{e\xi}\right)^{it} dt = O\left\{\frac{T^\alpha}{\log(\xi/T)}\right\},$$

as in the proof of (4), and (5) again follows. Similarly if $\xi > T' - T^{\frac{3}{5}}$.

3. LEMMA 2. *We have*

$$Z(s) = O(|t|^{1-\sigma+\delta}) \tag{8}$$

uniformly for $-\delta \leqslant \sigma \leqslant 1+\delta$, *and*

$$f(s) = O(|t|^{\frac{1}{2}+\delta}) \tag{9}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1+\delta$.

Clearly $Z(1+\delta+it)$ is bounded for a fixed positive δ; and, by the functional equation,

$$|Z(-\delta+it)| < A\left|\frac{\Gamma(1+\delta-it)}{\Gamma(-\delta+it)}\, Z(1+\delta-it)\right| < A\,|t|^{1+2\delta}$$

for $|t| > A$. Since $Z(s)$ is of finite order* in $-\delta \leqslant \sigma \leqslant 1+\delta$, (8) now follows from the usual Phragmén-Lindelöf argument. Since

$$e^{\frac{1}{2}\pi i(\frac{1}{2}-s)}(2\pi/\sqrt{\Delta})^{-s}\,\Gamma(s) = O(|t|^{\sigma-\frac{1}{2}}),$$

(9) also follows.

4. LEMMA 3. *If $R(x)$ is the number of lattice-points inside or on the ellipse*

$$a(u-u_0)^2+b(u-u_0)(v-v_0)+c(v-v_0)^2 = x,$$

where a, b, c, u_0, and v_0 are fixed, then

$$R(x) = 2\pi(4ac-b^2)^{-\frac{1}{2}}\,x+O(\sqrt{x}).$$

The proof given in Landau's *Vorlesungen*, 2, 186, for a circle, applies in the general case.

5. LEMMA 4. *If α is irrational,*

$$\sum_{n=N}^{N'} e^{2\pi i(\alpha n^2+\beta n)} = o(N'-N),$$

as $N'-N \to \infty$, uniformly with respect to β and N.

This is a well-known theorem of Hardy and Littlewood†. We give a short proof, substantially equivalent to Weyl's. For any real function‡ $f(n)$,

$$\sum_{n=N}^{N'} e^{2\pi i f(n)} = O\left(\frac{N'-N}{\sqrt{\rho}}\right)+O\left\{\frac{N'-N}{\rho}\sum_{r=1}^{\rho-1}\left|\sum_{n=N}^{N'-r} e^{2\pi i\{f(n+r)-f(n)\}}\right|\right\}^{\frac{1}{2}},$$

* See, *e.g.*, Potter, *loc. cit.*, Theorem 2.

† G. H. Hardy and J. E. Littlewood, *Acta Math.*, 37 (1914), 193–238; and H. Weyl, *Math. Annalen*, 77 (1915), 313–352.

‡ See E. C. Titchmarsh, *Quart. J. of Math.* (Oxford), 2 (1931), 161–173.

where $1 \leqslant \rho \leqslant N'-N$. Taking $f(n) = \alpha n^2+\beta n$, we have

$$\left| \sum_{n=N}^{N'-r} e^{2\pi i\{f(n+r)-f(n)\}} \right| = \left| \sum_{n=N}^{N'-r} e^{4\pi i\alpha nr} \right| \leqslant \operatorname{cosec} 2\pi\alpha r,$$

which is finite for every r, since α is irrational. Hence

$$\sum_{n=N}^{N'} e^{2\pi i(\alpha n^2+\beta n)} = O\left(\frac{N'-N}{\sqrt{\rho}}\right) + O\left\{\left(\frac{N'-N}{\rho}\sum_{r=1}^{\rho-1} |\operatorname{cosec} 2\pi\alpha r|\right)^{\frac{1}{2}}\right\}.$$

The result now follows if ρ is a sufficiently slowly increasing function of $N'-N$; *e.g.* let

$$\Omega(\rho) = \sum_{r=1}^{\rho-1} |\operatorname{cosec} 2\pi\alpha r|,$$

and let ρ be the greatest integer for which $\Omega(\rho) \leqslant N'-N-1$. Then ρ tends to infinity with $N'-N$, and, since $\Omega(\rho) \geqslant \rho-1$, $\rho \leqslant N'-N$. Hence

$$\sum_{n=N}^{N'} e^{2\pi i(\alpha n^2+\beta n)} = O\left(\frac{N'-N}{\sqrt{\rho}}\right) + O\left(\frac{N'-N}{\sqrt{\rho}}\right) = o(N'-N).$$

6. LEMMA 5. *If $a/\sqrt{\Delta}$ or $c/\sqrt{\Delta}$ is irrational (in particular if a, b, and c are integers and Δ is not a square), and $T^{\frac{1}{2}}\chi(T) \leqslant H \leqslant T$, where $\chi(T)$ is any positive function of T which tends to infinity with T, then*

$$\int_T^{T+H} W(t)\,dt = o(H) + O(T^{\frac{1}{2}+\delta})$$

for every positive δ.

Since $f(s)$ is regular in the rectangle formed by the lines $\sigma = \frac{1}{2}$, $\sigma = 1+\delta$, $t = T$, and $t = T+H$, we have

$$\begin{aligned}\int_T^{T+H} W(t)\,dt &= -i\int_{\frac{1}{2}+iT}^{\frac{1}{2}+i(T+H)} f(s)\,ds \\ &= -i\left\{\int_{\frac{1}{2}+iT}^{1+\delta+iT} + \int_{1+\delta+iT}^{1+\delta+i(T+H)} + \int_{1+\delta+i(T+H)}^{\frac{1}{2}+i(T+H)}\right\} f(s)\,ds.\end{aligned}$$

By Lemma 2, the first and third terms on the right-hand side are $O(T^{\frac{1}{2}+\delta})$. On $\sigma = 1+\delta$,

$$e^{\frac{1}{2}\pi i(\frac{1}{2}-s)}(2\pi/\sqrt{\Delta})^{-s}\,\Gamma(s) = Ct^{\frac{1}{2}+\delta}\left(\frac{t\sqrt{\Delta}}{2\pi e}\right)^{it} + O(t^{-\frac{1}{2}+\delta}),$$

where $C = (2\pi)^{-\frac{1}{2}-\delta}\Delta^{\frac{1}{2}+\frac{1}{2}\delta}$. The O-term gives

$$\int_T^{T+H} \sum_{\phi} O(t^{-\frac{1}{2}+\delta}/\phi^{1+\delta})\,dt = O(HT^{-\frac{1}{2}+\delta}) = O(T^{\frac{1}{2}+\delta}).$$

If we write $K = \sqrt{\Delta}/(2\pi)$, the main term is

$$C \sum_{\phi}' \frac{1}{\phi^{1+\delta}} \int_T^{T+H} t^{\frac{1}{2}+\delta} \left(\frac{Kt}{e\phi}\right)^{it} dt.$$

We divide the ϕ-range into the intervals

$$0,\ K(\sqrt{T}-1)^2,\ K(\sqrt{T}+1)^2,\ K\{\sqrt{(T+H)}-1\}^2,\ K\{\sqrt{(T+H)}+1\}^2,\ \infty,$$

each closed on the left. Let $\Sigma_1, \ldots, \Sigma_5$ be the corresponding parts of the above sum. Then

$$\Sigma_1 = \sum_{\nu < \sqrt{K}(\sqrt{T}-1)} \sum_{(\nu-1)^2 \leqslant \phi < \nu^2} \frac{1}{\phi^{1+\delta}} \int_T^{T+H} t^{\frac{1}{2}+\delta} \left(\frac{Kt}{e\phi}\right)^{it} dt.$$

By Lemma 3, the number of terms in the inner sum is $O(\nu)$; and, by Lemma 1 (4), each term is

$$O\left\{\frac{1}{\nu^{2+2\delta}} \frac{T^{\frac{1}{2}+\delta}}{\log(KT/\phi)}\right\} = O\left\{\frac{T^{\frac{1}{2}+\delta}}{\nu^{2+2\delta} \log(KT/\nu^2)}\right\}.$$

Hence

$$\Sigma_1 = O\left\{T^{\frac{1}{2}+\delta} \sum_{\nu < \sqrt{K}(\sqrt{T}-1)} \frac{1}{\nu^{1+2\delta} \log(KT/\nu^2)}\right\}.$$

The terms with $\nu < \frac{1}{2}\sqrt{(KT)}$ give

$$O\left(T^{\frac{1}{2}+\delta} \sum_{\nu < \frac{1}{2}\sqrt{(KT)}} \nu^{-1-2\delta}\right) = O(T^{\frac{1}{2}+\delta}).$$

If we write $M = [\sqrt{K}(\sqrt{T}-1)]$, the other terms are

$$O\left\{\sum_{m < \frac{1}{2}\sqrt{(KT)}} \frac{1}{\log\{(M+\sqrt{K})/(M-m)\}}\right\}$$

$$= O\left(\sum_{m < \frac{1}{2}\sqrt{(KT)}} \frac{M}{m+\sqrt{K}}\right) = O(T^{\frac{1}{2}} \log T).$$

Hence $$\Sigma_1 = O(T^{\frac{1}{2}+\delta}).$$

A similar argument applies to Σ_5. Next, by Lemma 1 (7), each term of Σ_2 is

$$O(T^{1+\delta} \phi^{-1-\delta}) = O(1),$$

and by Lemma 3 the number of terms is $O(T^{\frac{1}{2}})$. Hence

$$\Sigma_2 = O(T^{\frac{1}{2}}),$$

and a similar result holds for Σ_4.

Finally, by Lemma 1 (5),

$$\Sigma_3 = (2\pi)^{\frac{1}{2}} C e^{\frac{1}{4}i\pi} K^{-1-\delta} \Sigma e^{-2\pi i\phi/\sqrt{\Delta}} + O(T^{\frac{3}{10}+\delta} \Sigma \phi^{-1-\delta}) \tag{10}$$

$$+ O\left\{T^{\frac{1}{2}+\delta} \Sigma \frac{\phi^{-1-\delta}}{\log(\phi/KT)}\right\} + O\left\{T^{\frac{1}{2}+\delta} \Sigma \frac{\phi^{-1-\delta}}{\log\{K(T+H)/\phi\}}\right\},$$

each sum going from $K(\sqrt{T}+1)^2$ to $K\{\sqrt{(T+H)}-1\}^2$. The number of terms is

$$AKH + O(\sqrt{T}) = O(H),$$

so that the second term on the right of (10) is

$$O(T^{\frac{3}{10}+\delta} H T^{-1-\delta}) = O(HT^{-\frac{1}{10}}) = o(H).$$

If we write $N = [\sqrt{K}(\sqrt{T}+1)]+1$, the third term on the right of (10) is

$$O\left\{T^{-\frac{1}{2}} \Sigma \frac{1}{\log(\phi/KT)}\right\}$$

$$= O\left\{T^{-\frac{1}{2}} \sum_{\sqrt{K}(\sqrt{T}+1) \leqslant \nu < \sqrt{K}\{\sqrt{(T+H)}-1\}} \sum_{\nu^2 \leqslant \phi < (\nu+1)^2} \frac{1}{\log(\nu^2/KT)}\right\}$$

$$= O\left\{\sum_{0 \leqslant m < A\sqrt{T}} \frac{1}{\log\{(N+m)/(N-\sqrt{K})\}}\right\}$$

$$= O\left(\sum_{0 \leqslant m < A\sqrt{T}} \frac{N}{m+\sqrt{K}}\right)$$

$$= O(T^{\frac{1}{2}} \log T).$$

A similar argument applies to the fourth term on the right of (10).

Suppose that $c/\sqrt{\Delta}$ is irrational. Then

$$|\Sigma e^{-2\pi i\phi/\sqrt{\Delta}}| = \left|\sum_m \sum_n e^{-2\pi i(am^2+bmn+cn^2)/\sqrt{\Delta}}\right| \leqslant \sum_m \left|\sum_n e^{-2\pi i(bmn+cn^2)/\sqrt{\Delta}}\right|.$$

Since $H/\sqrt{T} \to \infty$, it is easily seen that the range of values of n consists of one or two intervals, the length of each of which tends to infinity. Hence, by Lemma 4,

$$\sum_n e^{-2\pi i(bmn+cn^2)/\sqrt{\Delta}} = o\left(\sum_n 1\right),$$

uniformly with respect to m. Hence

$$\Sigma e^{-2\pi i\phi/\sqrt{\Delta}} = o\left(\sum_m \sum_n 1\right) = o\{R(T+H)-R(T)\}$$

$$= o\{AH+O(\sqrt{T})\} = o(H)+o(\sqrt{T}).$$

If $a/\sqrt{\Delta}$ is irrational, a similar argument with m and n interchanged holds. This proves the lemma.

7. LEMMA 6. *Let λ be the least value other than 0 taken by $\phi(m, n)$, and $r(\lambda)$ the number of integer solutions of $\phi(m, n) = \lambda$. Then*

$$\int_T^{T+H} \lambda^{\frac{1}{2}+it} Z(\tfrac{1}{2}+it)\, dt = r(\lambda) H + O(T^{\frac{1}{2}+\delta})$$

for $0 < H \leqslant T$ and every positive δ.

We have

$$\int_{\frac{1}{2}+iT}^{\frac{1}{2}+i(T+H)} \lambda^s Z(s)\, ds = \left\{\int_{\frac{1}{2}+iT}^{2+iT} + \int_{2+iT}^{2+i(T+H)} + \int_{2+i(T+H)}^{\frac{1}{2}+i(T+H)}\right\} \lambda^s Z(s)\, ds = J_1 + J_2 + J_3,$$

say. By Lemma 2,

$$J_1 = O(T^{\frac{1}{2}+\delta}), \quad J_3 = O(T^{\frac{1}{2}+\delta}).$$

Also

$$\begin{aligned} J_2 &= \int_{2+iT}^{2+i(T+H)} r(\lambda)\, ds + \sum_{\phi<\lambda} \int_{2+iT}^{2+i(T+H)} \left(\frac{\lambda}{\phi}\right)^s ds \\ &= iHr(\lambda) + \sum_{\phi>\lambda} \left[\frac{(\lambda/\phi)^s}{\log(\lambda/\phi)}\right]_{2+iT}^{2+i(T+H)} \\ &= iHr(\lambda) + O(1). \end{aligned}$$

Hence the result.

8. THEOREM 1. *If $a/\sqrt{\Delta}$ or $c/\sqrt{\Delta}$ is irrational, the function $Z(s)$ has an infinity of zeros on the line $\sigma = \frac{1}{2}$. If $N_0(T)$ is the number of such zeros in $0 < t \leqslant T$, then*

$$N_0(T) > KT^{\frac{1}{2}-\epsilon}$$

for every positive ϵ and $T > T_0(\epsilon)$, $K = K(\epsilon)$.

If $W(t)$ is of constant sign over the interval T, $T+H$,

$$\left|\int_T^{T+H} W(t)\, dt\right| = \int_T^{T+H} |W(t)|\, dt.$$

By Lemma 5, the left-hand side is of the form

$$o(H) + O(T^{\frac{1}{2}+\delta})$$

for every positive δ; and, by Lemma 6,

$$\int_T^{T+H} |W(t)|\,dt > A\int_T^{T+H} |Z(\tfrac{1}{2}+it)|\,dt$$

$$> A\left|\int_T^{T+H} \lambda^{\frac{1}{2}+it} Z(\tfrac{1}{2}+it)\,dt\right| > AH+O(T^{\frac{1}{2}+\delta}).$$

This involves a contradiction of $H = T^{\frac{1}{2}+\epsilon}$ and $\delta < \epsilon$. Hence $W(t)$ has a zero in every interval $(T,\ T+T^{\frac{1}{2}+\epsilon})$ with T large enough. The theorem now follows.

A similar argument also gives the following result:

THEOREM 2. *The Dedekind zeta-function of any imaginary-quadratic corpus $P(\sqrt{-k})$ has an infinity of zeros on $\sigma = \frac{1}{2}$.*

9. Suppose next that a, b, and c are integers, which we may suppose, without loss of generality, to be prime to one another, and let

$$\sqrt{\Delta} = \sqrt{(4ac-b^2)} = k$$

be a positive integer. Then

$$4ac = b^2+k^2.$$

Since an odd square is congruent to 1 (mod 4), b and k must both be even. We write $b = 2b'$, $k = 2k'$.

LEMMA 7. *Let*

$$S(x) = \sum_{\phi \leqslant x} e^{-2\pi i\phi/k},$$

and

$$S_k = \sum_{m=0}^{k-1}\sum_{n=0}^{k-1} e^{-2\pi i\phi/k}.$$

Then

$$S(x) = 2\pi S_k k^{-3}x+O(\sqrt{x}).$$

Since $m \equiv \mu \pmod k$, $n \equiv \nu \pmod k$ implies that

$$\phi(m,\ n) \equiv \phi(\mu,\ \nu) \pmod k,$$

we have

$$S(x) = \sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1} \sum_{\phi \leqslant x,\ m\equiv\mu,\ n\equiv\nu \pmod k} e^{-2\pi i\phi(m,\ n)/k}$$

$$= \sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1} e^{-2\pi i\phi(\mu,\ \nu)/k} \sum_{\phi \leqslant x,\ m\equiv\mu,\ n\equiv\nu \pmod k} 1.$$

The inner sum is equal to the number of lattice-points inside or on the ellipse

$$a(uk+\mu)^2+b(uk+\mu)(vk+\nu)+c(vk+\nu)^2=x,$$

and so, by Lemma 3, is equal to

$$2\pi x\,k^{-2}(4ac-b^2)^{-\frac{1}{2}}+O(\sqrt{x})=2\pi x\,k^{-3}+O(\sqrt{x}),$$

the constant implied in the O depending only on a, b, and c (since $0\leqslant\mu\leqslant k-1$, $0\leqslant\nu\leqslant k-1$). Hence the result.

10. LEMMA 8. *If $k=2k'$, where k' is odd, then $S_k=0$. If k' is even, then $S_k=k\sigma_k$, where*

$$|\sigma_k|^2=\Sigma\Sigma e^{-\pi i\phi(m,\,n)/k'}$$

summed over values of m and n such that

$$0\leqslant m\leqslant k-1,\quad 0\leqslant n\leqslant k-1,$$

$$am+b'n\equiv 0\pmod{k'},\quad b'm+cn\equiv 0\pmod{k'}.$$

Since b is even, and $(a,\,b,\,c)=1$, one or other of a and c is odd, say a. Hence, if k' is odd,

$$e^{-\pi i\phi(m+k',\,n)/k'}=e^{-\pi i\{\phi(m,\,n)+2amk'+ak'^2+bnk'\}/k'}=-e^{-\pi i\phi(m,\,n)/k'}.$$

Hence

$$\sum_{m=0}^{k-1}e^{-\pi i\phi(m,\,n)/k}=0,$$

and so $S_k=0$.

In any case we have

$$\begin{aligned}|S_k|^2&=\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}\sum_{p=0}^{k-1}\sum_{q=0}^{k-1}e^{-\pi i\{\phi(p,\,q)-\phi(\mu,\,\nu)\}/k'}\\&=\sum_{m=0}^{k-1}\sum_{n=0}^{k-1}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}e^{-\pi i\{\phi(m+\mu,\,n+\nu)-\phi(\mu,\,\nu)\}/k'}\\&=\sum_{m=0}^{k-1}\sum_{n=0}^{k-1}e^{-\pi i\phi(m,\,n)/k'}\sum_{\mu=0}^{k-1}e^{-2\pi i(am+b'n)\mu/k'}\sum_{\nu=0}^{k-1}e^{-2\pi i(b'm+cn)\nu/k'}.\end{aligned}$$

Now

$$\begin{aligned}\sum_{\mu=0}^{k-1}e^{-2\pi i(am+b'n)\mu/k'}&=k\quad[am+b'n\equiv 0\pmod{k'}]\\&=0\text{ otherwise,}\end{aligned}$$

and similarly for the ν-sum. Hence the second result.

11. THEOREM 3. *If a, b, and c are relatively prime integers, and $k=\sqrt{(4ac-b^2)}$ is rational and congruent to* 2 (mod 4), *then*

$$N_0(T) > KT^{\frac{1}{2}-\epsilon} \quad [T > T_0(\epsilon)].$$

The same result holds when $k \equiv 0 \pmod 4$, *provided that*

$$\frac{r(\lambda)}{\sqrt{\lambda}} > \frac{2\pi|\sigma_k|}{\sqrt{k}}. \tag{11}$$

The analysis of Lemma 5 now shows that

$$\begin{aligned}\int_T^{T+H} W(t)\,dt &= (2\pi)^{\frac{1}{2}} K^{-1-\delta} Ce^{\frac{1}{4}i\pi}\left\{S[K\{\sqrt{(T+H)}-1\}^2]-S\{K(\sqrt{T}+1)^2\}\right\} \\ &\quad +o(H)+O(T^{\frac{1}{2}+\delta}) \\ &= (2\pi)^{\frac{3}{2}} Ce^{\frac{1}{4}i\pi} S_k k^{-3} K^{-\delta} H+o(H)+O(T^{\frac{1}{2}+\delta}) \\ &= 2\pi e^{\frac{1}{4}i\pi} S_k k^{-2} H+o(H)+O(T^{\frac{1}{2}+\delta})\end{aligned}$$

by Lemma 7. The result now follows as before if $k \equiv 2 \pmod 4$; and if $k \equiv 0 \pmod 4$ a similar proof holds provided that (11) is satisfied.

For example, we obtain the desired result for $\phi(m, n) = m^2+9n^2$ $(k=6)$, but not for $\phi(m, n) = m^2+4n^2$, where $k=4$, $\lambda=1$, $r(\lambda)=2$, and $|\sigma_4|^2=8$. In the latter case, however, we have

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{1}{(m^2+4n^2)^s} = 2\{1-2^{-s}+2^{1-2s}\} \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$$

for $\sigma > 1$, so that the result holds for this function also, since it is true for $\zeta(s)$.

12. THEOREM 4. *Let $N(T)$ be the number of zeros of $Z(s)$ for $0 < t \leqslant T$. Then*

$$N(T) = \frac{1}{\pi} T \log T - \frac{1}{\pi}\left(1+\log\frac{2\pi\lambda}{\sqrt{\Delta}}\right) T + O(\log T),$$

where λ is the smallest value other than 0 *taken by $\phi(m, n)$.*

The proof is similar to Backlund's proof* of the corresponding theorem for $\zeta(s)$. The only new point is that here the variation of am $Z(s)$ from σ_1 to σ_1+iT is, if σ_1 is large enough,

$$-T\log\lambda+O(1),$$

and not merely $O(1)$ as in the $\zeta(s)$ case.

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* See E. C. Titchmarsh, *The zeta-function of Riemann*, 4–5.

13. THEOREM 5. *Let $N(\sigma', T)$ denote the number of zeros of $Z(s)$ for $\sigma > \sigma'$, $0 < t < T$. Then*

$$N(\sigma, T) = O(T)$$

for any fixed $\sigma > \frac{1}{2}$, provided that the coefficients a, b, c of the quadratic form are integers.

This result may be deduced from a theorem due to Littlewood*. As we have already seen, $Z(s)$ is regular except for a simple pole at $s = 1$, and is of finite order in the region $\sigma \geqslant \frac{1}{2}$. The only other condition required by Littlewood's theorem is that the mean value of $|Z(s)|^2$ should exist for $\sigma > \frac{1}{2}$. But according to a theorem of Carlson† this is so, essentially because $Z(s)$ satisfies the functional equation (1) and‡

$$r(n) = O(n^{\epsilon})$$

for any $\epsilon > 0$, where $r(n)$ denotes the number of integer solutions of $\phi(x, y) = n$.

It follows at once from Theorems 4 and 5 that almost all the zeros of $Z(s)$ lie in the immediate neighbourhood of the line $\sigma = \frac{1}{2}$.

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* See E. C. Titchmarsh, *Theory of functions* (Oxford, 1932), 9, 623, or J. E. Littlewood, *Proc. Camb. Phil. Soc.*, 22 (1924), 295–318.

† F. Carlson, *Arkiv för Mat. Astr. och Fysik*, 19 (1926), No. 25, 12.

‡ E. Landau, *Göttinger Nachrichten* (1912), 750, Lemma 14.

THE DISTRIBUTION OF VELOCITIES IN A SLIGHTLY NON-UNIFORM GAS

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The determination of the law of distribution of velocities in a slightly non-uniform gas has been carried out by two methods which appear at first sight to be entirely unconnected. The first attack on the problem was made by Boltzmann, who showed that the distribution functions must satisfy a certain integral equation*, but the solution of this equation proved cumbersome and was completed satisfactorily only in 1917 by Enskog†. The second method is due to Chapman, and depends on the consideration of Maxwell's equations of transfer of certain "molecular properties" through a small volume of the gas; in the case of a simple gas in which the temperature and mean molecular velocity vary slightly from point to point, he shows that the number of molecules with velocities between (U, V, W) and $(U+dU, V+dV, W+dW)$ is $f_0(1+F)\,dU\,dV\,dW$, where

$$f_0 = \left(\frac{hm}{\pi}\right)^{\frac{3}{2}} \exp\{-hm(U^2+V^2+W^2)\},$$

the ordinary distribution function for a uniform gas, and

$$F = \frac{1}{T}\left(U\frac{\partial T}{\partial x}+V\frac{\partial T}{\partial y}+W\frac{\partial T}{\partial z}\right) P(C^2)$$
$$+\{c_{11}U^2+c_{22}V^2+c_{33}W^2+c_{23}VW+c_{31}WU+c_{12}UV\}\,Q(c^2),$$

* *Cf.* Jeans, *Dynamical theory of gases*, Ch. 8.

† I have not been able to consult this work and quote from references given by Chapman, *Phil. Trans. Royal Soc.* (A), 216 (1916), 279–348, 217 (1917), 115–197; these papers contain a bibliography of the subject up to 1917.

c_{11}, c_{22}, etc., being certain constants depending on the gradient of the mean velocity, T the temperature, and $P(C^2)$, $Q(C^2)$ integral functions of $C^2 = U^2+V^2+W^2$. If expansions $P(C^2) = \Sigma \beta_{r-1} C^{2r}$, $Q(C^2) = \Sigma \gamma_{r-1} C^{2r}$ are assumed and the rate of transfer of the "molecular properties" UC^{2s} and $U^2 C^{2s}$ is calculated, linear equations are obtained of the type

$$\sum_{r=0}^{\infty} \beta_r b_{rs} = 1, \quad \sum_{r=0}^{\infty} \gamma_r c_{rs} = 1,$$

which must be true for all values of s, whence β_r and γ_r are determined as the ratios of infinite determinants. The similar but more complex calculations for a mixture of two gases are given in the second of the two papers referred to above.

It is found, however, that the computation of these coefficients β_r and γ_r is very heavy, and even after the determinants are differenced the values obtained converge very slowly and the amount of work required to get good approximations is almost prohibitive. Fortunately, the main purpose of Chapman's papers is the calculation of the coefficients of viscosity, thermal conductivity, and diffusion, which depend on the sums of the coefficients, and the ratios of determinants which represent these sums converge so rapidly that the fourth approximations involve errors of less than one part in 1000. The method is also open to the objection that the investigation of the convergence of the series obtained appears to be somewhat difficult; Prof. Chapman informs me that the later paper on this part of the problem, which is foreshadowed in Note C of his 1917 paper, has not yet been published.

An alternative method of carrying out the calculations is given in this paper, by means of which these difficulties are avoided; the distribution functions $P(C^2)$, $Q(C^2)$ are supposed expanded, not in powers of C^2, but in series of Sonine's polynomials

$$\Sigma(-1)^k \Gamma(k+\tfrac{5}{2}) b_k T_{\frac{3}{2}}^k(hmC^2) \quad \text{and} \quad \Sigma(-1)^k \Gamma(k+\tfrac{7}{2}) c_k T_{\frac{5}{2}}^k(hmC^2),$$

and the equations of transfer of the "molecular properties" $UT_{\frac{3}{2}}^n(hmC^2)$, $U^2 T_{\frac{5}{2}}^n(hmC^2)$ are considered. The case of Maxwellian molecules is particularly simple on account of certain orthogonal properties of these polynomials, and in the general case, where the molecules exert forces on one another inversely proportional to the n-th power of the distance, the calculations can be carried out without the elaborate expansions necessary in Chapman's method; the investigation of the convergence of the series obtained presents no difficulties and the computation, although essentially the same as in Chapman's papers, is materially lightened by the form in

which the results are obtained. The solution of Boltzmann's equation involves essentially the same analysis and leads to the same equations as are obtained by the method of transfer; the two methods of solution are, in fact, identical.

The paper is divided into two parts, the first giving the required properties of Sonine's polynomials, while the second contains the main results given by Chapman in his two classical papers, obtained by the solution of Boltzmann's equation and by the method of transfer.

PART I.

1.1. If n is an integer, Sonine's polynomial $T_m{}^n(x)$ is defined by the expansion

$$(1+s)^{-m-1}\exp\left(\frac{xs}{1+s}\right)=\sum_{n=0}^{\infty}\Gamma(m+n+1)\,s^n\,T_m{}^n(x). \qquad (1.11)$$

Hence

$$T_m{}^n(x)=\frac{1}{2\pi i\,\Gamma(m+n+1)}\int\frac{ds}{s^{n+1}(1+s)^{m+1}}\exp\left(\frac{xs}{1+s}\right),$$

taken positively round any suitable contour enclosing the origin but not the point $s=-1$, and this can be used as the definition of $T_m{}^n(x)$ when n is not a positive integer, provided that $\arg s$ is restricted so as to make the integrand one-valued; but only integral values of n are considered in this paper and this will not be stated explicitly when the functions are used. The known result

$$\int_0^{\infty}T_m{}^n(x)\,T_m{}^{n_1}(x)\,x^m e^{-x}\,dx=0 \qquad (n_1\neq n) \quad (1.12)$$

$$=\frac{1}{\Gamma(n+1)\,\Gamma(m+n+1)} \qquad (n_1=n)$$

follows directly from either definition. It is found that the analysis in distribution function problems develops in terms of the functions $(-1)^n T_m{}^n(x)$ more naturally than when the functions are all taken positively and it is convenient to define

$$S_m{}^n(x)=(-1)^n\,T_m{}^n(x),$$

which is equal to the coefficient of s^n in the expansion of

$$\frac{(1-s)^{-m-1}}{\Gamma(m+n+1)}\exp\left(-\frac{xs}{1-s}\right), \qquad (1.13)$$

and hence

$$S_m^{(n)}(x) = \frac{1}{2\pi i\,\Gamma(m+n+1)} \int \frac{ds}{s^{n+1}(1-s)^{m+1}} \exp\left(-\frac{xs}{1-s}\right)$$

round a contour enclosing $s=0$ but not $s=1$, with a restriction as before on $\arg s$ if n is not an integer. The result (1.12) stated above is evidently true when the T functions are replaced by S functions; and it can be generalized as follows.

Let two bodies, initially at an infinitely great distance apart, approach one another, interact (the force which they exert on one another being for the moment an arbitrary function of their distance apart), and then again separate to an infinite distance. As is usual in this connection, the interaction between the two bodies will be referred to as a "collision" without any assumption that they actually come into contact. Following Chapman's notation, let the masses of the bodies be m_1 and m_2, and put $m_1+m_2=m_0$, $m_1=\mu_1 m_0$, $m_2=\mu_2 m_0$, so that $\mu_1+\mu_2=1$; also let the initial velocities of the bodies be C_1 and C_2 with components U_1, V_1, W_1 and U_2, V_2, W_2, and their final velocities C_1' and C_2' with components U_1', V_1', W_1' and U_2', V_2', W_2'. The velocity of the centre of gravity is C_0 and the relative velocity of the bodies, which is altered in direction but not in magnitude by the collision, is $(\mu_1\mu_2)^{-\frac{1}{2}} C_R$; also let ϵ be the angle between the plane containing the directions of C_0 and C_R before collision and the plane of the relative motion, and consider all the collisions in which the deflection of the relative motion is χ; then the integrals

$$\int_0^{2\pi} d\epsilon \iiint\iiint F \exp(-hm_1 C_1^2 - hm_2 C_2^2)\, dU_1\, dV_1\, dW_1\, dU_2\, dV_2\, dW_2,$$

taken over all values of the velocity components, where F may have any of the values

$$S_{\frac{1}{2}}^{k}(hm_1 C_1'^2)\, S_{\frac{1}{2}}^{n}(hm_1 C_1^2), \quad S_{\frac{1}{2}}^{k}(hm_1 C_1'^2)\, S_{\frac{1}{2}}^{n}(hm_2 C_2^2),$$

$$U_1' U_1 S_{\frac{3}{2}}^{k}(hm_1 C_1'^2)\, S_{\frac{3}{2}}^{n}(hm_1 C_1^2), \quad U_1' U_2 S_{\frac{3}{2}}^{k}(hm_1 C_1'^2)\, S_{\frac{3}{2}}^{n}(hm_2 C_2^2),$$

$$U_1'^2(3U_1^2 - C_1^2)\, S_{\frac{5}{2}}^{k}(hm_1 C_1'^2)\, S_{\frac{5}{2}}^{n}(hm_1 C_1^2),$$

$$U_1'^2(3U_2^2 - C_2^2)\, S_{\frac{5}{2}}^{k}(hm_1 C_1'^2)\, S_{\frac{5}{2}}^{n}(hm_2 C_2^2),$$

and similar functions with the suffixes 1 and 2 interchanged or with the dashed letters replaced by undashed ones, all vanish unless $k=n$, in which case their values can be expressed simply as the coefficients in certain binomial expansions. The results may be obtained by replacing

the Sonine functions by exponentials from the definition (1.13) and evaluating the integrals by Chapman's method of taking as variables of integration the components of C_0 and C_R; but the solution of Boltzmann's equation requires the values of the integrals with respect to U_2, V_2, W_2, and ϵ, *i.e.* for fixed values of U_1, V_1, W_1, so that it is convenient to determine these first and deduce the others by subsequent integration with respect to U_1, V_1, W_1.

1.2. Consider the integral

$$\int_0^{2\pi} d\epsilon \iiint S_{\frac{1}{2}}^{k}(hm_1 C_1'^2) \exp(-hm_2 C_2^2)\, dU_2 dV_2 dW_2$$

= coefficient of s^k in the expansion in powers of s of

$$\frac{(1-s)^{-\frac{3}{2}}}{\Gamma(k+\frac{3}{2})} \int_0^{2\pi} d\epsilon \iiint \exp(-ahm_1 C_1'^2 - hm_2 C_2^2)\, dU_2 dV_2 dW_2, \quad (1.211)$$

where $a = s/(1-s)$.

Now, if $(\mu_1\mu_2)^{-\frac{1}{2}} U_R$, $(\mu_1\mu_2)^{-\frac{1}{2}} V_R$, $(\mu_1\mu_2)^{-\frac{1}{2}} W_R$ are the components of the velocity of the second body relative to the first*, $U_2 = U_1 + (\mu_1\mu_2)^{-\frac{1}{2}} U_R$, ..., and $dU_1 dV_1 dW_1 = (\mu_1\mu_2)^{-\frac{3}{2}} dU_R dV_R dW_R$. Also

$$U_1' = U_1 + 2\sqrt{\left(\frac{\mu_2}{\mu_1}\right)} U_R \sin^2 \tfrac{1}{2}\chi - \sqrt{\left(\frac{\mu_2}{\mu_1}\right)} \sqrt{(V_R^2 + W_R^2)} \sin\chi \cos(\epsilon + \phi_0),$$

with similar expressions for V_1' and W_1', from which it follows that

$$\mu_1 C_1'^2 = \mu_1 C_1^2 + 4\mu_2 C_R^2 \sin^2 \tfrac{1}{2}\chi + 4(\mu_1\mu_2)^{\frac{1}{2}} C_1 C_R \cos\theta_{1R} \sin^2 \tfrac{1}{2}\chi$$
$$- 2(\mu_1\mu_2)^{\frac{1}{2}} C_1 C_R \sin\chi \sin\theta_{1R} \cos\epsilon, \quad (1.212)$$

θ_{1R} being the angle between the directions of C_1 and C_R before the collision. Further,

$$C_2^2 = C_1^2 + (\mu_1\mu_2)^{-1} C_R^2 + 2(\mu_1\mu_2)^{-\frac{1}{2}} C_1 C_R \cos\theta_{1R},$$

so that the integral in (1.211) can be written as

$$(\mu_1\mu_2)^{-\frac{3}{2}} \int_0^{2\pi} d\epsilon \iiint \exp[-hm_0\{aC_1^2 + bC_R^2 + 2C_1 C_R(c\cos\theta_{1R} + d\sin\theta_{1R}\cos\epsilon)\}]$$
$$\times C_R^2 dC_R \sin\theta_{1R} d\theta_{1R} d\phi_{1R},$$

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* In Chapman's work, $(\mu_1\mu_2)^{-\frac{1}{2}} C_R$ is the velocity of the first body relative to the second, but it has been found convenient to change the sign; otherwise Chapman's notation is followed rather than that of Jeans.

on using the direction of C_1 as the axis of the new polar coordinates; the values of the constants are

$$\left.\begin{aligned} a=\mu_2+\alpha\mu_1, \quad b=\frac{1}{\mu_1}(1+4\alpha\mu_1\mu_2\sin^2\tfrac{1}{2}\chi), \\ c=\sqrt{\left(\frac{\mu_2}{\mu_1}\right)}(1+2\alpha\mu_1\sin^2\tfrac{1}{2}\chi), \quad d=-(\mu_1\mu_2)^{\frac{1}{2}}\alpha\sin\chi. \end{aligned}\right\} \quad (1.213)$$

The integrand is independent of ϕ_{1R}, so that this integration can be carried out at once; further, if

$$c=\gamma\cos\eta, \quad d=\gamma\sin\eta, \quad \cos\vartheta=\cos\theta_{1R}\cos\eta+\sin\theta_{1R}\sin\eta\cos\epsilon,$$

the integration with respect to θ_{1R} and ϵ is equivalent to integration with respect to ϑ and a new azimuthal angle, and the integral is equal to the difference of the integrals

$$\frac{2\pi^2(\mu_1\mu_2)^{-\frac{3}{2}}}{\gamma hm_0 C_1}\int_0^\infty \exp\left[-hm_0\{aC_1^2+bC_R^2\pm 2\gamma C_1 C_R\}\right] C_R dC_R$$

$$=\frac{2\pi^{\frac{5}{2}}(\mu_1\mu_2)^{-\frac{3}{2}}}{(hm_0)^{\frac{3}{2}}b^{\frac{3}{2}}}\exp\left\{-hm_0 C_1^2\left(a-\frac{\gamma^2}{b}\right)\right\}.$$

Similarly,

$$\int_0^{2\pi} d\epsilon \iiint \exp\left(-ahm_1 C_1'^2-hm_2 C_2^2\right) C_R^p\, dU_2 dV_2 dW_2$$

$$=\frac{2\pi^2(\mu_1\mu_2)^{-\frac{3}{2}}}{\gamma hm_0 C_1}\int_0^\infty \exp\left\{-hm_0(aC_1^2+bC_R^2)\right\} C_R^{p+1}\sum_{n=0}^{\infty}\frac{(2hm_0\gamma C_1 C_R)^{2n+1}}{(2n+1)!}dC_R$$

$$=\frac{2\pi^{\frac{5}{2}}(\mu_1\mu_2)^{-\frac{3}{2}}}{(hm_0 b)^{\frac{1}{2}(p+3)}}\exp\left(-ahm_0 C_1^2\right)\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2}p+\frac{3}{2})}{n!\,\Gamma(n+\frac{3}{2})}\left(\frac{hm_0\gamma^2 C_1^2}{b}\right)^n,$$

or it may be written symbolically as

$$\frac{2\pi^{\frac{5}{2}}(\mu_1\mu_2)^{-\frac{3}{2}}}{(hm_0)^{\frac{3}{2}}}\left(-\frac{1}{hm_0}\frac{\partial}{\partial b}\right)^{\frac{1}{2}p}\frac{\exp\left[-hm_0 C_1^2\{a-(\gamma^2/b)\}\right]}{b^{\frac{3}{2}}}.$$

It is convenient to express these results in terms of Sonine polynomials as follows. Let

$$\frac{ab-\gamma^2}{b}=\frac{\mu_1 x}{1-x},$$

so that

$$x=\frac{ab-\gamma^2}{(a+\mu_1)b-\gamma^2}$$

and $$b = \frac{1-x}{\mu_1}\{(a+\mu_1)b-\gamma^2\};$$

then

$$\frac{1}{b^{\frac{3}{2}}}\exp[-hm_0 C_1^2\{a-(\gamma^2/b)\}]$$

$$= \frac{\mu_1^{\frac{3}{2}}}{\{(a+\mu_1)b-\gamma^2\}^{\frac{3}{2}}}\sum_{n=0}^{\infty}\Gamma(n+\tfrac{3}{2})x^n S_{\frac{1}{2}}^n(hm_1 C_1^2)$$

$$= \mu_1^{\frac{3}{2}}\sum_{n=0}^{\infty}\Gamma(n+\tfrac{3}{2})(ab-\gamma^2)^n\{(a+\mu_1)b-\gamma^2\}^{-n-\frac{3}{2}}S_{\frac{1}{2}}^n(hm_1 C_1^2),$$

and, on substituting for a, b, γ, we find that

$$ab-\gamma^2 = a(1-4\mu_1\mu_2\sin^2\tfrac{1}{2}\chi) = \frac{s}{1-s}(1-4\mu_1\mu_2\sin^2\tfrac{1}{2}\chi), \quad (1.214)$$

$$(a+\mu_1)b-\gamma^2 = \frac{1}{1-s}.$$

It follows that

$$\frac{1}{b^{\frac{3}{2}}}\exp\left\{-hm_0 C_1^2\left(a-\frac{\gamma^2}{b}\right)\right\}$$

$$= \mu_1^{\frac{3}{2}}(1-s)^{\frac{3}{2}}\sum_{n=0}^{\infty}\Gamma(n+\tfrac{3}{2})s^n(1-4\mu_1\mu_2\sin^2\tfrac{1}{2}\chi)^n S_{\frac{1}{2}}^n(hm_1 C_1^2)$$

and

$$\int_0^{2\pi}d\epsilon\iiint S_{\frac{1}{2}}^k(hm_1 C_1'^2)\exp(-hm_2 C_2^2)\,dU_2\,dV_2\,dW_2$$

$$= \frac{2\pi^{\frac{5}{2}}}{(hm_2)^{\frac{3}{2}}}(1-4\mu_1\mu_2\sin^2\tfrac{1}{2}\chi)^k S_{\frac{1}{2}}^k(hm_1 C_1^2). \quad (1.21)$$

Hence

$$\int_0^{2\pi}d\epsilon\iiiint\!\!\!\!\iint S_{\frac{1}{2}}^k(hm_1 C_1'^2)\,S_{\frac{1}{2}}^n(hm_1 C_1^2)\exp(-hm_1 C_1^2-hm_2 C_2^2)$$
$$\times dU_1\,dV_1\,dW_1\,dU_2\,dV_2\,dW_2$$

$$= \frac{8\pi^{\frac{7}{2}}}{(hm_2)^{\frac{3}{2}}}(1-4\mu_1\mu_2\sin^2\tfrac{1}{2}\chi)^k\int_0^{\infty}S_{\frac{1}{2}}^k(hm_1 C_1^2)\,S_{\frac{1}{2}}^n(hm_1 C_1^2)$$
$$\times\exp(-hm_1 C_1^2)\,C_1^2\,dC_1$$

$$= 0 \qquad (k\neq n)$$

$$= \frac{4\pi^{\frac{7}{2}}}{(h^2 m_1 m_2)^{\frac{3}{2}}}\frac{(1-4\mu_1\mu_2\sin^2\tfrac{1}{2}\chi)^n}{n!\,\Gamma(n+\frac{3}{2})} \quad (k=n), \qquad (1.22)$$

by an obvious application of (1.12).

Similarly,

$$\int_0^{2\pi} d\epsilon \iiint C_R{}^p S_{\frac{1}{2}}{}^k(hm_1 C_1'^2) \exp(-hm_2 C_2^2)\, dU_2 dV_2 dW_2$$

$$= \frac{2\pi^{\frac{5}{2}}}{(hm_2)^{\frac{3}{2}}\Gamma(k+\frac{3}{2})} \times \text{coefficient of } s^k \text{ in } (1-s)^{-\frac{3}{2}}\left(-\frac{1}{hm_0}\frac{\partial}{\partial b}\right)^{\frac{1}{2}p}$$

$$\times \sum_{n=0}^{\infty} \Gamma(n+\tfrac{3}{2})(ab-\gamma^2)^n \{(a+\mu_1)b-\gamma^2\}^{-n-\frac{3}{2}} S_{\frac{1}{2}}{}^n(hm_1 C_1^2)$$

$$= \frac{2\pi^{\frac{5}{2}}\Gamma(\frac{3}{2})}{(hm_2)^{\frac{3}{2}}\Gamma(k+\frac{3}{2})} \sum_{n=0}^{\infty} n!\, S_{\frac{1}{2}}{}^n(hm_1 C_1^2) \times \text{coefficient of } s^k t^n \text{ in}$$

$$(1-s)^{-\frac{3}{2}}\left(-\frac{1}{hm_0}\frac{\partial}{\partial b}\right)^{\frac{1}{2}p} \{(a+\mu_1)b-\gamma^2-(ab-\gamma^2)t\}^{-\frac{3}{2}}$$

$$= \frac{2\pi^{\frac{5}{2}}\Gamma(\frac{1}{2}p+\frac{3}{2})}{(hm_2)^{\frac{3}{2}}\Gamma(k+\frac{3}{2})} \frac{1}{(hm_0)^{\frac{1}{2}p}} \sum_{n=0}^{\infty} n!\, S_{\frac{1}{2}}{}^n(hm_1 C_1^2) \times \text{coefficient of } s^k t^n \text{ in}$$

$$(1-s)^{-\frac{3}{2}}\{a+\mu_1-at\}^{\frac{1}{2}p} \{(a+\mu_1)b-\gamma^2-(ab-\gamma^2)t\}^{-\frac{1}{2}(p+3)},$$

or, on substituting for a, b, γ, we have

$$\frac{2\pi^{\frac{5}{2}}\Gamma(\frac{1}{2}p+\frac{3}{2})}{(hm_2)^{\frac{3}{2}}(hm_0)^{\frac{1}{2}p}\Gamma(k+\frac{3}{2})} \sum_{n=0}^{\infty} n!\, S_{\frac{1}{2}}{}^n(hm_1 C_1^2) \times \text{coefficient of } s^k t^n \text{ in}$$

$$\{1-\mu_2(s+t)+(\mu_2-\mu_1)st\}^{\frac{1}{2}p} \{1-st+4\mu_1\mu_2 st \sin^2\tfrac{1}{2}\chi\}^{-\frac{1}{2}(p+3)}. \quad (1.23)$$

Also

$$\int_0^{2\pi} d\epsilon \iiiint\!\!\iiint C_R{}^p S_{\frac{1}{2}}{}^k(hm_1 C_1'^2) S_{\frac{1}{2}}{}^n(hm_1 C_1^2) \exp(-hm_1 C_1{}^2 - hm_2 C_2{}^2)$$

$$\times dU_1 dV_1 dW_1 dU_2 dV_2 dW_2$$

$$= \frac{4\pi^{\frac{7}{2}}\Gamma(\frac{1}{2}p+\frac{3}{2})}{(h^2 m_1 m_2)^{\frac{3}{2}}(hm_0)^{\frac{1}{2}p}\Gamma(k+\frac{3}{2})\Gamma(n+\frac{3}{2})} \times \text{coefficient of } s^k t^n \text{ in}$$

$$\{1-\mu_2(s+t)+(\mu_2-\mu_1)st\}^{\frac{1}{2}p} \{1-st+4\mu_1\mu_2 st \sin^2\tfrac{1}{2}\chi\}^{-\frac{1}{2}p-\frac{3}{2}} \quad (1.24)$$

as above in (1.22).

The corresponding integrals with $S_{\frac{1}{2}}{}^k(hm_1 C_1'^2)$ replaced by $S_{\frac{1}{2}}{}^k(hm_2 C_2'^2)$ are found similarly. Conservation of energy in a collision requires

that $\mu_1 C_1'^2+\mu_2 C_1'^2 = \mu_1 C_1^2+\mu_2 C_2^2$, and so, on using the value of $C_1'^2$ already found in (1.212),

$$\mu_2 C_2'^2 = \mu_2 C_2^2 - 4\mu_2 C_R^2 \sin^2 \tfrac{1}{2}\chi - 4(\mu_1\mu_2)^{\frac{1}{2}} C_1 C_R \cos\theta_{1R} \sin^2 \tfrac{1}{2}\chi + 2(\mu_1\mu_2)^{\frac{1}{2}} C_1 C_R \sin\theta_{1R} \sin\chi \cos\epsilon,$$

and the results are the same as before with appropriate values for a, b, c, d: they are

$$a = \mu_2(1+\alpha), \quad b = \frac{1}{\mu_1}(1+\alpha-4\alpha\mu_1\mu_2 \sin^2 \tfrac{1}{2}\chi),$$

$$c = \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}}(1+\alpha-2\alpha\mu_1 \sin^2 \tfrac{1}{2}\chi), \quad d = (\mu_1\mu_2)^{\frac{1}{2}}\alpha \sin\chi,$$

so that

$$ab-\gamma^2 = 4\frac{s}{1-s}\mu_1\mu_2 \sin^2 \tfrac{1}{2}\chi, \quad (a+\mu_1)b-\gamma^2 = \frac{1}{1-s},$$

$$a+\mu_1-at = \frac{1-\mu_1 s-\mu_2 t}{1-s},$$

and

$$\int_0^{2\pi} d\epsilon \iiint C_R^p S_{\frac{1}{2}}^k(hm_2 C_2'^2) \exp(-hm_2 C_2^2) dU_2 dV_2 dW_2$$

$$= \frac{2\pi^{\frac{5}{2}}\Gamma(\frac{1}{2}p+\frac{3}{2})}{(hm_2)^{\frac{3}{2}}(hm_0)^{\frac{1}{2}p}\Gamma(k+\frac{3}{2})} \sum_{n=0}^{\infty} n! S_{\frac{1}{2}}^n(hm_1 C_1^2) \times \text{coefficient of } s^k t^n \text{ in}$$

$$\{1-\mu_1 s-\mu_2 t\}^{\frac{1}{2}p} \{1-4\mu_1\mu_2 st \sin^2 \tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{3}{2})}, \quad (1.25)$$

$$\int_0^{2\pi} d\epsilon \iiiint\!\!\iiint C_R^p S_{\frac{1}{2}}^k(hm_2 C_2'^2) S_{\frac{1}{2}}^n(hm_1 C_1^2) \exp(-hm_1 C_1^2 - hm_2 C_2^2) \times dU_1 dV_1 dW_1 dU_2 dV_2 dW_2$$

$$= \frac{4\pi^{\frac{7}{2}}\Gamma(\frac{1}{2}p+\frac{3}{2})}{(h^2 m_1 m_2)^{\frac{3}{2}}(hm_0)^{\frac{1}{2}p}\Gamma(k+\frac{3}{2})\Gamma(n+\frac{3}{2})} \times \text{coefficient of } s^k t^n \text{ in}$$

$$\{1-\mu_1 s-\mu_2 t\}^{\frac{1}{2}p} \{1-4\mu_1\mu_2 st \sin^2 \tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{3}{2})}. \quad (1.26)$$

1.3. As above,

$$\int_0^{2\pi} d\epsilon \iiint U_1' S_{\frac{3}{2}}^k(hm_1 C_1'^2) \exp(-hm_2 C_2^2) dU_2 dV_2 dW_2$$

$$= \text{coefficient of } s^k \text{ in } \frac{(1-s)^{-\frac{5}{2}}}{\Gamma(k+\frac{5}{2})} \int_0^{2\pi} d\epsilon \iiint U_1' \times \exp(-ahm_1 C_1'^2 - hm_2 C_2^2) dU_2 dV_2 dW_2.$$

Now*

$$U_1' = U_1 + 2\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}} C_R \cos\theta_{XR} \sin^2 \tfrac{1}{2}\chi - \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}} C_R \sin\theta_{XR} \sin\chi \cos(\epsilon+\phi_0)$$

$$= U_1 + \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}} C_R\{\cos\theta_{XR} - \cos\theta'_{XR}\}, \qquad (1.311)$$

where θ_{XR}, θ'_{XR} are the angles which the directions of C_R before and after the collision make with the axis of x.

Let X, C_R, C_R', C_1 (Fig. 1) be points on a unit sphere representing respectively the directions of the x axis, the velocities C_R before and after the collision, and the velocity of the first body before the collision; then

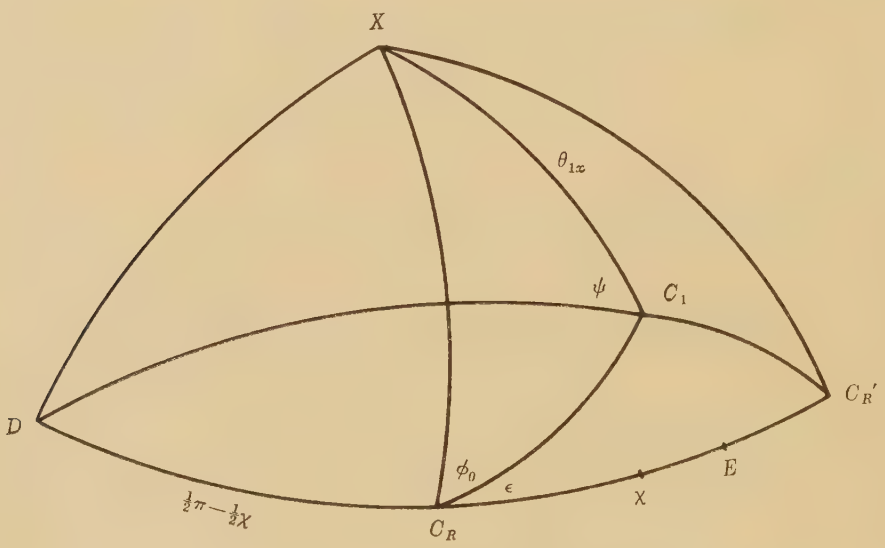

Fig. 1.

$C_R C_R' = \chi$, $XC_R = \theta_{XR}$, $XC_R' = \theta'_{XR}$, $XC_R C_1 = \phi_0$, $C_1 C_R C_R' = \epsilon$. Also let D be the mid-point of the arc of the great circle joining C_R to the antipodal point of C_R', so that $C_R D = \frac{1}{2}(\pi-\chi)$, $C_R' D = \frac{1}{2}(\pi+\chi)$; then

$$\cos\theta_{XR} = \cos XD \sin\tfrac{1}{2}\chi + \sin XD \cos\tfrac{1}{2}\chi \cos XDC_R,$$

$$\cos\theta_{XR} = -\cos XD \sin\tfrac{1}{2}\chi + \sin XD \cos\tfrac{1}{2}\chi \cos XDC_R,$$

* *Cf.* Chapman, *loc. cit.* (1916), 293.

so that

$$\cos\theta_{XR}-\cos\theta'_{XR} = 2\cos XD\sin\tfrac{1}{2}\chi$$
$$= 2\sin\tfrac{1}{2}\chi(\cos\theta_{1X}\cos C_1D+\sin\theta_{1X}\sin C_1D\cos\psi). \qquad (1.312)$$

Now, if θ_{1R}, χ, ϵ are constant and ϕ_{1R} varies from 0 to 2π, the triangle $C_1C_RC_R'$ turns about C_1 remaining invariant in form, and carries the point D with it, so that C_1D remains constant in length while the angle ψ, which differs from ϕ_{1R} by a constant, takes all values between 0 and 2π. Hence, on integration with respect to ϕ_{1R}, the term involving $\cos\psi$ disappears and

$$\int_0^{2\pi}d\epsilon\iiint U_1'\exp(-ahm_1C_1'^2-hm_2C_2^2)\,dU_2\,dV_2\,dW_2$$
$$=\int_0^{2\pi}d\epsilon\iiint\exp(-ahm_1C_1'^2-hm_2C_2^2)$$
$$\times\left\{U_1+2\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}}C_R\sin\tfrac{1}{2}\chi\cos\theta_{1X}\right.$$
$$\left.\times(\cos\theta_{1R}\sin\tfrac{1}{2}\chi-\sin\theta_{1R}\cos\tfrac{1}{2}\chi\cos\epsilon)\right\}dU_2\,dV_2\,dW_2,$$

on making an obvious substitution for $\cos C_1D$. Hence, from the results of the previous paragraph, the integral is

$$\frac{2\pi^{\frac{5}{2}}(\mu_1\mu_2)^{-\frac{3}{2}}}{(hm_0)^{\frac{3}{2}}}\left\{U_1-\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}}\frac{\sin\frac{1}{2}\chi\cos\theta_{1X}}{hm_0C_1}\right.$$
$$\left.\times\left(\sin\tfrac{1}{2}\chi\frac{\partial}{\partial c}-\cos\tfrac{1}{2}\chi\frac{\partial}{\partial d}\right)\right\}\frac{\exp[-hm_0C_1^2\{a-(\gamma^2/b)\}]}{b^{\frac{3}{2}}},$$

and, on performing the differentiations and simplifying as before, it is

$$\frac{2\pi^{\frac{5}{2}}(\mu_1\mu_2)^{-\frac{3}{2}}}{(hm_0)^{\frac{3}{2}}}U_1\left\{b-2\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}}\right.$$
$$\left.\times\sin\tfrac{1}{2}\chi(c\sin\tfrac{1}{2}\chi-d\cos\tfrac{1}{2}\chi)\right\}\frac{\exp[-hm_0C_1^2\{a-(\gamma^2/b)\}]}{b^{\frac{5}{2}}}$$
$$=\frac{2\pi^{\frac{5}{2}}}{(hm_2)^{\frac{3}{2}}}U_1(\mu_1+\mu_2\cos\chi)(1-s)^{\frac{5}{2}}$$
$$\times\sum_{n=0}^{\infty}\Gamma(n+\tfrac{5}{2})(1-4\mu_1\mu_2\sin^2\tfrac{1}{2}\chi)^n s^n S_{\frac{3}{2}}^n(hm_1C_1^2),$$

so that

$$\int_0^{2\pi} d\epsilon \iiint U_1' S_{\frac{3}{2}}^k(hm_1 C_1'^2) \exp(-hm_2 C_2^2)\, dU_2 dV_2 dW_2$$

$$= \frac{2\pi^{\frac{5}{2}}}{(hm_2)^{\frac{3}{2}}} U_1(\mu_1+\mu_2\cos\chi)(1-4\mu_1\mu_2\sin^2\tfrac{1}{2}\chi)^k S_{\frac{3}{2}}^k(hm_1 C_1^2) \quad (1.31)$$

and

$$\int_0^{2\pi} d\epsilon \iiiint\!\!\iiint U_1 U_1' S_{\frac{3}{2}}^k(hm_1 C_1'^2)\, S_{\frac{3}{2}}^n(hm_1 C_1^2)$$

$$\times \exp(-hm_1 C_1^2 - hm_2 C_2^2)\, dU_1 dV_1 dW_1 dU_2 dV_2 dW_2$$

$$= \frac{2\pi^{\frac{5}{2}}}{(hm_2)^{\frac{3}{2}}} (\mu_1+\mu_2\cos\chi)(1-4\mu_1\mu_2\sin^2\tfrac{1}{2}\chi)^k \iiint U_1^2 S_{\frac{3}{2}}^k(hm_1 C_1^2)\, S_{\frac{3}{2}}^n(hm_1 C_1^2)$$

$$\times \exp(-hm_1 C_1^2)\, dU_1 dV_1 dW_1$$

$$= \frac{2\pi^{\frac{5}{2}}}{(hm_2)^{\frac{3}{2}}} (\mu_1+\mu_2\cos\chi)(1-4\mu_1\mu_2\sin^2\tfrac{1}{2}\chi)^k \frac{4\pi}{3}$$

$$\times \int_0^\infty C_1^4 S_{\frac{3}{2}}^k(hm_1 C_1^2)\, S_{\frac{3}{2}}^n(hm_1 C_1^2) \exp(-hm_1 C_1^2)\, dC_1$$

$$= \frac{4\pi^{\frac{7}{2}}}{3h^4 m_1^{\frac{5}{2}} m_2^{\frac{3}{2}}} \frac{(\mu_1+\mu_2\cos\chi)(1-4\mu_1\mu_2\sin^2\tfrac{1}{2}\chi)^n}{n!\,\Gamma(n+\frac{5}{2})} \quad (k=n)$$

$$= 0 \quad (k\neq n). \quad (1.32)$$

Similarly,

$$\int_0^{2\pi} d\epsilon \iiint U_1' C_R^p \exp(-a hm_1 C_1'^2 - hm_2 C_2^2)\, dU_2 dV_2 dW_2$$

$$= \frac{2\pi^{\frac{5}{2}}(\mu_1\mu_2)^{-\frac{3}{2}}}{(hm_0)^{\frac{3}{2}}} U_1\left(-\frac{1}{hm_0}\frac{\partial}{\partial b}\right)^{\frac{1}{2}p}$$

$$\times \left\{b - 2\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}} \sin\tfrac{1}{2}\chi(c\sin\tfrac{1}{2}\chi - d\cos\tfrac{1}{2}\chi)\right\} \frac{\exp[-hm_1 C_1^2\{a-(\gamma^2/b)\}]}{b^{\frac{5}{2}}}$$

$$= \frac{2\pi^{\frac{5}{2}}\Gamma(\frac{5}{2})}{(hm_2)^{\frac{3}{2}}(hm_0)^{\frac{1}{2}p}} \mu_1 U_1\left(-\frac{\partial}{\partial b}\right)^{\frac{1}{2}p}$$

$$\times \left\{b - 2\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}} \sin\tfrac{1}{2}\chi(c\sin\tfrac{1}{2}\chi - d\cos\tfrac{1}{2}\chi)\right\} \sum_{n=0}^{\infty} n!\, S_{\frac{3}{2}}^n(hm_1 C_1^2)$$

$$\times \text{the coefficient of } t^n \text{ in } \{(a+\mu_1)b - \gamma^2 - (ab-\gamma^2)t\}^{-\frac{5}{2}},$$

so that

$$\int_0^{2\pi} d\epsilon \iiint U_1' C_R^p S_{\frac{3}{2}}^k(hm_1 C_1'^2) \exp(-hm_2 C_2^2)\, dU_2 dV_2 dW_2$$

$$= \frac{2\pi^{\frac{5}{2}}\Gamma(\frac{5}{2})}{(hm_2)^{\frac{3}{2}}(hm_0)^{\frac{1}{2}p}} \mu_1 U_1 \sum_{n=0}^{\infty} n!\, S_{\frac{3}{2}}^n(hm_1 C_1^2) \times \text{coefficient of } s^k t^n \text{ in}$$

$$\frac{(1-s)^{-\frac{5}{2}}}{\Gamma(k+\frac{5}{2})} \left(-\frac{\partial}{\partial b}\right)^{\frac{1}{2}p} \left\{b - 2\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}} \sin\tfrac{1}{2}\chi\,(c\sin\tfrac{1}{2}\chi - d\cos\tfrac{1}{2}\chi)\right\}$$

$$\times \{(a+\mu_1)b - \gamma^2 - (ab-\gamma^2)t\}^{-\frac{5}{2}}$$

$$= \frac{2\pi^{\frac{5}{2}}\Gamma(\frac{1}{2}p+\frac{3}{2})}{(hm_2)^{\frac{3}{2}}(hm_0)^{\frac{1}{2}p}\Gamma(k+\frac{5}{2})} U_1 \sum_{n=0}^{\infty} n!\, S_{\frac{3}{2}}^n(hm_1 C_1^2) \times \text{coefficient of } s^k t^n \text{ in}$$

$$(\tfrac{1}{2}p+\tfrac{3}{2})(\mu_1+\mu_2\cos\chi)\{1-\mu_2(s+t)+(\mu_2-\mu_1)st\}^{\frac{1}{2}p}\{1-st+4\mu_1\mu_2 st\sin^2\tfrac{1}{2}\chi\}^{-\frac{1}{2}p-\frac{5}{2}}$$

$$-\mu_1 \tfrac{1}{2}p\{1-\mu_2(s+t)+(\mu_2-\mu_1)st\}^{\frac{1}{2}p-1}\{1-st+4\mu_1\mu_2 st\sin^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{3}{2})}, \quad (1.33)$$

on performing the differentiations and substituting for a, b, c, d from (1.213) as in (1.214). Also

$$\int_0^{2\pi} d\epsilon \iiint\!\!\iint U_1 U_1' C_R^p S_{\frac{3}{2}}^k(hm_1 C_1'^2) S_{\frac{3}{2}}^n(hm_1 C_1^2)$$

$$\times \exp(-hm_1 C_1^2 - hm_2 C_2^2)\, dU_1 dV_1 dW_1 dU_2 dV_2 dW_2$$

$$= \frac{4\pi^{\frac{7}{2}}\Gamma(\frac{1}{2}p+\frac{3}{2})}{3h^4 m_1^{\frac{3}{2}} m_2^{\frac{3}{2}}(hm_0)^{\frac{1}{2}p}\Gamma(k+\frac{5}{2})\Gamma(n+\frac{5}{2})}$$

$$\times \text{coefficient of } s^k t^n \text{ in the same expression.} \quad (1.34)$$

The discussion of

$$\int_0^{2\pi} d\epsilon \iiint U_2' C_R^p S_{\frac{3}{2}}^k(hm_2 C_2'^2)\, dU_2 dV_2 dW_2$$

is similar; for, since $\mu_1 U_1' + \mu_2 U_2' = \mu_1 U_1 + \mu_2 U_2$ by the principle of conservation of momentum,

$$\mu_2 U_2' = \mu_2 U_2 - (\mu_1\mu_2)^{\frac{1}{2}} C_R(\cos\theta_{XR} - \cos\theta'_{XR})$$

$$= \mu_2 U_1 + \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}} C_R(\mu_2\cos\theta_{XR} + \mu_1\cos\theta'_{XR}).$$

Let $C_R C_R'$ (Fig. 1) be divided at E so that

$$\mu_2 \sin C_R E = \mu_1 \sin C_R' E. \quad (1.351)$$

Then

$$\mu_2\cos\theta_{XR} + \mu_1\cos\theta'_{XR} = \cos XE(\mu_2\cos C_R E + \mu_1\cos C_R' E),$$

and, as above, integration with respect to ϕ_{1R} shows that

$$\int_0^{2\pi} d\epsilon \iiint U_2' C_R^p \exp[-hm_0\{aC_1^2+bC_R^2+2C_1C_R(c\cos\theta_{1R}+d\sin\theta_{1R}\cos\epsilon)\}] \times dU_2 dV_2 dW_2$$
$$= \int_0^{2\pi} d\epsilon \iiint C_R^p \exp[-hm_0\{aC_1^2+bC_R^2+2C_1C_R(c\cos\theta_{1R}+d\sin\theta_{1R}\cos\epsilon)\}]$$
$$\times\{U_1+(\mu_1\mu_2)^{-\frac{1}{2}} C_R(\mu_2\cos C_R E+\mu_1\cos C_R' E)\cos\theta_{1X}\cos C_1 E\} dU_2 dV_2 dW_2.$$

Now, on account of (1.351), μ_1, μ_2, $C_R E$, and $C_R' E$ are related as in the plane triangle of Fig. 2, so that

$$\mu_2\cos C_R E+\mu_1\cos C_R' E = (\mu_1^2+\mu_2^2+2\mu_1\mu_2\cos\chi)^{\frac{1}{2}},$$
$$(\mu_2\cos C_R E+\mu_1\cos C_R' E)\cos C_R E = \mu_2+\mu_1\cos\chi,$$
$$(\mu_2\cos C_R E+\mu_1\cos C_R' E)\sin C_R E = \mu_1\sin\chi,$$

and the integral is equal to

$$\int_0^{2\pi} d\epsilon \iiint C_R^p \exp[-hm_0\{aC_1^2+bC_R^2+2C_1C_R(c\cos\theta_{1R}+d\sin\theta_{1R}\cos\epsilon)\}]$$
$$\times[U_1+(\mu_1\mu_2)^{-\frac{1}{2}} C_R\cos\theta_{1X}\{(\mu_2+\mu_1\cos\chi)\cos\theta_{1R} +\mu_1\sin\chi\sin\theta_{1R}\cos\epsilon\}] dU_2 dV_2 dW_2$$
$$= \frac{2\pi^{\frac{5}{2}}(\mu_1\mu_2)^{-\frac{3}{2}} U_1}{(hm_0)^{\frac{3}{2}}}\left(-\frac{1}{hm_0}\frac{\partial}{\partial b}\right)^{\frac{1}{2}p}[b-(\mu_1\mu_2)^{-\frac{1}{2}}\{(\mu_2+\mu_1\cos\chi)c+\mu_1\sin\chi d\}] \times\frac{\exp[-hm_0 C_1^2\{a-(\gamma^2/b)\}]}{b^{\frac{3}{2}}}$$

by differentiating the integral of § 1.2 with respect to c and d.

When a, b, c, d are given the appropriate values (1.251), it is found that

$$\int_0^{2\pi} d\epsilon \iiint U_2' C_R^p \exp(-hm_2 C_2^2) S_{\frac{3}{2}}^k(hm_2 C_2'^2)\, dU_2 dV_2 dW_2$$
$$= \frac{2\pi^{\frac{5}{2}}\mu_1 U_1}{(hm_2)^{\frac{3}{2}}(hm_0)^{\frac{1}{2}p}}\frac{\Gamma(\frac{1}{2}p+\frac{3}{2})}{\Gamma(k+\frac{5}{2})}\sum_{n=0}^{\infty} n!\, S_{\frac{3}{2}}^n(hm_1 C_1^2)\times\text{coefficient of } s^k t^n \text{ in}$$
$$(p+3)\sin^2\tfrac{1}{2}\chi\{1-\mu_1 s-\mu_2 t\}^{\frac{1}{2}p}\{1-4\mu_1\mu_2 st\sin^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{5}{2})}$$
$$-\tfrac{1}{2}p(1-\mu_1 s-\mu_2 t)^{\frac{1}{2}p-1}\{1-4\mu_1\mu_2 st\sin^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{3}{2})} \quad (1.35)$$

and

$$\int_0^{2\pi} d\epsilon \iiint\iiint U_1 U_2' C_R^p S_{\frac{3}{2}}^k(hm_2 C_2'^2) S_{\frac{3}{2}}^n(hm_1 C_1^2) \times\exp(-hm_1 C_1^2-hm_2 C_2^2)\, dU_1 dV_1 dW_1 dU_2 dV_2 dW_2$$
$$= \frac{4\pi^{\frac{7}{2}}\mu_1 U_1}{3h^4 m_1^{\frac{5}{2}} m_2^{\frac{3}{2}}(hm_0)^{\frac{1}{2}p}}\frac{\Gamma(\frac{1}{2}p+\frac{3}{2})}{\Gamma(k+\frac{3}{2})\Gamma(n+\frac{3}{2})} \times\text{coefficient of } s^k t^n \text{ in this same expression.} \quad (1.36)$$

1.4. Since, from (1.311) and (1.312),

$$U_1'^2 = U_1^2 + 4\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}} U_1 C_R \sin\tfrac{1}{2}\chi \cos XD + 4\frac{\mu_2}{\mu_1} C_R^2 \sin^2\tfrac{1}{2}\chi \cos^2 XD,$$

where
$$\cos XD = \cos\theta_{1X}\cos C_1 D + \sin\theta_{1X}\sin C_1 D\cos\psi,$$
$$\cos C_1 D = \cos\theta_{1R}\sin\tfrac{1}{2}\chi - \sin\theta_{1R}\cos\tfrac{1}{2}\chi\cos\epsilon,$$

we have

$$\int_0^{2\pi} d\epsilon \iiint U_1'^2 \exp(-ahm_1 C_1'^2 - hm_2 C_2^2)\, dU_2 dV_2 dW_2$$

$$= \int_0^{2\pi} d\epsilon \iiint \exp(-ahm_1 C_1'^2 - hm_2 C_2^2)$$

$$\times\Big[U_1^2 + 4\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}} U_1 C_R \sin\tfrac{1}{2}\chi\cos\theta_{1X}\cos C_1 D$$

$$+ 4\frac{\mu_2}{\mu_1} C_R^2 \sin^2\tfrac{1}{2}\chi\{\cos^2\theta_{1X}\cos^2 C_1 D + \tfrac{1}{2}\sin^2\theta_{1X}\sin^2 C_1 D\}\Big]\, dU_2 dV_2 dW_2$$

$$= \frac{2\pi^{\frac{5}{2}}(\mu_1\mu_2)^{-\frac{3}{2}}}{(hm_0)^{\frac{3}{2}}}\Big[U_1^2 - 2\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}}\frac{U_1\sin\frac{1}{2}\chi\cos\theta_{1X}}{hm_0 C_1}\left(\sin\tfrac{1}{2}\chi\frac{\partial}{\partial c} - \cos\tfrac{1}{2}\chi\frac{\partial}{\partial d}\right)$$

$$+ 4\frac{\mu_2}{\mu_1}\sin^2\tfrac{1}{2}\chi\left\{\frac{3\cos^2\theta_{1X}-1}{2(2hm_0 C_1)^2}\left(\sin\tfrac{1}{2}\chi\frac{\partial}{\partial c} - \cos\tfrac{1}{2}\chi\frac{\partial}{\partial d}\right)^2 - \frac{\sin^2\theta_{1X}}{2hm_0}\frac{\partial}{\partial b}\right\}\Big]$$

$$\times \frac{\exp[-hm_0 C_1^2\{a-(\gamma^2/b)\}]}{b^{\frac{3}{2}}}$$

$$= \frac{2\pi^{\frac{5}{2}}(\mu_1\mu_2)^{-\frac{3}{2}}}{(hm_0)^{\frac{3}{2}}}\Big[U_1^2\left\{b^2 - 4\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}} b\sin\tfrac{1}{2}\chi(c\sin\tfrac{1}{2}\chi - d\sin\tfrac{1}{2}\chi)\right\}$$

$$+ 2\frac{\mu_2}{\mu_1}\sin^2\tfrac{1}{2}\chi\left\{(3U_1^2 - C_1^2)(c\sin\tfrac{1}{2}\chi - d\cos\tfrac{1}{2}\chi)^2 + \frac{3\cos^2\theta_{1X}-1}{2hm_0}b\right.$$

$$\left.+ \frac{\sin^2\theta_{1X}}{hm_0}\tfrac{3}{2}b + \sin^2\theta_{1X}\gamma^2 C_1^2\right\}\Big]\frac{\exp[-hm_0 C_1^2\{a-(\gamma^2/b)\}]}{b^{\frac{7}{2}}}$$

$$= \frac{2\pi^{\frac{5}{2}}\mu_1^2}{(hm_2)^{\frac{3}{2}}}\Big[U_1^2\left\{b - 2\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}}\sin\tfrac{1}{2}\chi(c\sin\tfrac{1}{2}\chi - d\cos\tfrac{1}{2}\chi)\right\}^2$$

$$+ 2\frac{\mu_2}{\mu_1}\sin^2\tfrac{1}{2}\chi(C_1^2 - U_1^2)(c\cos\tfrac{1}{2}\chi + d\sin\tfrac{1}{2}\chi)^2\Big]$$

$$\times \sum_{n=0}^{\infty} n!\, S_{\frac{5}{2}}^n (hm_1 C_1^2) \times \text{coefficient of } t^n \text{ in } \{(a+\mu_1)b - \gamma^2 - (ab-\gamma^2)t\}^{-\frac{5}{2}}.$$

If this result is operated on by the symbolic fractional differential operator $\left(-\frac{1}{hm_0}\frac{\partial}{\partial b}\right)^{\frac{1}{2}p}$, as before, and the appropriate values of a, b, c, d

and substituted from (1.213) and (1.221), it is found that

$$\int_0^{2\pi} d\epsilon \iiint U_1'^2\, C_R^{\,p}\, S_{\frac{5}{2}}^{\,k}(hm_1 C_1'^2) \exp(-hm_2 C_2^{\,2})\, dU_2\, dV_2\, dW_2$$

$$= \frac{2\pi^{\frac{5}{2}}\,\Gamma(\frac{1}{2}p+\frac{3}{2})}{(hm_2)^{\frac{3}{2}}\,(hm_0)^{\frac{1}{2}p}\,\Gamma(k+\frac{7}{2})} \sum_{n=0}^{\infty} n!\, S_{\frac{5}{2}}^{\,n}(hm_1 C_1^{\,2}) \times \text{coefficient of } s^k t^n \text{ in}$$

$$(\tfrac{1}{2}p+\tfrac{3}{2})(\tfrac{1}{2}p+\tfrac{5}{2})\{U_1^{\,2}(\mu_1+\mu_2\cos\chi)^2+2(C_1^{\,2}-U_1^{\,2})\,\mu_2^{\,2}\sin^2\tfrac{1}{2}\chi\cos^2\tfrac{1}{2}\chi)\}$$

$$\times\{1-\mu_2(s+t)+(\mu_2-\mu_1)\,st\}^{\frac{1}{2}p}\{1-st+4\mu_1\mu_2\,st\sin^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{7}{2})}$$

$$-\mu_1 p(\tfrac{1}{2}p+\tfrac{3}{2})\,U_1^{\,2}(\mu_1+\mu_2\cos\chi)\{1-\mu_2(s+t)+(\mu_2-\mu_1)\,st\}^{\frac{1}{2}p-1}$$

$$\times\{1-st+4\mu_1\mu_2\,st\sin^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{5}{2})}$$

$$+\mu_1^{\,2}\tfrac{1}{2}p(\tfrac{1}{2}p-1)\,U_1^{\,2}\{1-\mu_2(s+t)+(\mu_2-\mu_1)\,st\}^{\frac{1}{2}p-2}$$

$$\times\{1-st+4\mu_1\mu_2\,st\sin^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{3}{2})}; \quad (1.41)$$

$$\int_0^{2\pi} d\epsilon \iiint\!\!\iint\!\!\iint U_1'^2(3U_1^{\,2}-C_1^{\,2})\, C_R^{\,p}\, S_{\frac{5}{2}}^{\,k}(hm_1 C_1'^2)\, S_{\frac{5}{2}}^{\,n}(hm_1 C_1^{\,2})$$

$$\times\exp(-hm_1 C_1^{\,2}-hm_2 C_2^{\,2})\, dU_1\, dV_1\, dW_1\, dU_2\, dV_2\, dW_2$$

$$= \frac{16\pi^{\frac{7}{2}}\,\Gamma(\frac{1}{2}p+\frac{3}{2})}{15h^5 m_1^{\frac{7}{2}} m_2^{\frac{3}{2}}\,(hm_0)^{\frac{1}{2}p}\,\Gamma(k+\frac{7}{2})\,\Gamma(n+\frac{7}{2})}$$

$$\times\text{coefficient of } s^k t^n U_1^{\,2} \text{ in this expression,} \quad (1.42)$$

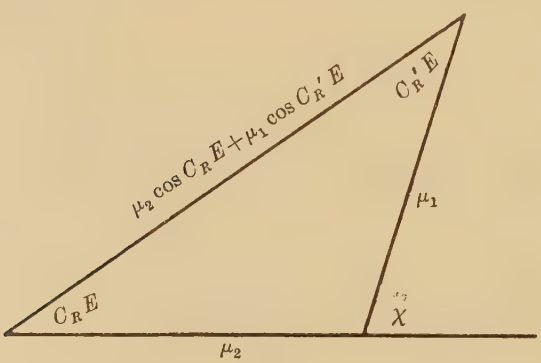

Fig. 2.

on making use of the obvious results

$$\iiint U_1^{\,2} f(C_1^{\,2})\, dU_1\, dV_1\, dW_1 = \tfrac{1}{3}\iiint C_1^{\,2} f(C_1^{\,2})\, dU_1\, dV_1\, dW_1,$$

$$\iiint U_1^{\,4} f(C_1^{\,2})\, dU_1\, dV_1\, dW_1 = \tfrac{1}{5}\iiint C_1^{\,4} f(C_1^{\,2})\, dU_1\, dV_1\, dW_1.$$

Also

$$\int_0^{2\pi} d\epsilon \iiint U_2'^2\, C_R{}^p\, S_{\frac{5}{2}}{}^k(hm_2 C_2'^2) \exp(-hm_2 C_2{}^2)\, dU_2 dV_2 dW_2$$

$$= \frac{2\pi^{\frac{5}{2}}\Gamma(\frac{1}{2}p+\frac{3}{2})\,\mu_1{}^2}{(hm_2)^{\frac{3}{2}}(hm_0)^{\frac{1}{2}p}\,\Gamma(k+\frac{7}{2})} \sum_{n=0}^{\infty} n!\, S_{\frac{5}{2}}{}^n(hm_1 C_1{}^2) \times \text{coefficient of } s^k t^n \text{ in}$$

$$(\tfrac{1}{2}p+\tfrac{3}{2})(\tfrac{1}{2}p+\tfrac{5}{2})\{4U_1{}^2 \sin^4\tfrac{1}{2}\chi + 2(C_1{}^2 - U_1{}^2)\sin^2\tfrac{1}{2}\chi \cos^2\tfrac{1}{2}\chi\}$$

$$\times \{1-\mu_1 s-\mu_2 t\}^{\frac{1}{2}p}\{1-4\mu_1\mu_2 st \sin^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{7}{2})}$$

$$-p(p+3)\,U_1{}^2 \sin^2\tfrac{1}{2}\chi\{1-\mu_1 s-\mu_2 t\}^{\frac{1}{2}p-1}\{1-4\mu_1\mu_2 st \sin^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{5}{2})}$$

$$+\tfrac{1}{2}p(\tfrac{1}{2}p-1)\,U_1{}^2\{1-\mu_1 s-\mu_2 t\}^{\frac{1}{2}p-2}\{1-4\mu_1\mu_2 st \sin^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{3}{2})}; \quad (1.43)$$

$$\int_0^{2\pi} d\epsilon \iiiint\!\!\iint U_2'^2(3U_1{}^2 - C_1{}^2)\, C_R{}^p\, S_{\frac{5}{2}}{}^k(hm_2 C_2'^2)\, S_{\frac{5}{2}}{}^n(hm_1 C_1{}^2)$$

$$\times \exp(-hm_1 C_1{}^2 - hm_2 C_2{}^2)\, dU_1 dV_1 dW_1 dU_2 dV_2 dW_2$$

$$= \frac{16\pi^{\frac{7}{2}}\Gamma(\frac{1}{2}p+\frac{3}{2})\,\mu_1{}^2}{15h^5 m_1{}^{\frac{3}{2}} m_2{}^{\frac{3}{2}}(hm_0)^{\frac{1}{2}p}\,\Gamma(k+\frac{7}{2})\,\Gamma(n+\frac{7}{2})} \times \text{coefficient of } s^k t^n \text{ in}$$

$$(\tfrac{1}{2}p+\tfrac{3}{2})(\tfrac{1}{2}p+\tfrac{5}{2})\,4\sin^2\tfrac{1}{2}\chi\{\sin^2\tfrac{1}{2}\chi - \tfrac{1}{2}\cos^2\tfrac{1}{2}\chi\}$$

$$\times \{1-\mu_1 s-\mu_2 t\}^{\frac{1}{2}p}\{1-4\mu_1\mu_2 st \sin^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{7}{2})}$$

$$-p(p+3)\sin^2\tfrac{1}{2}\chi\{1-\mu_1 s-\mu_2 t\}^{\frac{1}{2}p-1}\{1-4\mu_1\mu_2 st \sin^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{5}{2})}$$

$$+\tfrac{1}{2}p(\tfrac{1}{2}p-1)\{1-\mu_1 s-\mu_2 t\}^{\frac{1}{2}p-2}\{1-4\mu_1\mu_2 st \sin^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{3}{2})}. \quad (1.44)$$

$$\int_0^{2\pi} d\epsilon \iiint U_1' V_1'\, C_R{}^p S_{\frac{5}{2}}{}^k(hm_1 C_1'^2) \exp(-hm_2 C_2{}^2)\, dU_2 dV_2 dW_2$$

$$= \frac{2\pi^{\frac{5}{2}}\Gamma(\frac{1}{2}p+\frac{3}{2})}{(hm_2)^{\frac{3}{2}}(hm_0)^{\frac{1}{2}p}\,\Gamma(k+\frac{7}{2})}\, U_1 V_1 \sum_{n=0}^{\infty} n!\, S_{\frac{5}{2}}{}^n(hm_1 C_1{}^2)\, B_{kn}, \quad (1.45)$$

where B_{kn} is the same coefficient as occurs in (1.42); and

$$\int_0^{2\pi} d\epsilon \iiint U_2' V_2'\, C_R{}^p\, S_{\frac{5}{2}}{}^k(hm_2 C_2'^2) \exp(-hm_2 C_2{}^2)\, dU_2 dV_2 dW_2$$

$$= \frac{2\pi^{\frac{5}{2}}\Gamma(\frac{1}{2}p+\frac{3}{2})\,\mu_1{}^2}{(hm_2)^{\frac{3}{2}}(hm_0)^{\frac{1}{2}p}\,\Gamma(k+\frac{7}{2})}\, U_1 V_1 \sum_{n=0}^{\infty} n!\, S_{\frac{5}{2}}{}^n(hm_1 C_1{}^2)\, B'_{kn}, \quad (1.46)$$

B'_{kn} being the coefficient in (1.44).

The corresponding integrals taken over all values of the velocities of both bodies evidently vanish.

When $p=0$ the binomial expressions obtained in (1.24), (1.26), (1.34), (1.36), (1.42), (1.44) all become functions of st only, so that these integrals all vanish unless $k=n$. The corresponding results, with unaccented letters written in place of the accented ones, are got (*cf.* Chapman) by putting $\chi=0$, since they refer to collisions in which there is no deflection*.

1.5. *Convergence of the series* $\sum_{k=0}^{\infty} \Gamma(k+m+1)\, b_k\, S_m{}^k(x^2)$.

In the series which occur in this paper the coefficients b_k are such that $|b_k|k^{-r}$ tends to a finite limit as k tends to infinity, so that the behaviour of $S_m{}^k(x^2)$ for large values of k has to be investigated. The results for Weber's parabolic cylinder function have already been obtained by Adamoff and Watson, and $S_m{}^k(x^2)$ is connected with Weber's function by the known formulae†

$$\Gamma(m+\tfrac{1}{2})\, S_m{}^k(x^2) = \int_0^{\pi} S^k_{-\frac{1}{2}}(x^2\cos^2\theta)\sin^{2m}\theta\, d\theta,$$

$$S^k_{-\frac{1}{2}}(x^2) = \frac{2^{-\frac{1}{2}k}}{\Gamma(k+1)\,\Gamma(k+\frac{1}{2})}\, e^{\frac{1}{2}x^2}\, D_k(x\sqrt{2}).$$

Hence

$$\begin{aligned}
&\Gamma(m+\tfrac{1}{2})\, S_m{}^k(x^2) \\
&\quad = \frac{2^{-\frac{1}{2}k}}{\Gamma(k+1)\,\Gamma(k+\frac{1}{2})} \int_0^{\pi} \exp(\tfrac{1}{2}x^2\cos^2\theta)\, D_k(x\cos\theta\sqrt{2})\sin^{2m}\theta\, d\theta \\
&\quad \sim \frac{2^{-\frac{1}{2}k}}{\Gamma(k+1)\,\Gamma(k+\frac{1}{2})} \left(\frac{k}{e}\right)^{\frac{1}{2}k} 2^{\frac{1}{2}} \\
&\qquad\qquad \times \int_0^{\pi} \exp(\tfrac{1}{2}x^2\cos^2\theta)\sin^{2m}\theta\cos\{x\cos\theta\sqrt{(2k)} - \tfrac{1}{2}k\pi\}\, d\theta,
\end{aligned}$$

from the asymptotic expressions for Weber's function obtained by Adamoff and Watson. If $m > -\frac{1}{2}$ and x is not zero, an application of Stirling's formula and the principle of stationary phase shows that $\Gamma(k+m+1)\, S_m{}^k(x^2)$ tends to zero as k increases. If x is zero,

$$\Gamma(k+m+1)\, S_m{}^k(x^2)$$

* It should be noticed that Jean's angle of deflection θ' is the supplement of the χ of Chapman's papers.

† *Cf.* Whittaker and Watson, *Modern analysis*, ch. 16, and examples 8, 19 at the end of the chapter; also Bateman, *Electrical and optical wave motion* (1915), 100.

becomes the coefficient of s^k in the expansion in powers of s of $(1-s)^{-m-1}$, which is known to be of the order of k^m when k is large. Hence the series $\sum_{k=0}^{\infty} \Gamma(k+m+1)\, b_k\, S_m{}^k(x^2)$ certainly converges uniformly with respect to x if $|b_k k^{m+1+\epsilon}|$ tends to a finite or zero limit as k tends to infinity (ϵ being, as usual, an arbitrarily small positive number), since then the series converges at least as rapidly as $\Sigma k^{-1-\epsilon}$; more generally, if $\Sigma b_k B_k$ converges and B_k is of order not less than k^m when k is large, then $\sum_{k=0}^{\infty} \Gamma(k+m+1)\, b_k S_m{}^k(x^2)$ converges uniformly in x. It is also evident that, if $|b_k k^{1+\epsilon}|$ tends to a finite or zero limit as k tends to infinity, then

$$\sum_{k=0}^{\infty} \Gamma(k+m+1)\, b_k\, S_m{}^k(x^2)$$

converges save when $x=0$.

1.6. *Expansion of an arbitrary function as a series of Sonine polynomials*.*

If
$$f(x) = \sum_{k=0}^{\infty} \Gamma(k+m+1)\, a_k\, S_m{}^k(x),$$

assuming for the moment the validity of the expansion, then

$$a_k = k! \int_0^{\infty} e^{-t} t^m\, S_m{}^k(t) f(t)\, dt$$

by (1.12). Hence, if C is a contour of suitable type enclosing $s=0$ but not $s=1$,

$$\frac{\Gamma(k+m+1)}{k!}\, a_k = \frac{1}{2\pi i} \int_0^{\infty} e^{-t} t^m f(t)\, dt \int_C \frac{ds}{s^{k+1}(1-s)^{m+1}} \exp\left(-\frac{st}{1-s}\right)$$

$$= \frac{1}{2\pi i} \int_C \frac{ds}{s^{k+1}(1-s)^{m+1}} \int_0^{\infty} \exp\left(-\frac{t}{1-s}\right) t^m f(t)\, dt,$$

assuming the legitimacy of the change in the order of integration. Hence, if

$$\int_0^{\infty} \exp\left(-\frac{t}{1-s}\right) t^m f(t)\, dt$$

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* The restrictions to be imposed on the given function could be investigated rigorously on the lines followed by Watson, *Proc. London Math. Soc.* (2), 8 (1910), 417, in the analogous problem of expansion in terms of Weber's parabolic cylinder functions, but the present brief discussion appears to be sufficient for the functions which occur in distribution function problems.

converges ($|s|<1$ may be assumed) and a_k satisfies the conditions (1.5) above, the series converges and the expansion is valid. The cases which occur in this paper are $m=\frac{3}{2}$, $f(x)=x^{\pm\frac{1}{2}}$, and for these the series are easily found.

Case I. $f(x)=x^{\frac{1}{2}}$:

$$\frac{\Gamma(k+\frac{5}{2})}{k!}a_k=\frac{1}{2\pi i}\int_C\frac{ds}{s^{k+1}(1-s)^{\frac{5}{2}}}\,2(1-s)^3$$

$$=2\times\text{coefficient of } s^k \text{ in the expansion of } (1-s)^{\frac{1}{2}}$$

$$=-\frac{\Gamma(k-\frac{1}{2})}{k!\,\Gamma(\frac{1}{2})},$$

so that
$$a_k=-\frac{1}{\sqrt{\pi}}\frac{\Gamma(k-\frac{1}{2})}{\Gamma(k+\frac{5}{2})}.$$

Case II. $f(x)=x^{-\frac{1}{2}}$:

$$\frac{\Gamma(k+\frac{5}{2})}{k!}a_k=\text{coefficient of } s^k \text{ in the expansion of} (1-s)^{-\frac{1}{2}},$$

so that
$$a_k=\frac{1}{\sqrt{\pi}}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+\frac{5}{2})}.$$

Hence
$$(\pi x)^{\frac{1}{2}}=-\sum_{k=0}^{\infty}\Gamma(k-\tfrac{1}{2})\,S_{\frac{3}{2}}{}^k(x), \tag{1.61}$$

$$\left(\frac{\pi}{x}\right)^{\frac{1}{2}}=\sum_{k=0}^{\infty}\Gamma(k+\tfrac{1}{2})\,S_{\frac{3}{2}}{}^k(x), \tag{1.62}$$

this latter series, of course, being divergent when $x=0$.

PART II.

2.1. In order to use the results of Part I for the calculation of the distribution functions for a gas, let (U_1, V_1, W_1) and (U_2, V_2, W_2) be the component velocities of molecules, of masses m_1 and m_2, relative to the mass motion (u_0, v_0, w_0); u_0, v_0, w_0 will therefore not appear in the equations, though their derivatives will. Also let the number of molecules of the first kind per unit volume be ν_1 and the number with velocities relative to the mass motion in the range bounded by (U_1, V_1, W_1) and $(U_1+dU_1,\ V_1+dV_1,\ W_1+dW_1)$ be $\nu_1 f_0(1+F_1)\,dU_1\,dV_1\,dW_1$, where

$f_0 = (hm_1/\pi)^{\frac{3}{2}} \exp(-hm_1 C_1^2)$ is the distribution function for a uniform gas. Similar letters with suffixes 2 instead of 1 will refer to the corresponding quantities for molecules of the second kind. Boltzmann's equation* for a gas in which the molecules m_1 and m_2 repel one another with a force $Km_1 m_2 r^{-(p-5)/(p-1)}$, $0 \leqslant p \leqslant 1$, is equivalent to

$$\Sigma \nu_2 [(m_1+m_2)K]^{-\frac{1}{2}(p-1)} \left(\frac{hm_2}{\pi}\right)^{\frac{3}{2}} \int_0^\infty a da \{I_1(\chi)+I_2(\chi)-I_1(0)-I_2(0)\}$$

$$= (1-\tfrac{2}{3}hm_1 C_1^2)\left\{\frac{1}{\nu_1}\frac{\partial \nu_1}{\partial t}+\tfrac{2}{3}\frac{1}{h}\frac{\partial h}{\partial t}\right\} - 2h(U_1 X+V_1 Y+W_1 Z)$$

$$+\left(U_1\frac{\partial}{\partial x}+V_1\frac{\partial}{\partial y}+W_1\frac{\partial}{\partial z}\right)\log \nu_1+\left(\frac{3}{2h}-m_1 C_1^2\right)\left(U_1\frac{\partial h}{\partial x}+V_1\frac{\partial h}{\partial y}+W_1\frac{\partial h}{\partial z}\right)$$

$$+\tfrac{2}{3}hm_1\Big[(3U_1^2-C_1^2)\frac{\partial u_0}{\partial x}+(3V_1^2-C_1^2)\frac{\partial v_0}{\partial y}+(3W_1^2-C_1^2)\frac{\partial w_0}{\partial z}$$

$$+3V_1 W_1\left(\frac{\partial v_0}{\partial z}+\frac{\partial w_0}{\partial y}\right)+3W_1 U_1\left(\frac{\partial w_0}{\partial x}+\frac{\partial u_0}{\partial z}\right)+3U_1 V_1\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)\Big], \tag{2.11}$$

where X, Y, Z are the components of the external force, a and χ are connected by the relation

$$\pi-\chi = 2\int_0^{\eta_0} \frac{d\eta}{\{1-\eta^2-\frac{1}{2}(1-p)(\eta/a)^{4/(1-p)}\}^{\frac{1}{2}}}, \tag{2.111}$$

η_0 being the positive zero of the denominator,

$$I_1(\chi) = (\mu_1\mu_2)^{-\frac{1}{2}p}\int_0^{2\pi} d\epsilon \iiint F(U_1', V_1', W_1') \times \exp(-hm_2 C_2^2) C_R^p \, dU_2 dV_2 dW_2, \tag{2.112}$$

$$I_2(\chi) = (\mu_1\mu_2)^{-\frac{1}{2}p}\int_0^{2\pi} de \iiint F(U_2', V_2', W_2') \times \exp(-hm_2 C_2^2) C_R^p \, dU_2 dV_2 dW_2; \tag{2.113}$$

* *Cf.* Jeans, *loc. cit.*, but there are errors in signs (3rd edition, equation 537) which vitiate a number of the subsequent results of the chapter; it should be noted that the a of the present paper is the same as in Jeans and in Chapman (1917) but differs from the a of Chapman (1916) by a factor $\{2/(1-p)\}^{\frac{1}{2}(1-p)}$.

as in Part I, $(\mu_1\mu_2)^{-\frac{1}{2}} C_R$ is the relative velocity of the second molecule with respect to the first before the collision,

$$\mu_1 = \frac{m_1}{m_1+m_2}, \quad \mu_2 = \frac{m_2}{m_1+m_2}.$$

The summation on the left-hand side of (2.11) is taken over all the different kinds of molecules in the gas, including those of the first kind (m_1), in which case the suffixes 2 have to be replaced by 1 after the integrations are performed. The law of force between the molecules of the different constituents is not necessarily the same, so that p, K, χ may be different for different pairs of colliding molecules; they will be distinguished when necessary by different suffixes.

Consider first the case when the gas is a simple one, *i.e.* consisting of one type of molecule only: then* the equation of continuity,

$$\frac{1}{\nu}\frac{\partial \nu}{\partial t} = -\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}+\frac{\partial w_0}{\partial z}\right),$$

the three equations of conservation of momentum,

$$\nu X = \frac{1}{2h}\frac{\partial \nu}{\partial x} - \frac{\nu}{2h^2}\frac{\partial h}{\partial x}, \ldots,$$

and the equation of conservation of energy,

$$\frac{1}{h}\frac{\partial h}{\partial t} = \tfrac{2}{3}\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}+\frac{\partial w_0}{\partial z}\right) = -\frac{2}{3\nu}\frac{\partial \nu}{\partial t},$$

simplify the right-hand side of equation (2.11) to

$$\begin{aligned}
&\frac{5}{2h}(1-\tfrac{2}{5}hm_1 C_1^2)\left(U_1\frac{\partial h}{\partial x}+V_1\frac{\partial h}{\partial y}+W_1\frac{\partial h}{\partial z}\right)\\
&\qquad +\tfrac{2}{3}hm_1\{c_{11}U_1^2+c_{22}V_1^2+c_{33}W_1^2+C_{23}V_1W_1+c_{31}W_1U_1+c_{12}U_1V_1\}\\
&\quad = \frac{5}{2h}\left(U_1\frac{\partial h}{\partial x}+V_1\frac{\partial h}{\partial y}+W_1\frac{\partial h}{\partial z}\right)\Gamma(\tfrac{5}{2})S_{\frac{5}{2}}^{1}(hm_1C_1^2)\\
&\qquad +\tfrac{3}{2}hm_1\{c_{11}U_1^2+c_{22}V_1^2+c_{33}W_1^2+c_{23}V_1W_1+c_{31}W_1U_1+c_{12}U_1V_1\}\\
&\qquad\qquad \times\Gamma(\tfrac{7}{2})S_{\frac{5}{2}}^{0}(hm_1C_1^2), \quad (2.12)
\end{aligned}$$

* *Cf.* Chapman (1916), equations 20, 25, 28.

where

$$c_{11} = 2\frac{\partial u_0}{\partial x} - \frac{\partial v_0}{\partial y} - \frac{\partial w_0}{\partial z}, \quad c_{23} = 3\left(\frac{\partial v_0}{\partial z} + \frac{\partial w_0}{\partial y}\right), \tag{2.13}$$

with similar meanings for c_{22}, c_{33}, c_{31}, c_{12}.

Now it appears from (1.3) and (1.4) that, if $F(U, V, W)$ is of the form $U\Sigma\Gamma(k+\frac{5}{2})b_k S_{\frac{3}{2}}^k(hm C^2)$, the left-hand side of (2.11) is also of this form after the integrations are performed, while if $F(u, v, w)$ contains squares or products of U, V, W, multiplied by a function of C^2, so does the integrated expression; hence the parts of the distribution function which are linear and quadratic in U, V, W may be determined separately by equating corresponding terms on the two sides of Boltzmann's equation. The same property has been shown by Chapman to hold when the distribution functions are calculated by the method of transfer.

2.2. To determine the part of the distribution function which is linear in U, V, W, assume

$$F(U, V, W) = \frac{1}{h}\left(U\frac{\partial h}{\partial x} + V\frac{\partial h}{\partial y} + W\frac{\partial h}{\partial z}\right)\sum_{k=0}^{\infty}\Gamma(k+\tfrac{5}{2})b_k S_{\frac{3}{2}}^k(hm C^2), \tag{2.21}$$

where the coefficients b_k have to be determined.

This function evidently satisfies the conditions

$$\iiint F\exp(-hm C^2)\,dU\,dV\,dW = 0, \quad \iiint F C^2\exp(-hm C^2)\,dU\,dV\,dW = 0,$$

and it must also satisfy $\iiint F U\exp(-hm C^2)\,dU\,dV\,dW = 0$ and two symmetric equations; these require that

$$\sum_{k=0}^{\infty}\Gamma(k+\tfrac{5}{2})b_k\iiint U^2 S_{\frac{3}{2}}^k(hm C^2)\exp(-hm C^2)\,dU\,dV\,dW$$

and two symmetric expressions should vanish, *i.e.*

$$\sum_{k=0}^{\infty}\Gamma(k+\tfrac{5}{2})b_k\iiint C^4 S_{\frac{3}{2}}^k(hm C^2)\exp(-hm C^2)\,dC = 0.$$

Now this integral vanishes by (1.12) unless $k = 0$, so that the condition requires

$$b_0 = 0. \tag{2.22}$$

The values of $I_1(\chi)$ and $I_2(\chi)$ are found at once from (1.33), (1.35) by putting $m_1 = m_2 = \frac{1}{2}m_0$, $\mu_1 = \mu_2 = \frac{1}{2}$, since the collisions occur between

equal molecules; hence

$$I_1(\chi) = \frac{1}{h}\left(U_1\frac{\partial h}{\partial x}+V_1\frac{\partial h}{\partial y}+W_1\frac{\partial h}{\partial z}\right)\frac{\pi^{\frac{5}{2}}\Gamma(\frac{1}{2}p+\frac{3}{2})\,2^p}{(hm_1)^{\frac{5}{2}}(2hm_1)^{\frac{1}{2}p}}\sum_{k,n} b_k n!\, S_{\frac{3}{2}}^{n}(hm_1 C_1^2)$$

$$\times \text{coefficient of } s^k t^n \text{ in}$$

$$(p+3)\cos^2\tfrac{1}{2}\chi\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p}\{1-st\cos^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{5}{2})}$$

$$-\tfrac{1}{2}p\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p-1}\{1-st\cos^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{3}{2})},$$

and $I_2(\chi)$ is got by replacing $\cos\frac{1}{2}\chi$ by $\sin\frac{1}{2}\chi$ throughout. The corresponding term on the right-hand side of the equation is, from (2.12),

$$\frac{1}{h}\left(U_1\frac{\partial h}{\partial x}+V_1\frac{\partial h}{\partial y}+W_1\frac{\partial h}{\partial z}\right)\Gamma(\tfrac{7}{2})\,S_{\frac{3}{2}}^{1}(hm_1 C_1^2),$$

so that the coefficients b_k are determined by the equations

$$\sum_{k=1}^{\infty} b_k B_{k1} = -1, \quad \sum_{k=1}^{\infty} b_k B_{kn} = 0 \quad (n>1), \qquad (2.23)$$

where

$$B_{kn} = \pi\nu_1\frac{\Gamma(\frac{1}{2}p+\frac{3}{2})}{(hm_1)^{\frac{1}{2}p}\Gamma(\frac{7}{2})}\,2^{\frac{1}{2}}(Km_1)^{\frac{1}{2}-\frac{1}{2}p}\int_0^\infty a\,da \times \text{coefficient of } s^k t^n \text{ in}$$

$$(p+3)\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p}\,[(1-st)^{-(\frac{1}{2}p+\frac{5}{2})}-\cos^2\tfrac{1}{2}\chi(1-st\cos^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{5}{2})}$$

$$-\sin^2\tfrac{1}{2}\chi(1-st\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{5}{2})}]$$

$$-\tfrac{1}{2}p\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p-1}[1+(1-st)^{-(\frac{1}{2}p+\frac{3}{2})}-(1-st\cos^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{3}{2})}$$

$$-(1-st\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{3}{2})}]. \qquad (2.24)$$

To obtain this result by Chapman's method, consider the rate of transfer of the "molecular property"

$$U_1 S_{\frac{3}{2}}^{n}(hm_1 C_1^2) = U_1\sum_{r=0}^{n}\frac{(-hm_1 C_1^2)^r}{\Gamma(r+\frac{5}{2})\,r!\,(n-r)!}.$$

Chapman shows that the rate of transfer of $U_1(hm_1 C_1^2)^r$, due to the external forces and to variations in temperature and density, is

$$-\frac{2\nu_1}{3hm_1}\frac{r\Gamma(r+\frac{5}{2})}{\Gamma(\frac{1}{2})}\frac{1}{h}\frac{\partial h}{\partial x},$$

and so the rate of transfer of $U_1 S_{\frac{3}{2}}^n(hm_1 C_1^2)$ is

$$-\sum_{r=0}^{n} \frac{2\nu_1}{3hm_1\sqrt{\pi}} \frac{1}{h}\frac{\partial h}{\partial x}\frac{(-1)^r}{(r-1)!\,(n-r)!} = 0 \qquad (n>1),$$

$$= \frac{2\nu_1}{3hm_1\sqrt{\pi}} \frac{1}{h}\frac{\partial h}{\partial x} \quad (n=1). \quad (2.241)$$

Also the rate of transfer due to collisions is*

$$2{\nu_1}^2 \left(\frac{hm_1}{\pi}\right)^3 \frac{1}{h}\frac{\partial h}{\partial x} (\tfrac{1}{2}Km_1)^{-(\frac{1}{2}p-\frac{1}{2})} \int_0^\infty \alpha\, d\alpha \int_0^{2\pi} d\epsilon$$

$$\times \iiint\iiint \{U_1' S_{\frac{3}{2}}^n(hm_1 C_1'^2) - U_1 S_{\frac{3}{2}}^n(hm_1 C_1^2)\}$$

$$\times \{U_1 \Sigma\Gamma(k+\tfrac{5}{2}) b_k S_{\frac{3}{2}}^k(hm_1 C_1^2) + U_2 \Sigma\Gamma(k+\tfrac{5}{2}) b_k S_{\frac{3}{2}}^k(hm_2 C_2^2)\}$$

$$\times \exp(-hm_1 C_1^2 - hm_2 C_2^2)\, C_R^p\, dU_1 dV_1 dW_1 dU_2 dV_2 dW_2,$$

where the suffixes 2 are to be replaced by 1 after the integrations are performed. Hence, if

$$I_{11}(\chi) = \Sigma\Gamma(k+\tfrac{5}{2}) b_k \int_0^{2\pi} d\epsilon \iiint\iiint U_1' U_1 C_R^p S_{\frac{3}{2}}^n(hm_1 C_1'^2) S_{\frac{3}{2}}^k(hm_1 C_1^2)$$

$$\times \exp(-hm_1 C_1^2 - hm_2 C_2^2)\, dU_1 dV_1 dW_1 dU_2 dV_2 dW_2,$$

$$I_{12}(\chi) = \Sigma\Gamma(k+\tfrac{5}{2}) b_k \int_0^{2\pi} d\epsilon \iiint\iiint U_1' U_2 C_R^p S_{\frac{3}{2}}^n(hm_1 C_1'^2) S_{\frac{3}{2}}^k(hm_2 C_2^2)$$

$$\times \exp(-hm_1 C_1^2 - hm_2 C_2^2)\, dU_1 dV_1 dW_1 dU_2 dV_2 dW_2,$$

the rate of transfer is

$$2{\nu_1}^2 \left(\frac{hm_1}{\pi}\right)^3 \frac{1}{h}\frac{\partial h}{\partial x} (\tfrac{1}{2}Km_1)^{-(\frac{1}{2}p-\frac{1}{2})} \int_0^\infty \alpha\, d\alpha \{I_{11}(\chi) + I_{12}(\chi) - I_{11}(0) - I_{12}(0)\}$$

$$= \frac{1}{h}\frac{\partial h}{\partial x} \frac{2\nu_1}{3hm_1\sqrt{\pi}} \Sigma b_k B'_{kn}, \qquad (2.242)$$

where B'_{kn}, by (1.34) and (1.36), differs from the value of B_{kn} found above by a factor $\Gamma(\frac{7}{2})/\Gamma(n+\frac{5}{2})$, so that $B'_{k1} = B_{k1}$.

Hence, from (2.241) and (2.242),

$$\Sigma b_k B'_{kn} = 1 \quad (n=1),$$

$$= 0 \quad (n>1),$$

which are essentially the same equations as (2.23).

* *Cf.* Chapman (1916), (63) and (182), with the definition of α as modified above.

In the particular case of Maxwellian molecules, the inter-molecular force is proportional to the inverse fifth power of the distance, so that $p=0$ and $B_{kn}=0$ unless $k=n$. Hence $b_1=-(1/B_{11})$ and $b_k=0$, $k>1$. Also

$$B_{11}=+\pi\nu_1\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{7}{2})}(2Km_1)^{\frac{1}{2}}\int_0^\infty \alpha\,d\alpha\,\tfrac{1}{2}^5(1-\cos^4\tfrac{1}{2}\chi-\sin^4\tfrac{1}{2}\chi)$$

$$=\pi\nu_1(2Km_1)^{\frac{1}{2}}\int_0^\infty \alpha\,d\alpha\sin^2\chi$$

$$=\nu_1(2Km_1)^{\frac{1}{2}}I_2$$

in Maxwell's notation.

Hence*

$$F(U_1, V_1, W_1)$$

$$=-\Gamma(\tfrac{7}{2})\frac{1}{\nu_1\sqrt{(2Km_1)}\,I_2}\frac{1}{h}\left(U_1\frac{\partial h}{\partial x}+V_1\frac{\partial h}{\partial y}+W_1\frac{\partial h}{\partial z}\right)S_{\frac{3}{2}}^1(hm_1C_1^2)$$

$$=-\frac{5}{2\nu_1\sqrt{(2Km_1)}\,I_2}\frac{1}{h}\left(U_1\frac{\partial h}{\partial x}+V_1\frac{\partial h}{\partial y}+W_1\frac{\partial h}{\partial z}\right)(1-\tfrac{2}{5}hm_1C_1^2). \quad (2.25)$$

In the general case, (2.23) gives an infinite number of linear simultaneous equations for the determination of the coefficients b_k, which can, therefore, be expressed as the ratios of infinite determinants in the form

$$b_k=(-1)^{k-1}\begin{vmatrix} B_{21} & B_{22} & B_{23} & \dots \\ B_{31} & B_{32} & B_{33} & \dots \\ B_{41} & B_{42} & B_{43} & \dots \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \end{vmatrix} \div \begin{vmatrix} B_{11} & B_{12} & B_{13} & \dots \\ B_{21} & B_{22} & B_{23} & \dots \\ B_{31} & B_{32} & B_{33} & \dots \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \end{vmatrix}. \quad (2.261)$$

It is evident from (2.24) that $B_{kn}=B_{nk}$, and it will now be proved that $B_{kn}\leqslant 0$, $k\neq n$, and $\sum_n B_{kn}>0$ for all k.

Let $B_{kn}=(p+3)C_{kn}-\frac{1}{2}pD_{kn}$, where, except for a positive factor

* When the difference in the definitions of the quantity σ, and therefore of I_2, is taken into account this result agrees with Chapman's (198) except for a factor π; this appears to be due to an error in his equation (192) which is not consistent with (170) and (189).

independent of k and n,

C_{kn} = coefficient of $s^k t^n$ in

$$\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p}\int_0^\infty a\,da\{(1-st)^{-(\frac{1}{4}p+\frac{5}{2})}-\cos^2\tfrac{1}{2}\chi(1-st\cos^2\tfrac{1}{2}\chi)^{-(\frac{1}{4}p+\frac{5}{2})} -\sin^2\tfrac{1}{2}\chi(1-st\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{4}p+\frac{5}{2})}\}, \quad (2.262)$$

D_{kn} = coefficient of $s^k t^n$ in

$$\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p-1}\int_0^\infty a\,da\{1+(1-st)^{-(\frac{1}{4}p+\frac{3}{2})}-(1-st\cos^2\tfrac{1}{2}\chi)^{-(\frac{1}{4}p+\frac{3}{2})} -(1-st\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{4}p+\frac{3}{2})}\}. \quad (2.263)$$

Hence, assuming $k>n$ and neglecting a factor as before,

$$C_{kn}=\int_0^\infty a\,da\Big[\binom{-\frac{1}{2}p-\frac{5}{2}}{n}(-1)^n(1-\cos^{2n+2}\tfrac{1}{2}\chi-\sin^{2n+2}\tfrac{1}{2}\chi) \times\text{coefficient of } s^{k-n} \text{ in } \{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p}$$
$$+\binom{-\frac{1}{2}p-\frac{5}{2}}{n-1}(-1)^{n-1}(1-\cos^{2n}\tfrac{1}{2}\chi-\sin^{2n}\tfrac{1}{2}\chi) \times\text{coefficient of } s^{k-n+1}t \text{ in } \{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p}+\ldots\Big],$$

each term of which is negative, since $0<p<1$; and

$$D_{kn}=\int_0^\infty a\,da\Big[\binom{-\frac{1}{2}p-\frac{3}{2}}{n}(-1)^n(1-\cos^{2n}\tfrac{1}{2}\chi-\sin^{2n}\tfrac{1}{2}\chi) \times\text{coefficient of } s^{k-n} \text{ in } \{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p-1}+\ldots\Big],$$

each term of which is positive, and so

$$B_{kn}<0. \quad (2.264)$$

Further, leaving out a positive factor as before, $\sum_n C_{kn}$ is the coefficient of s^k in

$$\Big(\frac{1-s}{2}\Big)^{\frac{1}{2}p}\int_0^\infty a\,da\{(1-s)^{-(\frac{1}{4}p+\frac{5}{2})}-\cos^2\tfrac{1}{2}\chi(1-s\cos^2\tfrac{1}{2}\chi)^{-(\frac{1}{4}p+\frac{5}{2})} -\sin^2\tfrac{1}{2}\chi(1-s\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{4}p+\frac{5}{2})}\}$$
$$=2^{-\frac{1}{2}p}(1-s)^{\frac{1}{4}p-1}\int_0^\infty a\,da\{(1-s)^{-(\frac{1}{4}p+\frac{3}{2})}-(1-s\cos^2\tfrac{1}{2}\chi)^{-(\frac{1}{4}p+\frac{3}{2})} -(1-s\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{4}p+\frac{3}{2})}+\sin^2\tfrac{1}{2}\chi(1-s\cos^2\tfrac{1}{2}\chi)^{-(\frac{1}{4}p+\frac{5}{2})} +\cos^2\tfrac{1}{2}\chi(1-s\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{4}p+\frac{5}{2})}\},$$

while $\sum_n D_{kn}$ is the coefficient of s^k in

$$2^{1-\frac{1}{2}p}(1-s)^{\frac{1}{2}p-1}\int_0^\infty a\,da\{1+(1-s)^{-(\frac{1}{2}p+\frac{3}{2})}-(1-s\cos^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{3}{2})}$$
$$-(1-s\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{3}{2})}\}.$$

Hence $\sum_n B_{kn}$ is the coefficient of s^k in

$$2^{-\frac{1}{2}p}(1-s)^{\frac{1}{2}p-1}\int_0^\infty a\,da[3\{1+(1-s)^{-(\frac{1}{2}p+\frac{3}{2})}-(1-s\cos^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{3}{2})}$$
$$-(1-s\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{3}{2})}\}-(p+3)\{1-\cos^2\tfrac{1}{2}\chi(1-s\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{5}{2})}$$
$$-\sin^2\tfrac{1}{2}\chi(1-s\cos^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{3}{2})}\}]. \qquad (2.265)$$

The coefficient of each power of s in the expansion of the integrand is evidently positive, as is every coefficient in the expansion of $(1-s)^{\frac{1}{2}p-1}$ since $\frac{1}{2}p<1$; it follows that $\sum_n B_{kn}$ is positive for all values of k. (2.266)

Hence, by the theorem on determinants proved in the appendix, each b_k is negative.

Also, since

$$1-\cos^{2m}\tfrac{1}{2}\chi-\sin^{2m}\tfrac{1}{2}\chi>1-\cos^4\tfrac{1}{2}\chi-\sin^4\tfrac{1}{2}\chi \quad (m>2),$$

the coefficient of s^k in the integrand in (2.265) is greater than

$$\int_0^\infty a\,da(1-\cos^4\tfrac{1}{2}\chi-\sin^4\tfrac{1}{2}\chi)\times\text{coefficient of } s^k \text{ in } (1-s)^{-(\frac{1}{2}p+\frac{3}{2})},$$

and so (2.265) is greater than

$$2^{-\frac{1}{2}p}\int_0^\infty a\,da(1-\cos^4\tfrac{1}{2}\chi-\sin^4\tfrac{1}{2}\chi)\times\text{coefficient of } s^k \text{ in } (1-s)^{-\frac{5}{2}},$$

i.e. it is of the order of $\Gamma(k+\frac{5}{2})/k!$, *i.e.* of the order of $k^{\frac{3}{2}}$, at least; therefore $\sum B_{kn}$ is of the order of $k^{\frac{3}{2}}$ at least, and $\sum_k b_k\sum_n B_{kn}=-1$ from (2.23). Hence, by §1.5, $\sum_k \Gamma(k+\frac{5}{2})\,b_k\,S_{\frac{3}{2}}{}^k(hm\,C^2)$ converges uniformly.

In the case of molecules which are rigid elastic spheres of radius σ, the

$$(\tfrac{1}{2}Km_1)^{-(\frac{1}{2}p-\frac{1}{2})}\int_0^\infty a\,da$$

of the previous formula must be replaced by

$$\sigma^2\int_0^\pi \sin\chi\,d\chi$$

and p by unity*. Hence, in this case, (2.24) gives

$$B_{kn} = \frac{\pi \nu_1 \sigma^2}{\Gamma(\frac{7}{2})} \sqrt{\left(\frac{2}{hm_1}\right)} \times \text{coefficient of } s^k t^n \text{ in}$$

$$\int_0^\pi \sin\chi \, d\chi \left[4\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}} \left\{ \frac{1}{(1-st)^3} - \frac{\cos^2 \tfrac{1}{2}\chi}{(1-st\cos^2\tfrac{1}{2}\chi)^3} - \frac{\sin^2\tfrac{1}{2}\chi}{(1-st\sin^2\tfrac{1}{2}\chi)^3} \right\} \right.$$

$$\left. -\tfrac{1}{2}\{1-\tfrac{1}{2}(s+t)\}^{-\frac{1}{2}} \left\{ 1 + \frac{1}{(1-st)^2} - \frac{1}{(1-st\cos^2\tfrac{1}{2}\chi)^2} - \frac{1}{(1-st\sin^2\tfrac{1}{2}\chi)^2} \right\} \right]$$

$$= \tfrac{8}{15} \nu_1 \sigma^2 \sqrt{\left(\frac{2\pi}{hm_1}\right)} \times \text{coefficient of } s^k t^n \text{ in}$$

$$\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}} \frac{st}{(1-st)^3} - \tfrac{1}{4}\{1-\tfrac{1}{2}(s+t)\}^{-\frac{1}{2}} \frac{s^2t^2}{(1-st)^2}, \quad (2.271)$$

on performing the integrations. The coefficients can now be easily written down; the first few are

$$B_{11} = \frac{8}{15} \nu_1 \sigma^2 \sqrt{\left(\frac{2\pi}{hm_1}\right)}, \quad B_{12} = -\frac{1}{2^2} B_{11}, \quad B_{13} = -\frac{1}{2^5} B_{11}, \quad B_{14} = -\frac{1}{2^7} B_{11},$$

$$B_{22} = \frac{45}{2^4} B_{11}, \quad B_{23} = -\frac{103}{2^4} B_{11}, \quad B_{24} = -\frac{59}{2^9} B_{11},$$

$$B_{33} = \frac{5657}{2^{10}} B_{11}, \quad B_{34} = -\frac{6783}{2^{12}} B_{11},$$

$$B_{44} = \frac{1\,49749}{2^{14}} B_{11}.$$

These values should be compared with the constituents of Chapman's differenced determinant (221): it can easily be shown by considering the equations of viscosity and thermal conduction, as in Chapman (1916), § 11, that the first coefficient b_1 in the distribution function

$$\Sigma \Gamma(k+\tfrac{5}{2}) b_k S_{\frac{3}{2}}^k (hm C^2)$$

corresponds to $\Sigma \beta_r$ in Chapman's notation, so that the similarity of the determinants is not surprising.

The approximate values of the coefficients calculated from (2.261) with the above values for B_{kn} are

$$B_{11} b_1 = -1{\cdot}0251, \quad B_{11} b_2 = -0{\cdot}0976, \quad B_{11} b_3 = -0{\cdot}0218, \quad B_{11} b_4 = -0{\cdot}0061.$$

* *Cf.* Chapman (1916), § 9 (D), (201).

2.7. To determine the part of the distribution function which contains terms in U^2, UV, etc., assume it to be $f_0 F$, where

$$F = \{c_{11}U^2+c_{22}V^2+c_{33}W^2+c_{23}VW+c_{31}WU+c_{12}UV\} \times \Sigma\,\Gamma(k+\tfrac{7}{2})\,b_k\,S_{\frac{5}{2}}^{k}(hm\,C^2).$$

The conditions

$$\iiint FU\exp(-hmC^2)\,dU\,dV\,dW = \iiint FV\exp(-hmC^2)\,dU\,dV\,dW = \iiint FW\exp(-hmC^2)\,dU\,dV\,dW = 0$$

are evidently satisfied; also, since

$$\iiint U^2 f(C^2)\,dU\,dV\,dW = \iiint V^2 f(C^2)\,dU\,dV\,dW = \iiint W f(C^2)\,dU\,dV\,dW,$$

and

$$\iiint VW f(C^2)\,dU\,dV\,dW$$

and similar expressions vanish where $f(C^2)$ is any rational integral function of C^2, it follows that

$$\iiint F\exp(-hmC^2)\,dU\,dV\,dW = (c_{11}+c_{22}+c_{33})\iiint U^2 f(C^2)\,dU\,dV\,dW = 0$$

since $c_{11}+c_{22}+c_{33}=0$ from (2.13), and, similarly,

$$\iiint FC^2\exp(-hmC^2)\,dU\,dV\,dW = 0.$$

In this case, putting $\mu_1=\mu_2=\frac{1}{2}$ in (1.41), (1.43), (1.45), and (1.46), and adding similar expressions, it is found that

$$I_1(\chi) = \{c_{11}U_1^2+c_{22}V_1^2+c_{33}W_1^2+c_{23}V_1W_1+c_{31}W_1U_1+c_{12}U_1V_1\}\,2^{\frac{1}{2}p+1} \times \frac{\pi^{\frac{5}{2}}\Gamma(\frac{1}{2}p+\frac{3}{2})}{(hm_1)^{\frac{1}{2}p+\frac{3}{2}}} \sum_{k,\,n} n!\,b_k\,S_{\frac{5}{2}}^{n}(hm_1C_1^2) \times \text{coefficient of } s^k t^n \text{ in}$$

$$(\tfrac{1}{2}p+\tfrac{3}{2})(\tfrac{1}{2}p+\tfrac{5}{2})\cos^2\tfrac{1}{2}\chi(\cos^2\tfrac{1}{2}\chi-\tfrac{1}{2}\sin^2\tfrac{1}{2}\chi)\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p}\{1-st\cos^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{3}{2})}$$
$$-\tfrac{1}{2}p(\tfrac{1}{2}p+\tfrac{3}{2})\cos^2\tfrac{1}{2}\chi\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p-1}\{1-st\cos^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{5}{2})}$$
$$+\tfrac{1}{8}p(\tfrac{1}{2}p-1)\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p-2}\{1-st\cos^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{3}{2})} \qquad (2.73)$$

and $I_2(\chi)$ is got by interchanging $\cos\frac{1}{2}\chi$ and $\sin\frac{1}{2}\chi$ throughout. Hence, by equating corresponding terms in Boltzmann's equation (2.11) and (2.12),

$$\sum_{k,n} n!\, b_k\, B_{kn}\, S_{\frac{5}{2}}^{n}(hm_1 C_1^2) = -S_{\frac{5}{2}}^{0}(hm_1 C_1^2),$$

i.e.

$$\sum_k b_k B_{k0} = -1, \quad \sum_k b_k B_{kn} = 0 \quad (n>0), \tag{2.74}$$

where

$$B_{kn} = \pi\nu_1 \frac{\Gamma(\frac{1}{2}p+\frac{3}{2})}{(hm_1)^{\frac{1}{2}p}\,\Gamma(\frac{7}{2})} \frac{3}{hm_1}\, 2^{\frac{1}{2}}(Km_1)^{-\frac{1}{2}(p-1)} \int_0^\infty a\,da \times \text{coefficient of } s^k t^n \text{ in}$$

$$(\tfrac{1}{2}p+\tfrac{3}{2})(\tfrac{1}{2}p+\tfrac{5}{2})\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p}[(1-st)^{-(\frac{1}{2}p+\frac{7}{2})} - \cos^2\tfrac{1}{2}\chi(\cos^2\tfrac{1}{2}\chi - \tfrac{1}{2}\sin^2\tfrac{1}{2}\chi)$$
$$\times(1-st\cos^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{7}{2})} - \sin^2\tfrac{1}{2}\chi(\sin^2\tfrac{1}{2}\chi - \tfrac{1}{2}\cos^2\tfrac{1}{2}\chi)(1-st\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{7}{2})}]$$
$$-(\tfrac{1}{2}p)(\tfrac{1}{2}p+\tfrac{3}{2})\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p-1}[(1-st)^{-(\frac{1}{2}p+\frac{5}{2})} - \cos^2\tfrac{1}{2}\chi(1-st\cos^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{5}{2})}$$
$$-\sin^2\tfrac{1}{2}\chi(1-st\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{5}{2})}]$$
$$+\tfrac{1}{8}p(\tfrac{1}{2}p-1)\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p-2}[1+(1-st)^{-(\frac{1}{2}p+\frac{3}{2})} - (1-st\cos^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{3}{2})}$$
$$-(1-st\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{3}{2})}]. \tag{2.75}$$

The result is obtained equally easily by Chapman's method; for the rate of transfer of the "molecular property" $U_1^2 S_{\frac{5}{2}}^{n}(hm_1 C_1^2)$ by molecular collisions is*

$$(\tfrac{1}{2}Km_1)^{-\frac{1}{2}(p-1)} c_{11}\nu_1^2 \left(\frac{hm_1}{\pi}\right)^3 \int_0^\infty a\,da\{I_{11}(\chi)+I_{12}(\chi)-I_{11}(0)-I_{12}(0)\},$$

where

$$I_{11}(\chi) = \sum_k \Gamma(k+\tfrac{7}{2})\, b_k \int_0^{2\pi} d\epsilon \iiint\!\!\iiint\!\!\iint U_1'^2(3U_1^2 - C_1^2)\, S_{\frac{5}{2}}^{k}(hm_1 C_1^2)\, S_{\frac{5}{2}}^{n}(hm_1 C_1^2)$$
$$\exp(-hm_1 C_1^2 - hm_2 C_2^2)\, dU_1 dV_1 dW_1 dU_2 dV_2 dW_2$$

$$I_{12}(\chi) = \sum_k \Gamma(k+\tfrac{7}{2})\, b_k \int_0^{2\pi} d\epsilon \iiint\!\!\iiint\!\!\iint U_1'^2(3U_2^2 - C_2^2)\, S_{\frac{5}{2}}^{k}(hm_2 C_2^2)\, S_{\frac{5}{2}}^{n}(hm_1 C_1'^2)$$
$$\times\exp(-hm_1 C_1^2 - hm_2 C_2^2)\, dU_1 dV_1 dW_1 dU_2 dV_2 dW_2.$$

Since

$$S_{\frac{5}{2}}^{n}(hm_1 C_1^2) = \sum_{r=0}^{n} \frac{(-hm_1 C_1^2)^r}{\Gamma(r+\frac{7}{2})\, r!\, (n-r)!},$$

---

* *Cf.* Chapman (1916), equation (66).

the rate of transfer is also*

$$\frac{8\nu_1 c_{11}}{45} \sum_{r=0}^{n} \frac{(-1)^r}{hm_1\sqrt{\pi}} \frac{1}{r!\,(n-r)!} = \frac{8\nu_1 c_{11}}{45hm_1\sqrt{\pi}} \quad (n=0),$$

$$= 0 \quad (n>0).$$

Hence, by equating the two expressions for the rate of transfer,

$$\Sigma b_k B'_{k0} = -1, \quad \Sigma b_k B'_{kn} = 0, \tag{2.76}$$

where B'_{kn} can be written down from the above expressions and (1.42), (1.44); it differs from B_{kn} above only by a factor $\Gamma(\frac{7}{2})/\Gamma(n+\frac{7}{2})$, so that equations (2.74) and (2.76) are equivalent.

In the case of Maxwellian molecules $p=0$, and it follows from (2.75) that $B_{kn}=0$ unless $k=n$: hence, from (2.74), $b_k=0$ unless $k=0$ and $b_0=-(1/B_{00})$, where

$$B_{00} = 3\pi\nu_1 \frac{(2Km_1)^{\frac{1}{2}}}{hm_1} \int_0^\infty \alpha\, d\alpha (1-\cos^4\tfrac{1}{2}\chi-\sin^4\tfrac{1}{2}\chi+\sin^2\tfrac{1}{2}\chi\cos^2\tfrac{1}{2}\chi)$$

$$= \tfrac{9}{4}\nu_1 \frac{(2Km_1)^{\frac{1}{2}}}{hm_1} I_2,$$

with the same definition of I_2 as before; hence

$$b_0 = -\frac{4hm_1}{9\nu_1\sqrt{(2Km_1)}\,I_2},$$

and this part of the distribution function is

$$-\frac{4hm_1}{9\nu_1\sqrt{(2Km_1)}\,I_2}(c_{11}U_1^2+c_{22}V_1^2 + c_{33}W_1^2+c_{23}V_1W_1+c_{31}W_1U_1+c_{12}U_1V_1),$$

agreeing with Chapman's result except for a factor π already noted (2.25) and a misprinted factor 2 in the product terms in his paper.

The investigation of the properties of the coefficients and of the convergence of the series in the general case follows the same lines as before; it is easily shown that B_{kn} is negative, $k\neq n$, that $\sum_n B_{kn}$ is positive for all values of k, and therefore (*cf.* Appendix) each b_k is negative; also, as before,

* Chapman, equation (30).

$\sum_n B_{kn}$ is of the order of $k^{\frac{5}{2}}$ at least when k is large, and so

$$\sum_k \Gamma(k+\tfrac{7}{2})\, b_k\, S_{\frac{5}{2}}{}^k(hm\, C^2)$$

converges uniformly in C^2.

If the molecules are rigid elastic spheres of radius σ, we have

$$B_{kn} = \frac{3\pi\nu_1\sigma^2}{\Gamma(\frac{7}{2})}\frac{2^{\frac{1}{2}}}{(hm_1)^{\frac{3}{2}}} \times \text{coefficient of } s^k t^n \text{ in}$$

$$\int_0^\pi \sin\chi\, d\chi \Big[6\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}} \Big\{ \frac{1}{(1-st)^4} - \frac{\cos^2\frac{1}{2}\chi(\cos^2\frac{1}{2}\chi - \frac{1}{2}\sin^2\frac{1}{2}\chi)}{(1-st\cos^2\frac{1}{2}\chi)^4} - \frac{\sin^2\frac{1}{2}\chi(\sin^2\frac{1}{2}\chi - \frac{1}{2}\cos^2\frac{1}{2}\chi)}{(1-st\sin^2\frac{1}{2}\chi)^4} \Big\}$$

$$- \{1-\tfrac{1}{2}(s+t)\}^{-\frac{1}{2}} \Big\{ \frac{1}{(1-st)^3} - \frac{\cos^2\frac{1}{2}\chi}{(1-st\cos^2\frac{1}{2}\chi)^3} - \frac{\sin^2\frac{1}{2}\chi}{(1-st\sin^2\frac{1}{2}\chi)^3} \Big\}$$

$$- \tfrac{1}{16}\{1-\tfrac{1}{2}(s+t)\}^{-\frac{3}{2}} \Big\{ 1 + \frac{1}{(1-st)^2} - \frac{1}{(1-st\cos^2\frac{1}{2}\chi)^2} - \frac{1}{(1-st\sin^2\frac{1}{2}\chi)^2} \Big\} \Big]$$

$$= \tfrac{48}{5}\frac{\nu_1\sigma^2}{hm_1}\sqrt{\left(\frac{2\pi}{hm_1}\right)} \times \text{coefficient of } s^k t^n \text{ in}$$

$$\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}}(1+\tfrac{2}{3}st+\tfrac{1}{3}s^2t^2)(1-st)^{-4} - \tfrac{1}{3}st\{1-\tfrac{1}{2}(s+t)\}^{-\frac{1}{2}}(1-st)^{-3}$$

$$- \tfrac{1}{48}s^2t^2\{1-\tfrac{1}{2}(s+t)\}^{-\frac{3}{2}}(1-st)^{-2},$$

on performing the integrations.

The values of the first few coefficients are

$$B_{00} = \frac{48}{5}\frac{\nu_1\sigma^2}{hm_1}\sqrt{\left(\frac{2\pi}{hm_1}\right)}, \quad B_{0k} = -\frac{\Gamma(k-\frac{1}{2})}{k!\,\Gamma(\frac{1}{2})\,2^{k+1}}\, B_{00},$$

$$B_{11} = \frac{205}{3\,.\,2^4}\, B_{00}, \quad B_{12} = -\frac{163}{2^4}\, B_{00}, \quad B_{13} = -\frac{287}{3\,.\,2^9}\, B_{00},$$

$$B_{22} = \frac{11889}{2^{10}}\, B_{00}, \quad B_{23} = -\frac{15043}{2^{12}}\, B_{00},$$

$$B_{33} = \frac{12\,20437}{3\,.\,2^{14}}\, B_{00},$$

and the approximate values of the coefficients in the series for the distribution function are

$$B_{00}b_0 = -1{\cdot}01600, \quad B_{00}b_1 = -0{\cdot}0626, \quad B_{00}b_2 = -0{\cdot}0108, \quad B_{00}b_3 = 0{\cdot}00232.$$

The values of the coefficients B_{kn} above agree with those given by Chapman (1916), equation (222), except B_{23} and B_{33}. I have, however,

recomputed the elements of Chapman's determinant $\nabla(c_{rs})$ (218) and I find that the element which he gives as $\frac{62\,02777\frac{2}{7}}{2^5 3^4 7^2 11}$ should be $\frac{20\,65837}{2^5 . 3^3 . 7^2 . 11}$; and, when this value is used, his determinant (222) becomes identical with that obtained in this paper. The effect of the error on his value of $\Sigma\gamma_r$, which is the same as the $-B_{00}b_0$ above, is quite small and his value 1·016 for this quantity is accurate. The identity of Chapman's $\Sigma\gamma_r$ with the $-B_{00}b_0$ of this paper is easily established by calculating the mean value of U^2, *cf.* Chapman, § 11.

2.3. *Mixture of two gases.*

Since it is clear from the preceding section that solutions by the method of transfer and by Boltzmann's equation are equivalent, only the latter method will be worked out here; and it will be sufficient to consider a gas composed of spherically symmetrical molecules of two different kinds, at the same temperature, and in proportions which do not vary with t, so that, if their densities are ν_1 and ν_2 per unit volume and $\nu_0 = \nu_1 + \nu_2$, then

$$\frac{1}{\nu_1}\frac{\partial \nu_1}{\partial t} = \frac{1}{\nu_2}\frac{\partial \nu_2}{\partial t} = \frac{1}{\nu_0}\frac{\partial \nu_0}{\partial t}. \tag{2.31}$$

Hence the equation of continuity* is

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z} = -\frac{1}{\nu_0}\frac{\partial \nu_0}{\partial t} = -\frac{1}{\nu_1}\frac{\partial \nu_1}{\partial t},$$

and Boltzmann's equation† takes the form

$$\begin{aligned}
&\left\{\frac{\partial \nu_1}{\partial t} + \tfrac{3}{2}\frac{\nu_1}{h}\frac{\partial h}{\partial t}\right\}(1 - \tfrac{2}{3}hm_1 C_1^2) + 2hm_1\nu_1\left(U_1\frac{\partial u_0}{\partial t} + V_1\frac{\partial v_0}{\partial t} + W_1\frac{\partial w_0}{\partial t}\right) \\
&\quad - 2h\nu_1(U_1 X_1 + V_1 Y_1 + W_1 Z_1) + \left(U_1\frac{\partial \nu_1}{\partial x} + V_1\frac{\partial \nu_1}{\partial y} + W_1\frac{\partial \nu_1}{\partial z}\right) \\
&\quad + \frac{3\nu_1}{2h}(1 - \tfrac{2}{3}hm_1 C_1^2)\left(U_1\frac{\partial h}{\partial x} + V_1\frac{\partial h}{\partial y} + W_1\frac{\partial h}{\partial z}\right) \\
&\quad + \tfrac{2}{3}hm_1\nu_1(c_{11}U_1^2 + c_{22}V_1^2 + c_{33}W_1^2 + c_{23}V_1 W_1 + c_{31}W_1 U_1 + c_{12}U_1 V_1) \\
&= \nu_1\nu_2\left(\frac{hm_2}{\pi}\right)^{\frac{3}{2}}\{K(m_1+m_2)\}^{-\frac{1}{2}(p-1)}\int_0^\infty \alpha\, d\alpha\{I_{12}(\chi) + I_{22}(\chi) - I_{12}(0) - I_{22}(0)\} \\
&+ \nu_1^2\left(\frac{hm_1}{\pi}\right)^{\frac{3}{2}}\{2K_1 m_1\}^{-\frac{1}{2}(p_1-1)}\int_0^\infty \alpha\, d\alpha\{I_{11}(\chi_1) + I_{12}(\chi_1) - I_{11}(0) - I_{12}(0)\},
\end{aligned} \tag{2.32}$$

* *Cf.* Chapman (1917), 2.013.

† *Cf.* Jeans, *loc. cit.*, ch. 8, and foot-note above, § 2.1.

where

$$I_{12}(\chi) = (\mu_1\mu_2)^{-\frac{1}{2}p}\int_0^{2\pi} d\epsilon \iiint F(U_1', V_1', W_1') \times \exp(-hm_2 C_2^2)\, C_R^p\, dU_2\, dV_2\, dW_2, \quad (2.321)$$

$$I_{22}(\chi) = (\mu_1\mu_2)^{-\frac{1}{2}p}\int_0^{2\pi} d\epsilon \iiint F(U_2', V_2', W_2') \times \exp(-hm_2 C_2^2) C_R^p\, dU_2\, dV_2\, dW_2, \quad (2.322)$$

and $I_{11}(\chi_1)$, $I_{12}(\chi_1)$ are similar expressions in which the suffixes 2 are replaced by 1 after the integrations are performed, so that $\mu_1 = \mu_2 = \frac{1}{2}$, ... in these integrals. K and p are constants which specify the law of force between two molecules of different kinds, which, at distance r apart, repel one another with a force $Km_1 m_2 r^{-(p-5)/(p-1)}$; K_1, p_1 are the corresponding quantities for two molecules of the first kind, and K_2, p_2 will be used below with reference to the force between two molecules of the second kind. The angles χ, χ_1, χ_2 are defined in terms of these constants by equations of the type (2.111).

Conservation of energy requires that*

$$0 = \frac{1}{\nu_1}\frac{\partial \nu_1}{\partial t} - \frac{1}{h}\frac{\partial h}{\partial t} - \tfrac{5}{3}\frac{1}{\nu_0}\frac{\partial \nu_0}{\partial t}$$

$$= -\tfrac{2}{3}\left(\frac{1}{\nu_1}\frac{\partial \nu_1}{\partial t} + \tfrac{3}{2}\frac{1}{h}\frac{\partial h}{\partial t}\right)$$

in this case, by virtue of (2.31), so that the first term on the left-hand side of (2.32) vanishes. Further, the terms involving U_1 are

$$\nu_1 U_1\left[2hm_1\frac{\partial u_0}{\partial t} - 2hX_1 + \frac{1}{\nu_1}\frac{\partial \nu_1}{\partial x} - \frac{1}{h}\frac{\partial h}{\partial x} + \frac{5}{2h}(1 - \tfrac{2}{5}hm_1 C_1^2)\frac{\partial h}{\partial x}\right]$$

$$= U_1\left[a_x \Gamma(\tfrac{5}{2})\, S_{\frac{3}{2}}^0(hm_1 C_1^2) + \frac{5\nu_1}{2h}\frac{\partial h}{\partial x}\Gamma(\tfrac{5}{2})\, S_{\frac{3}{2}}^1(hm_1 C_1^2)\right], \quad (2.323)$$

where a_x can be expressed with the help of Chapman's equation (2.04) of conservation of momentum in the form

$$a_x = \frac{2\nu_1\nu_2 h}{\nu_1 m_1 + \nu_2 m_2}(m_1 X_2 - m_2 X_1) - \tfrac{1}{2}h\frac{\nu_1 m_1 - \nu_2 m_2}{\nu_1 m_1 + \nu_2 m_2}\frac{\partial}{\partial x}\frac{\nu_1 + \nu_2}{h} + \tfrac{1}{2}h\frac{\partial}{\partial x}\frac{\nu_1 - \nu_2}{h}. \quad (2.324)$$

* *Cf.* Chapman (1917), 2.14, 2.15, 2.23.

Hence the left-hand side of equation (2.32) becomes

$$\{a_x U_1+a_y V_1+a_z W_1\}\,\Gamma(\tfrac{5}{2})\,S_{\frac{3}{2}}^{0}(hm_1 C_1^2)$$
$$+\tfrac{5}{2}\frac{\nu_1}{h}\left\{U_1\frac{\partial h}{\partial x}+V_1\frac{\partial h}{\partial y}+W_1\frac{\partial h}{\partial z}\right\}\Gamma(\tfrac{5}{2})\,S_{\frac{3}{2}}^{1}(hm_1 C_1^2)$$
$$+\tfrac{2}{3}hm_1\nu_1\{c_{11}U_1^2+c_{22}V_1^2+c_{33}W_1^2+c_{23}V_1W_1+c_{31}W_1U_1+c_{12}U_1V_1\}$$
$$\times\Gamma(\tfrac{7}{2})\,S_{\frac{3}{2}}^{0}(hm_1 C_1^2), \quad (2.325)$$

a_y and a_z being defined by equations similar to (2.324).

Let the parts of the distribution functions which contain U_1 and U_2 be

$$\nu_1\left(\frac{hm_1}{\pi}\right)^{\frac{3}{2}}\exp(-hm_1 C_1^2)\,F_1 \quad\text{and}\quad \nu_2\left(\frac{hm_2}{\pi}\right)^{\frac{3}{2}}\exp(-hm_2 C_2^2)\,F_2,$$

where

$$F_1=U_1\sum_k\Gamma(k+\tfrac{5}{2})\,b_k\,S_{\frac{3}{2}}^{k}(hm_1 C_1^2), \quad F_2=U_2\sum_k\Gamma(k+\tfrac{5}{2})\,\beta_k\,S_{\frac{3}{2}}^{k}(hm_2 C_2^2).$$

The conditions

$$\iiint F\exp(-hmC^2)\,dU\,dV\,dW=0,$$

$$\iiint FC^2\exp(-hmC^2)\,dU\,dV\,dW=0$$

are evidently satisfied for both types of molecules, while*

$$\left(\frac{hm_1}{\pi}\right)^{\frac{3}{2}}\iiint\nu_1 U_1 F_1\exp(-hm_1 C_1^2)\,dU_1\,dV_1\,dW_1$$
$$+\left(\frac{hm_2}{\pi}\right)^{\frac{3}{2}}\iiint\nu_2 U_2 F_2\exp(-hm_2 C_2^2)\,dU_2\,dV_2\,dW_2=0$$

leads, as in (2.22), to

$$\frac{\nu_1}{m_1}b_0+\frac{\nu_2}{m_2}\beta_0=0. \quad (2.331)$$

Equating the coefficients of $U_1 S_{\frac{3}{2}}^{0}(hm_1 C_1^2)$ and $U_1 S_{\frac{3}{2}}^{1}(hm_1 C_1^2)$ in (2.32), (2.325) gives

$$-a_x=\sum_k(B_{k0}b_k-C_{k0}\beta_k), \quad (2.332)$$

$$-\tfrac{5}{2}\frac{\nu_1}{h}\frac{\partial h}{\partial x}=\sum_k(B_{k1}b_k-C_{k1}\beta_k), \quad (2.333)$$

$$0=\sum_k(B_{kn}b_k-C_{kn}\beta_k)\quad(n>1), \quad (2.334)$$

* The velocities of the molecules are measured (as by Chapman) relative to a "mass motion of the composite gas", which is defined to be such that equal numbers of the two kinds of molecules cross a plane travelling with that velocity in opposite directions in any time. The above condition follows at once.

where B_{kn} is, from (1.33) and (1.35), the coefficient of $s^k t^n$ in

$$2\pi\nu_1\nu_2 \frac{\Gamma(\frac{1}{2}p+\frac{3}{2})\,K^{-\frac{1}{2}(p-1)}\,(m_1+m_2)^{\frac{1}{2}}}{(hm_1m_2)^{\frac{1}{2}p}\,\Gamma(\frac{5}{2})} \int_0^\infty a\,da$$

$$\times[(\tfrac{1}{2}p+\tfrac{3}{2})\{1-\mu_2(s+t)+(\mu_2-\mu_1)\,st\}^{\frac{1}{2}p}\{(1-st)^{-(\frac{1}{2}p+\frac{5}{2})}-(\mu_1+\mu_2\cos\chi)$$

$$\times(1-st+4\mu_1\mu_2\,st\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{5}{2})}\}$$

$$-\mu_1\tfrac{1}{2}p\{1-\mu_2(s+t)+(\mu_2-\mu_1)\,st\}^{\frac{1}{2}p-1}$$

$$\times\{(1-st)^{-(\frac{1}{2}p+\frac{3}{2})}-(1-st+4\mu_1\mu_2\,st\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{3}{2})}\}]$$

$$+2\pi{\nu_1}^2 \frac{\Gamma(\frac{1}{2}p_1+\frac{3}{2})\,K_1^{-\frac{1}{2}(p_1-1)}\,(2m_1)^{\frac{1}{2}}}{(hm_1^2)^{\frac{1}{2}p}\,\Gamma(\frac{5}{2})} \int_0^\infty a\,da$$

$$\times[(\tfrac{1}{2}p_1+\tfrac{3}{2})\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p_1}\{(1-st)^{-(\frac{1}{2}p_1+\frac{5}{2})}-\cos^2\tfrac{1}{2}\chi_1(1-st\cos^2\tfrac{1}{2}\chi_1)^{-(\frac{1}{2}p_1+\frac{5}{2})}$$

$$-\sin^2\tfrac{1}{2}\chi_1(1-st\sin^2\tfrac{1}{2}\chi_1)^{-(\frac{1}{2}p_1+\frac{5}{2})}\}$$

$$-\tfrac{1}{4}p_1\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p_1-1}\{1+(1-st)^{-(\frac{1}{2}p_1+\frac{3}{2})}-(1-st\cos^2\tfrac{1}{2}\chi_1)^{-(\frac{1}{2}p_1+\frac{3}{2})}$$

$$-(1-st\sin^2\tfrac{1}{2}\chi_1)^{-(\frac{1}{2}p_1+\frac{3}{2})}\}], \quad (2.335)$$

and C_{kn} is the coefficient of $s^k t^n$ in

$$2\pi\nu_1\nu_2 \frac{\Gamma(\frac{1}{2}p+\frac{3}{2})\,K^{-\frac{1}{2}(p-1)}(m_1+m_2)^{\frac{1}{2}}}{(hm_1m_2)^{\frac{1}{2}p}\,\Gamma(\frac{5}{2})}\,\mu_1 \int_0^\infty a\,da$$

$$\times[(\tfrac{1}{2}p+\tfrac{3}{2})\{1-\mu_1 s-\mu_2 t\}^{\frac{1}{2}p}\,2\sin^2\tfrac{1}{2}\chi(1-4\mu_1\mu_2\,st\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{5}{2})}$$

$$+\tfrac{1}{2}p\{1-\mu_1 s-\mu_2 t\}^{\frac{1}{2}p-1}\{1-(1-4\mu_1\mu_2\,st\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{3}{2})}\}]. \quad (2.336)$$

The corresponding equations obtained by considering the Boltzmann equation for the second type of molecule differ from (2.332), (2.333), (2.334) only by an interchange of the suffixes 1 and 2; this, by (2.324), changes the sign of a_x so that

$$-a_x = \sum_k (C'_{k0}\,b_k - B'_{k0}\beta_k), \quad (2.337)$$

$$\tfrac{5}{2}\frac{\nu_2}{h}\frac{\partial h}{\partial x} = \sum_k (C'_{k1}\,b_k - B'_{k1}\beta_k), \quad (2.338)$$

$$0 = \sum_k (C'_{kn}\,b_k - B'_{kn}\beta_k) \quad (n>1), \quad (2.339)$$

where B'_{kn}, C'_{kn} differ from B_{kn}, C_{kn} only by an interchange of the suffixes 1 and 2.

Now, from (2.335) and (2.336), B_{k0} and C_{k0} are the coefficients of s^k in

$$2\pi\nu_1\nu_2\frac{\Gamma(\frac{1}{2}p+\frac{3}{2})K^{-\frac{1}{2}(p-1)}(m_1+m_2)^{\frac{1}{2}}}{(hm_1m_2)^{\frac{1}{2}p}\Gamma(\frac{5}{2})}\frac{p+3}{2}\int_0^\infty a\,da \times\{1-(\mu_1+\mu_2\cos\chi)\}(1-\mu_2 s)^{\frac{1}{2}p}$$

$$=2\pi\nu_1\nu_2\frac{\Gamma(\frac{1}{2}p+\frac{3}{2})K^{-\frac{1}{2}(p-1)}(m_1+m_2)^{\frac{1}{2}}}{(hm_1m_2)^{\frac{1}{2}p}\Gamma(\frac{5}{2})}\frac{p+3}{2}\,2\mu_2(1-\mu_2 s)^{\frac{1}{2}p}\int_0^\infty a\sin^2\tfrac{1}{2}\chi\,da$$

and

$$2\pi\nu_1\nu_2\frac{\Gamma(\frac{1}{2}p+\frac{3}{2})K^{-\frac{1}{2}(p-1)}(m_1+m_2)^{\frac{1}{2}}}{(hm_1m_2)^{\frac{1}{2}p}\Gamma(\frac{5}{2})}\frac{p+3}{2}\,2\mu_1(1-\mu_1 s)^{\frac{1}{2}p}\int_0^\infty a\sin^2\tfrac{1}{2}\chi\,da,$$

respectively, from which it is clear that $B_{k0}=C'_{k0}$, $B'_{k0}=C_{k0}$. Hence equations (2.332) and (2.337) are identical, but (2.332)–(2.334), (2.338), (2.339) taken along with (2.331) are sufficient to determine b_k and β_k for all values of k. They may be modified by putting

$$b_k=a_x c_k+\tfrac{5}{2}\frac{\nu_0}{h}\frac{\partial h}{\partial x}d_k,\quad \beta_k=-a_x\gamma_k-\tfrac{5}{2}\frac{\nu_0}{h}\frac{\partial h}{\partial x}\delta_k, \tag{2.341}$$

where $\nu_0=\nu_1+\nu_2$ and [*cf.* (2.331)]

$$\frac{\nu_1 c_0}{m_1}-\frac{\nu_2\gamma_0}{m_2}=0, \tag{2.342}$$

$$\frac{\nu_1 d_0}{m_1}-\frac{\nu_2\delta_0}{m_2}=0. \tag{2.343}$$

The new coefficients c_k, γ_k, d_k, δ_k are determined by (2.342) and (2.343), together with

$$\Sigma(c_k B_{k0}+\gamma_k C_{k0})=-1, \tag{2.344}$$

$$\Sigma(c_k B_{kn}+\gamma_k C_{kn})=0,\quad \Sigma(c_k C'_{kn}+\gamma_k B'_{kn})=0\quad (n>0), \tag{2.345}$$

$$\Sigma(d_k B_{k0}+\delta_k C_{k0})=0, \tag{2.346}$$

$$\Sigma(d_k B_{k1}+\delta_k C_{k1})=-\frac{\nu_1}{\nu_0},\quad \Sigma(dk\,C'_{k1}+\delta_k B'_{k1})=\frac{\nu_2}{\nu_0}, \tag{2.347}$$

$$\Sigma(d_k B_{kn}+\delta_k C_{kn})=0,\quad \Sigma(d_k C'_{kn}+\delta_k B'_{kn})=0\quad (n>1). \tag{2.348}$$

In the particular case of Maxwellian molecules of different mass, p, p_1, and p_2 are all zero and B_{kn}, C_{kn}, B'_{kn}, C'_{kn} vanish unless $k=n$; hence, if $k>0$, c_k and γ_k are determined by the equations

$$B_{kk}c_k+C_{kk}\gamma_k=0,$$

$$C'_{kk}c_k+B'_{kk}\gamma_k=0,$$

and so $c_k = \gamma_k = 0$, $k > 0$; c_0 and γ_0 are given by

$$B_{00} c_0 + C_{00} \gamma_0 = -1,$$

$$\frac{\nu_1}{m_1} c_0 - \frac{\nu_2}{m_2} \gamma_0 = 0,$$

so that

$$c_0 = -\frac{\nu_2 m_1}{\nu_1 m_2 C_{00} + \nu_2 m_1 B_{00}}, \quad \gamma_0 = -\frac{\nu_1 m_2}{\nu_1 m_2 C_{00} + \nu_2 m_1 B_{00}}. \tag{2.351}$$

Similarly, $d_k = \delta_k = 0$ unless $k = 1$, in which case

$$B_{11} d_1 + C_{11} \delta_1 = -\frac{\nu_1}{\nu_0}, \quad C'_{11} d_1 + B'_{11} \delta_1 = \frac{\nu_2}{\nu_0},$$

and so

$$d_1 = \frac{\nu_1 B'_{11} + \nu_2 C_{11}}{\nu_0 (C_{11} C'_{11} - B_{11} B'_{11})}, \quad \delta_1 = \frac{\nu_1 C'_{11} + \nu_2 B_{11}}{\nu_0 (C_{11} C'_{11} - B_{11} B'_{11})}, \tag{2.352}$$

where
$$B_{00} = 4\pi \nu_1 \nu_2 K^{\frac{1}{2}} (m_1 + m_2)^{\frac{1}{2}} \mu_2 \int_0^\infty \alpha \sin^2 \tfrac{1}{2}\chi \, d\alpha$$

$$= \nu_1 \nu_2 m_2 \left(\frac{K}{m_1 + m_2}\right)^{\frac{1}{2}} I_1, \tag{2.353}$$

in Maxwell's notation,

$$C_{00} = \nu_1 \nu_2 m_1 \left(\frac{K}{m_1 + m_2}\right)^{\frac{1}{2}} I_1, \tag{2.354}$$

$$B_{11} = 5\pi \nu_1 \nu_2 \{K(m_1 + m_2)\}^{\frac{1}{2}} \int_0^\infty \alpha \, d\alpha \{1 - (\mu_1 + \mu_2 \cos\chi)(1 - 4\mu_1 \mu_2 \sin^2 \tfrac{1}{2}\chi)\}$$

$$+ 5\pi {\nu_1}^2 \{2K_1 m_1\}^{\frac{1}{2}} \int_0^\infty \alpha \, d\alpha \{1 - \cos^4 \tfrac{1}{2}\chi_1 - \sin^4 \tfrac{1}{2}\chi_1\}.$$

If the law of force is the same for all pairs of molecules, so that $K_1 = K$, $\chi_1 = \chi$,

$$B_{11} = 5\pi \nu_1 \nu_2 \{K(m_1 + m_2)\}^{\frac{1}{2}} \int_0^\infty \alpha \, d\alpha \{2\mu_2 (1 + 2\mu_1 - 4\mu_1 \mu_2) \sin^2 \tfrac{1}{2}\chi$$

$$+ 2\mu_1 {\mu_2}^2 \sin^2 \chi\} + 5\pi {\nu_1}^2 \{2Km_1\}^{\frac{1}{2}} \int_0^\infty \alpha \, d\alpha \tfrac{1}{2} \sin^2 \chi$$

$$= \tfrac{5}{2} \nu_1 \nu_2 m_2 \left\{\frac{K}{m_1 + m_2}\right\}^{\frac{1}{2}} \{(1 + 2\mu_1 - 4\mu_1 \mu_2) I_1 + 4\mu_1 \mu_2 I_2\}$$

$$+ \tfrac{5}{2} {\nu_1}^2 \{2Km_1\}^{\frac{1}{2}} I_2, \tag{2.355}$$

and
$$C_{11} = 10 \nu_1 \nu_2 m_2 \left\{\frac{K}{m_1 + m_2}\right\}^{\frac{1}{2}} {\mu_1}^2 (I_1 - I_2). \tag{2.356}$$

The discussion of the general properties of the coefficients on the lines of (2.26) requires the investigation of the signs of B_{kn}, C_{kn}, ..., and this seems to be somewhat long and intricate; the following results appear to hold:

(i) every c_k and every γ_k is positive and of order lower than $k^{-\frac{5}{2}}$ when k is large*:

(ii) c_0+d_0 and $\gamma_0+\delta_0$† are negative; c_k+d_k, $\gamma_k+\delta_k$ $(k>0)$ are positive and of lower order than $k^{-\frac{5}{2}}$ when k is large, so that d_k and δ_k are also positive when k is greater than zero and of lower order than $k^{-\frac{5}{2}}$ when k is large*.

The convergence of the series for the distribution functions now follows from §1.5.

Lorentz case: $m_2 \ll m_1$.

Although the direct solution of Boltzman's equation is quite simple in this case‡, it is of some interest to verify that it is also obtained as the limiting form of the general series. If we regard the molecules as rigid elastic spheres of masses m_1, m_2 and radii σ_1, σ_2, where $m_2 \ll m_1$, $\sigma_2 \ll \sigma_1$, the first approximations to the constants B_{kn}, C_{kn}, B'_{kn}, C'_{kn} are the coefficients of $s^k t^n$ in the following expressions:

for B_{kn}:

$$\frac{\pi\nu_1\nu_2{\sigma_1}^2}{(hm_2)^{\frac{1}{2}}\Gamma(\frac{5}{2})}\int_0^\pi \sin\chi\, d\chi\{\text{terms containing } \mu_2 \text{ as a factor}\}$$

$$+\frac{4\pi{\nu_1}^2{\sigma_1}^2}{\Gamma(\frac{5}{2})}\left(\frac{2}{hm_1}\right)^{\frac{1}{2}}\int_0^\pi \sin\chi\, d\chi\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}}$$

$$\times\{(1-st)^{-3}-\cos^2\tfrac{1}{2}\chi(1-st\cos^2\tfrac{1}{2}\chi)^{-3}-\sin^2\tfrac{1}{2}\chi(1-st\sin^2\tfrac{1}{2}\chi)^{-3})\}$$

$$-\tfrac{1}{8}\{1-\tfrac{1}{2}(s+t)\}^{-\frac{1}{2}}\{1+(1-st)^{-2}-(1-st\cos^2\tfrac{1}{2}\chi)^{-2}-(1-st\sin^2\tfrac{1}{2}\chi)^{-2}\}$$

$$=4{\nu_1}^2{\sigma_1}^2\left(\frac{2\pi}{hm_1}\right)^{\frac{1}{2}}\left[\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}}\frac{st}{(1-st)^3}-\tfrac{1}{4}\{1-\tfrac{1}{2}(s+t)\}^{-\frac{1}{2}}\frac{s^2t^2}{(1-st)^2}\right], \quad (2.371)$$

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* The masses m_1 and m_2 are supposed to be in a finite ratio; these results require modification in the Lorentz case, see below.

† It is found convenient to replace equations (2.346), (2.347), (2.348) by

$$\Sigma(c_k+d_k)B_{k0}+\Sigma(\gamma_k+\delta_k)C_{k0}=-1,$$

$$\Sigma(c_k+d_k)\left(B_{k0}-\frac{\nu_0}{\nu_1}B_{kn}\right)+\Sigma(\gamma_k+\delta_k)\left(C_{k0}-\frac{\nu_0}{\nu_1}C_{kn}\right)=0 \quad (n>0), \quad \ldots$$

which are of the standard type in which the constant term is zero in every equation except the first.

‡ *Cf.* Lorentz, *Theory of electrons*, 266; Jeans, *loc. cit.*, ch. 8; Burnett, *Proc. Camb. Phil. Soc.*, 27 (1931), 578.

as in (2.271), the first line being neglected since it contains a factor $\mu_2^{\frac{1}{2}}$;

for C_{kn}:

$$\frac{\pi\nu_1\nu_2{\sigma_1}^2}{(hm_2)^{\frac{1}{2}}\Gamma(\frac{5}{2})}\int_0^\pi \sin\chi\, d\chi(1-s)^{\frac{1}{2}}2\sin^2\tfrac{1}{2}\chi = \tfrac{8}{3}\nu_1\nu_2{\sigma_1}^2\left(\frac{\pi}{hm_2}\right)^{\frac{1}{2}}(1-s)^{\frac{1}{2}}; \quad (2.372)$$

for B'_{kn}:

$$\frac{\pi\nu_1\nu_2{\sigma_1}^2}{(hm_2)^{\frac{1}{2}}\Gamma(\frac{5}{2})}\int_0^\pi \sin\chi\, d\chi\{1-s-t+st\}^{\frac{1}{2}}(1-st)^{-3}(1-\cos\chi)$$
$$= \tfrac{8}{3}\nu_1\nu_2{\sigma_1}^2\left(\frac{\pi}{hm_2}\right)^{\frac{1}{2}}(1-s)^{\frac{1}{2}}(1-t)^{\frac{1}{2}}(1-st)^{-3}; \quad (2.373)$$

for C'_{kn}: terms containing $\mu_2^{\frac{1}{2}}$ as a factor. (2.374)

From (2.371) it follows that $B_{k0}=0$ for all values of k, and from (2.374) that $C'_{kn}=0$ for all values of k and n, to the order of approximation adopted.

In order to obtain the result in the form given by Lorentz, it is convenient to modify slightly the definition of the mass motion of the gas relative to which the velocities of the molecules are measured; let it be taken as the mass motion of the heavy molecules, so that equation (2.331) is replaced by

$$b_0 = 0. \quad (2.375)$$

Equations (2.332)–(2.334), (2.337)–(2.339) now become

$$+a_x = \Sigma\, C_{k0}\beta_k = \Sigma\, B'_{k0}\beta_k, \quad (2.376)$$

$$-\tfrac{5}{2}\frac{\nu_1}{h}\frac{\partial h}{\partial x} = \Sigma\, B_{k1}b_k, \quad -\tfrac{5}{2}\frac{\nu_2}{h}\frac{\partial h}{\partial x} = \Sigma\, B'_{k1}\beta_k, \quad (2.377)$$

$$0 = \Sigma\, B_{kn}b_k, \quad 0 = \Sigma\, B'_{kn}\beta_k \quad (n>1). \quad (2.378)$$

The values of the b_k are therefore the same as in 2.26, *i.e.* they are unaffected by the presence of the light molecules; and the β_k are uniquely determined by the equations on the right in (2.376)–(2.378). Hence each β_k can be expressed as the sum of two ratios of infinite determinants as above, but they can also be obtained in a finite form as follows. From (2.373),

$$\sum_{k=0}^{\infty} B'_{kn}s^{k-\frac{1}{2}}$$
$$= \tfrac{8}{3}\nu_1\nu_2{\sigma_1}^2\left(\frac{\pi}{hm_2}\right)^{\frac{1}{2}} \times \text{coefficient of } t^n \text{ in } s^{-\frac{1}{2}}(1-s)^{\frac{1}{2}}(1-t)^{\frac{1}{2}}(1-st)^{-3},$$

and so

$$\sum_{k=0}^{\infty} \frac{B'_{kn}}{(k+\frac{1}{2})(k+\frac{3}{2})} = \tfrac{8}{3}\nu_1\nu_2{\sigma_1}^2\left(\frac{\pi}{hm_2}\right)^{\frac{1}{2}} \times \text{coefficient of } t^n \text{ in}$$

$$(1-t)^{\frac{1}{2}}\int_0^1 dx \int_0^x ds\, s^{-\frac{1}{2}}(1-s)^{\frac{1}{2}}(1-st)^{-3}.$$

The double integral is equal to

$$\int_0^1 ds \int_s^1 dx\, s^{-\frac{1}{2}}(1-s)^{\frac{1}{2}}(1-st)^{-3} = \int_0^1 ds\, s^{-\frac{1}{2}}(1-s)^{\frac{3}{2}}(1-st)^{-3}$$

$$= \frac{3\pi}{8}(1-t)^{-\frac{1}{2}},$$

as is easily seen by making the substitution $z = (1-s)/(1-st)$.

Hence

$$\sum_{k=0}^{\infty} \frac{B'_{kn}}{(k+\frac{1}{2})(k+\frac{3}{2})} = \pi\nu_1\nu_2{\sigma_1}^2\left(\frac{\pi}{hm_2}\right)^{\frac{1}{2}} \quad (n=0)$$

$$= 0 \quad (n>0).$$

Similarly it may be shown that

$$\sum_{k=1}^{\infty} \frac{B'_{kn}}{(k-\frac{1}{2})(k+\frac{1}{2})(k+\frac{3}{2})} = \tfrac{8}{3}\nu_1\nu_2{\sigma_1}^2\left(\frac{\pi}{hm_2}\right)^{\frac{1}{2}} \times \text{coefficient of } t^n \text{ in}$$

$$(1-t)^{\frac{1}{2}}\{\tfrac{8}{3} - \tfrac{15}{16}\pi(1-t)^{\frac{1}{2}}\},$$

and so

$$\sum_{k=0}^{\infty} \frac{B'_{kn}}{(k-\frac{1}{2})(k+\frac{1}{2})(k+\frac{3}{2})} = \tfrac{8}{3}\nu_1\nu_2{\sigma_1}^2\left(\frac{\pi}{hm_2}\right)^{\frac{1}{2}} \times \text{coefficient of } t^n \text{ in}$$

$$\{-\tfrac{15}{16}\pi(1-t)\},$$

i.e. it is

$$-\tfrac{5}{2}\pi\nu_1\nu_2{\sigma_1}^2\left(\frac{\pi}{hm_2}\right)^{\frac{1}{2}} \quad (n=0),$$

$$+\tfrac{5}{2}\pi\nu_1\nu_2{\sigma_1}^2\left(\frac{\pi}{hm_2}\right)^{\frac{1}{2}} \quad (n=1),$$

$$0 \quad (n>1).$$

Hence the equations (2.376)–(2.378) for the β_k are satisfied by

$$\beta_k = \frac{1}{(k+\frac{1}{2})(k+\frac{3}{2})} \frac{(hm_2)^{\frac{1}{2}}}{\pi^{\frac{3}{2}}\nu_1\nu_2{\sigma_1}^2} \left\{a_x - \tfrac{5}{2}\frac{\nu_2}{h}\frac{\partial h}{\partial x}\right\}$$

$$- \frac{1}{(k-\frac{1}{2})(k+\frac{1}{2})(k+\frac{3}{2})} \frac{(hm_2)^{\frac{1}{2}}}{\pi^{\frac{3}{2}}\nu_1\nu_2{\sigma_1}^2} \frac{\nu_2}{h}\frac{\partial h}{\partial x},$$

and the distribution function is

$$\nu_2\left(\frac{hm_2}{\pi}\right)^{\frac{3}{2}} \exp\left(-hm_2 C_2^2\right)\{1+U_2 \Sigma\, \Gamma(k+\tfrac{5}{2})\, \beta_k\, S_{\frac{3}{2}}^k(hm_2 C_2^2)\}$$

$$= \nu_2\left(\frac{hm_2}{\pi}\right)^{\frac{3}{2}} \exp\left(-hm_2 C_2^2\right)$$

$$\times \left\{1+U_2 \Sigma\, \Gamma(k+\tfrac{1}{2})\, S_{\frac{3}{2}}^k(hm_2 C_2^2)\, \frac{(hm_2)^{\frac{1}{2}}}{\pi^{\frac{3}{2}} \nu_1 \nu_2 \sigma_1^2}\left(a_x - \tfrac{5}{2}\, \frac{\nu_2}{h}\, \frac{\partial h}{\partial x}\right)\right.$$

$$\left. - U_2 \Sigma\, \Gamma(k-\tfrac{1}{2})\, S_{\frac{3}{2}}^k(hm_2 C_2^2)\, \frac{(hm_2)^{\frac{1}{2}}}{\pi^{\frac{3}{2}} \nu_1 \nu_2 \sigma_1^2}\, \frac{\nu_2}{h}\, \frac{\partial h}{\partial x}\right\}$$

$$= \nu_2\left(\frac{hm_2}{\pi}\right)^{\frac{3}{2}} \exp\left(-hm_2 C_2^2\right)\left[1+\frac{U_2}{\pi \nu_1 \nu_2 \sigma_1^2 C_2}\left\{a_x - (\tfrac{5}{2} - hm_2 C_2^2)\, \frac{\nu_2}{h}\, \frac{\partial h}{\partial x}\right\}\right],$$

from (1.61), (1.62). This result, allowing for the difference in notation, agrees with that given by Lorentz.

To determine the part of the distribution function in the general case which is of the second degree in U, V, W, assume it to be of the form

$$F_1 = \nu_1\left(\frac{hm_1}{\pi}\right)^{\frac{3}{2}} \exp\left(-hm_1 C_1^2\right)(c_{11} U_1^2 + c_{22} V_1^2 + c_{33} W_1^2 + c_{23} V_1 W_1$$
$$+ c_{31} W_1 U_1 + c_{12} U_1 V_1) \Sigma\, \Gamma(k+\tfrac{7}{2})\, b_k\, S_{\frac{5}{2}}^k(hm_1 C_1^2)$$

for the first type of molecule, and

$$F_2 = \nu_2\left(\frac{hm_2}{\pi}\right)^{\frac{3}{2}} \exp\left(-hm_2 C_2^2\right)(c_{11} U_2^2 + c_{22} V_2^2 + c_{33} W_2^2 + c_{23} V_2 W_2$$
$$+ c_{31} W_2 U_2 + c_{12} U_2 V_2) \Sigma\, \Gamma(k+\tfrac{7}{2})\, \beta_k\, S_{\frac{5}{2}}^k(hm_2 C_2^2)$$

for the second type. The necessary conditions

$$\iiint U F\, dU\, dV\, dW = \iiint V F\, dU\, dV\, dW = \iiint W F\, dU\, dV\, dW = 0$$

are evidently satisfied for each type of molecule; and so are

$$\iiint F\, dU\, dV\, dW = 0, \quad \iiint C^2 F\, dU\, dV\, dW = 0,$$

in virtue of the relation

$$c_{11} + c_{22} + c_{23} = 0 \quad (cf.\ 2.13).$$

By equating corresponding coefficients in Boltzmann's equation (2.32), (2.325), and using (1.41), (1.43), (1.45), (1.46), we find that

$$\Sigma(B_{k0} b_k - C'_{k0} \beta_k) = -1,$$
$$\Sigma(B_{kn} b_k - C'_{kn} \beta_k) = 0 \quad (n > 0), \qquad (2.381)$$

where B_{kn} is the coefficient of $s^k t^n$ in

$$\frac{3\pi\nu_2}{hm_1}\frac{K^{-\frac{1}{2}(p-1)}(m_1+m_2)^{\frac{1}{2}}}{(hm_1m_2)^{\frac{1}{2}p}}\frac{\Gamma(\frac{1}{2}p+\frac{3}{2})}{\Gamma(\frac{7}{2})}$$

$$\times\int_0^\infty a\,da(\tfrac{1}{2}p+\tfrac{3}{2})(\tfrac{1}{2}p+\tfrac{5}{2})\{1-\mu_2(s+t)+(\mu_2-\mu_1)st\}^{\frac{1}{2}p}[(1-st)^{-(\frac{1}{2}p+\frac{7}{2})}$$

$$-\{(\mu_1+\mu_2\cos\chi)^2-2\mu_2{}^2\sin^2\tfrac{1}{2}\chi\cos^2\tfrac{1}{2}\chi\}\{1-st+4\mu_1\mu_2 st\sin^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{7}{2})}]$$

$$-\mu_1 p(\tfrac{1}{2}p+\tfrac{3}{2})\{1-\mu_2(s+t)+(\mu_2-\mu_1)st\}^{\frac{1}{2}p-1}$$

$$\times[(1-st)^{-(\frac{1}{2}p+\frac{5}{2})}-(\mu_1+\mu_2\cos\chi)\{1-st+4\mu_1\mu_2 st\sin^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{5}{2})}]$$

$$+\mu_1{}^2\tfrac{1}{2}p(\tfrac{1}{2}p-1)\{1-\mu_2(s+t)+(\mu_2-\mu_1)st\}^{\frac{1}{2}p-2}$$

$$\times[(1-st)^{-(\frac{1}{2}p+\frac{3}{2})}-\{1-st+4\mu_1\mu_2 st\sin^2\tfrac{1}{2}\chi\}^{-(\frac{1}{2}p+\frac{3}{2})}]$$

$$+\frac{3\pi\nu_1}{hm_1}\frac{K_1^{-\frac{1}{2}(p_1-1)}(2m_1)^{\frac{1}{2}}}{(hm_1{}^2)^{\frac{1}{2}p_1}}\frac{\Gamma(\frac{1}{2}p_1+\frac{3}{2})}{\Gamma(\frac{7}{2})}$$

$$\times\int_0^\infty a\,da(\tfrac{1}{2}p_1+\tfrac{3}{2})(\tfrac{1}{2}p_1+\tfrac{5}{2})\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p_1}$$

$$\times[(1-st)^{-(\frac{1}{2}p_1+\frac{7}{2})}-\cos^2\tfrac{1}{2}\chi_1(\cos^2\tfrac{1}{2}\chi_1-\tfrac{1}{2}\sin^2\tfrac{1}{2}\chi_1)(1-st\cos^2\tfrac{1}{2}\chi_1)^{-(\frac{1}{2}p_1+\frac{7}{2})}$$

$$-\sin^2\tfrac{1}{2}\chi_1(\sin^2\tfrac{1}{2}\chi_1-\tfrac{1}{2}\cos^2\tfrac{1}{2}\chi_1)(1-st\sin^2\tfrac{1}{2}\chi_1)^{-(\frac{1}{2}p_1+\frac{7}{2})}]$$

$$-\tfrac{1}{2}p_1(\tfrac{1}{2}p_1+\tfrac{3}{2})\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p_1-1}[(1-st)^{-(\frac{1}{2}p_1+\frac{5}{2})}$$

$$-\cos^2\tfrac{1}{2}\chi_1(1-st\cos^2\tfrac{1}{2}\chi_1)^{-(\frac{1}{2}p_1+\frac{5}{2})}-\sin^2\tfrac{1}{2}\chi_1(1-st\sin^2\tfrac{1}{2}\chi_1)^{-(\frac{1}{2}p_1+\frac{5}{2})}]$$

$$+\tfrac{1}{8}p_1(\tfrac{1}{2}p_1-1)\{1-\tfrac{1}{2}(s+t)\}^{\frac{1}{2}p_1-2}[1+(1-st)^{-(\frac{1}{2}p_1+\frac{3}{2})}$$

$$-(1-st\cos^2\tfrac{1}{2}\chi_1)^{-(\frac{1}{2}p_1+\frac{3}{2})}-(1-st\sin^2\tfrac{1}{2}\chi_1)^{-(\frac{1}{2}p_1+\frac{3}{2})}],$$

the quantities K_1, p_1, χ_1 being distinguished by suffixes to take account of the possibility of a law of force between two molecules of type 1 different from the law of force between molecules of different types; and C_{kn} is the coefficient of $s^k t^n$ in

$$\frac{3\pi\nu_2}{hm_1}\frac{K^{-\frac{1}{2}(p-1)}(m_1+m_2)^{\frac{1}{2}}}{(hm_1m_2)^{\frac{1}{2}p}}\frac{\Gamma(\frac{1}{2}p+\frac{3}{2})}{\Gamma(\frac{7}{2})}\mu_1{}^2\int_0^\infty a\,da$$

$$\times[(\tfrac{1}{2}p+\tfrac{3}{2})(\tfrac{1}{2}p+\tfrac{5}{2})\{4\sin^4\tfrac{1}{2}\chi-2\sin^2\tfrac{1}{2}\chi\cos^2\tfrac{1}{2}\chi\}$$

$$\times(1-\mu_1 s-\mu_2 t)^{\frac{1}{2}p}(1-4\mu_1\mu_2 st\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{7}{2})}$$

$$-p(p+3)\sin^2\tfrac{1}{2}\chi(1-\mu_1 s-\mu_2 t)^{\frac{1}{2}p-1}(1-4\mu_1\mu_2 st\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{5}{2})}$$

$$-\tfrac{1}{2}p(\tfrac{1}{2}p-1)(1-\mu_1 s-\mu_2 t)^{\frac{1}{2}p-2}\{1-(1-4\mu_1\mu_2 st\sin^2\tfrac{1}{2}\chi)^{-(\frac{1}{2}p+\frac{3}{2})}\}].$$

The corresponding integral equation for the second type of molecule gives similarly

$$\Sigma(C'_{k0}\, b_k - B'_{k0}\, \beta_k) = 1,$$

$$\Sigma(C'_{kn}\, b_k - B'_{kn}\, \beta_k) = 0 \quad (n > 0), \tag{2.382}$$

where B'_{kn} and C'_{kn} are got from B_{kn} and C_{kn} by interchanging the suffixes 1 and 2, and these equations together with (2.381) are sufficient to determine the b_k and the β_k.

The discussion of the specially simple cases treated by Maxwell and Lorentz follows the same lines as before, but they present no unusual features and are therefore not reproduced here.

Appendix.

The determinant which occurs in (2.261), ... is of Sylvester's unisignant type and is known to be positive*. The second property, viz. that every cofactor of such a determinant is also positive, I have not been able to find in Muir's writings, and I therefore prove here in a more general form for the determinant

$$\begin{vmatrix} A_{11} & -A_{12} & -A_{13} & \dots & -A_{1n} \\ -A_{21} & A_{22} & -A_{23} & \dots & -A_{2n} \\ \vdots & & & & \\ -A_{n1} & -A_{n2} & \dots & \dots & A_{nn} \end{vmatrix}, \tag{A}$$

where $A_{11}, A_{12}, \dots$ are all positive and such that the sum of the elements in each row (or each column, but not necessarily both) is positive. It is not necessary to assume axial symmetry. For a determinant of this type of order 2 or 3, it is readily seen that the determinant and all its cofactors are positive; assume that this is true for such determinants of order $n-1$ and $n-2$ and consider the determinant (A) of order n. It is equal to

$$\begin{vmatrix} S_1 & -A_{12} & -A_{13} & \dots & -A_{1n} \\ S_2 & A_{22} & -A_{23} & \dots & -A_{2n} \\ \vdots & & & & \\ S_n & -A_{n2} & -A_{n3} & \dots & A_{nn} \end{vmatrix}$$

$$= S_1 \Delta_1 + A_{12} S_2 \Delta_{22} - A_{13} S_2 \Delta_{23} - A_{12} S_3 \Delta_{32} \dots,$$

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* *Cf.* Muir, *Trans. Roy. Soc. South Africa*, 2 (1911), 187, or his *Contributions to the theory of determinants*, 1900–1920 (1930).

where S_k is the sum of the elements in the k-th row and is supposed positive; Δ_1 is the principal first minor and is a determinant of type (A) of order $n-1$, so that it is positive by hypothesis; Δ_{kl} is the second minor of (A) obtained by suppressing the first and k-th rows and the first and l-th columns, and is therefore a first minor of Δ_1 and its sign, by hypothesis, is that of $(-1)^{k+l}$. Hence each term of the above expansion is positive if the assumption concerning the determinant of order $n-1$ is correct.

The cofactor of $-A_{kl}$ may be obtained by writing (A) as

$$-\begin{vmatrix} -A_{kl} & A_{kk} & -A_{k1} & -A_{k2} & \cdots \\ A_{ll} & -A_{lk} & -A_{l1} & -A_{l2} & \cdots \\ -A_{1l} & -A_{1k} & A_{11} & -A_{12} & \cdots \\ -A_{2l} & -A_{2k} & -A_{21} & A_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

assuming that k and l are unequal and both greater than 2, but the modifications required in these cases are obvious. The required cofactor can therefore be expanded in the form

$$A_{lk}\Delta + A_{l1}A_{1k}\Delta_{11} - A_{l1}\Delta_{2k}\Delta_{12}\ldots,$$

where Δ is the determinant of order $n-2$, and Δ_{11}, Δ_{12}, ... are its first minors, and as before each term in this expression is positive by hypothesis. It follows by induction that the determinant of order n and all its cofactors are positive.

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ON THE LATTICE POINTS ON CURVES OF GENUS 1

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Suppose that

$$F(x, y) = a_0 x^3 + a_1 x^2 y + a_2 x y^2 + a_3 y^3$$

is a cubic binary form with integer coefficients, which is irreducible in the field of all rational numbers, and that $k \neq 0$ is an integer. A special case of Thue's famous theorem states that the equation

$$F(x, y) = k$$

has only a finite number $A(k)$ of solutions in integers x, y; more recent researches of the author have proved that this number $A(k)$ can be large only if the integer k is the product of a great number of equal or different prime factors. A result due to C. L. Siegel includes the inequality

$$\sum_{h=1}^{k} A(h) = O(k^{\frac{3}{4}}),$$

and therefore $A(k)$ must be zero for nearly all integers k.

Until recently, there did not seem to exist theorems in the other direction, *i.e.* whether the number $A(k)$ can, in fact, be very large for suitably chosen integers k, or whether $A(k)$ is absolutely bounded. I shall prove in this paper that $A(k)$ is greater than any given number t for certain integers k, and that there is even an infinity of integers k for which

$$A(k) \geqslant \sqrt[4]{\log k}.$$

I cannot prove similar results for the "primitive" solutions of $F(x, y) = k$, *i.e.* those solutions x, y for which x and y are coprime integers;

† This paper is an extension of one previously contributed to a Festschrift for Prof. Hellinger.

whether their number is bounded is still an open question, and I doubt whether it can be solved by a method similar to that of this paper.

The cubic equation $F(x, y) = k$ defines a cubic curve of genus 1 in the (x, y)-plane, and so such curves exist on which more than t lattice points lie, where t is an arbitrary integer. Now these cubic curves $F(x, y) = k$ are not the most general ones, since their invariant g_2 is zero. A special case of a theorem of Siegel shows that on every cubic curve of genus 1 which is defined by an equation with rational coefficients, there are only a finite number of lattice points. Two cubic curves which have the same absolute invariant J can be transformed into one another by a birational transformation, and this transformation has rational coefficients when there are points with rational coordinates on both curves; also the systems of these rational points are changed into each other by the transformation and therefore are invariants. It is interesting to observe that a corresponding result for lattice points on these curves does not exist, for I shall prove in this paper that, for any given integer t and for any given rational value of the invariant J, there exists a cubic curve defined by an equation with rational coefficients on which lie more than t lattice points.

The method of the text is useful for the study of all curves of genus 1 in two or more dimensions; I give a number of results obtained by it, which perhaps are not all new. It may be remarked that curves of genus 0 with only a finite number of lattice points may be treated also in this way; the reader will not find it difficult to establish, for example, the existence of rational numbers a such that there are at least t lattice points on the lemniscate

$$(x^2+y^2)^2+a(x^2-y^2) = 0,$$

where t is an arbitrarily large integer.

I.

1. Suppose that

$$F(x, y) = a_0 x^3+a_1 x^2 y+a_2 xy^2+a_3 y^3$$

is a cubic binary form with integer coefficients and with only simple linear factors. The equation

$$F(x, y) = 1$$

defines a cubic curve C without double points and therefore of genus 1 in the (x, y)-plane. Its three asymptotes may be denoted by Γ', Γ'', Γ'''.

If (x_0, y_0) is any point with integer coordinates not lying on Γ', Γ'', Γ''', and

$$F(x_0, y_0) = k,$$

then k is an integer and $k \neq 0$; there is one and only one curve

$$F(x, y) = k,$$

say $C(k)$, which goes through (x_0, y_0) and is similar and similarly situated to the curve C relative to the origin.

2. Since C is of genus 1, a uniformisation of C,

$$x = \phi(u), \quad y = \psi(u),$$

exists; here $\phi(u)$ and $\psi(u)$ are two elliptic functions of u of order 2 or 3 and with periods ω_1 and ω_2, say. Instead of the point (x, y) on C, we shall speak also of the point u, where u is the elliptic argument of (x, y). Then u and u' are the same point if and only if

$$u \equiv u' \pmod{\omega_1, \omega_2},$$

i.e. if

$$u = u' + h_1\omega_1 + h_2\omega_2$$

with two integers h_1 and h_2.

Now let us take any straight line Γ; it cuts C in three points u_1, u_2, u_3 which satisfy the congruence

$$u_1 + u_2 + u_3 \equiv C_0 \pmod{\omega_1, \omega_2}.$$

Here C_0 is a constant, not depending on Γ, which may be assumed, without loss of generality, to be zero; then the congruence takes the simpler form

$$u_1 + u_2 + u_3 \equiv 0 \pmod{\omega_1, \omega_2}.$$

If, in particular, Γ is a tangent with u_1 its point of contact and u_2 its other point of intersection with C, *i.e.* if u_2 is the tangential of u_1, then

$$2u_1 + u_2 \equiv 0 \pmod{\omega_1, \omega_2}.$$

3. Beginning with any point u on C we construct an infinite set U of points

$$u_1 = u, \ u_{-2}, \ u_4, \ u_{-5}, \ u_7, \ u_{-8}, \ \ldots, \ u_{3m+1}, \ u_{-3m-2}, \ \ldots$$

on C, given by their elliptic arguments

$$u_1 = u, \quad u_{-2} = -2u, \quad u_4 = 4u, \quad \ldots,$$

$$u_{3m+1} = (3m+1)\,u, \quad u_{-3m-2} = -(3m+2)\,u, \quad \ldots.$$

Since

$$2u_1 + u_{-2} \equiv 0 \pmod{\omega_1, \omega_2} \quad \text{and} \quad 2u_{-2} + u_4 \equiv 0 \pmod{\omega_1, \omega_2},$$

u_{-2} is the point of intersection of C with the tangent to C at u, and u_4 is the point of intersection of C with the tangent to C at u_{-2}. Further, since

$$u_{-2} + u_{-3(m-1)-2} + u_{3m+1} \equiv 0 \pmod{\omega_1, \omega_2} \quad (m = 2, 3, \ldots)$$

and

$$u_1 + u_{3m+1} + u_{-(3m+2)} \equiv 0 \pmod{\omega_1, \omega_2} \quad (m = 1, 2, \ldots),$$

the point

$$u_{3m+1} \quad (m = 2, 3, \ldots)$$

is the point of intersection with C of the straight line through u_{-2} and u_{-3m+1}; and the point

$$u_{-(3m+2)} \quad (m = 1, 2, \ldots)$$

is the point of intersection with C of the line through u_1 and u_{3m+1}.

This construction makes use only of such properties of C as remain unaltered by collineations, and is therefore invariant when we apply a transformation of this kind. Now the transformation

$$x \to k^{\frac{1}{3}} x, \quad y \to k^{\frac{1}{3}} y$$

changes C into $C(k)$, and so changes the set U of points

$$u_{3m+1}, \; u_{-3m-2} \quad (m = 0, 1, 2, \ldots)$$

on C into a set V of points

$$v_{3m+1}, \; v_{-3m-2} \quad (m = 0, 1, 2, \ldots)$$

on $C(k)$, which has the same properties with regard to $C(k)$ as U has with regard to C. It is obvious that the line joining the point u_λ and the corresponding point v_λ passes through the origin.

4. It is possible that some of the points of U coincide. Let us consider the $2n$ points

$$u_{3m+1}, \; u_{-3m-2} \quad (m = 0, 1, 2, \ldots, n-1)$$

of this set. If they are not all different, then one of the congruences

$$(3m_1+1)u \equiv (3m_2+1)u \pmod{\omega_1, \omega_2},$$

$$(3m_1+1)u \equiv -(3m_2+2)u \pmod{\omega_1, \omega_2},$$

$$-(3m_1+2)u \equiv -(3m_2+2)u \pmod{\omega_1, \omega_2}$$

$$(m_1,\ m_2 = 0,\ 1,\ \ldots,\ n-1;\ m_1 \neq m_2)$$

must be satisfied, and therefore u is of the form

$$u \equiv \frac{h_1\omega_1 + h_2\omega_2}{3N} \pmod{\omega_1, \omega_2},$$

where the denominator $3N$ is one of the numbers

$$3N = 3,\ 6,\ 9,\ \ldots,\ 3(2n-1),$$

and h_1 and h_2 are any two integers. For every N, this gives only $9N^2$ different points u, and so there are at most

$$\sum_{N=1}^{2n-1} 9N^2 \leqslant (2n-1)\,.\,9(2n-1)^2 < 72n^3$$

different positions of u on C, say

$$u_1',\ u_1'',\ \ldots,\ u_1^{(\mu)} \quad (\mu \leqslant 72n^3),$$

such that the $2n$ first points

$$u_{3m+1},\ u_{-3m-2} \quad (m = 0,\ 1,\ \ldots,\ n-1)$$

are not all different from one another.

5. Since C is a cubic curve, it has three points u', u'', u''' at infinity. It is possible that one of the first $2n$ points

$$u_{3m+1},\ u_{-3m-2} \quad (m = 0,\ 1,\ \ldots,\ n-1)$$

of the set U coincides with u' or u'' or u'''. Then one of the congruences

$$(3m+1)\,u \equiv u^{(i)} \pmod{\omega_1, \omega_2} \quad (i = 1,\ 2,\ 3;\ m = 0,\ 1,\ \ldots,\ n-1),$$

or one of the congruences

$$-(3m+2)\,u \equiv u^{(i)} \pmod{\omega_1, \omega_2} \quad (i = 1,\ 2,\ 3;\ m = 0,\ 1,\ \ldots,\ n-1),$$

must be satisfied, and therefore u has one of the values

$$u \equiv \frac{u^{(i)}+h_1\omega_1+h_2\omega_2}{3m+1} \pmod{\omega_1, \omega_2} \quad (i=1, 2, 3)$$

or

$$u \equiv \frac{-u^{(i)}+h_1\omega_1+h_2\omega_2}{3m+2} \pmod{\omega_1, \omega_2} \quad (i=1, 2, 3),$$

where h_1 and h_2 are two integers. For every m this gives only

$$3.(3m+1)^2+3.(3m+2)^2 \leqq 6.9.(m+1)^2$$

different points u. Hence there are at most

$$\sum_{m=0}^{n-1} 6.9.(m+1)^2 \leqq n.54n^2 = 54n^3$$

different positions of u on C, say

$$u_2', u_2'', \ldots, u_2^{(\nu)} \quad (\nu \leqq 54n^3),$$

such that one of the $2n$ points

$$u_{3m+1}, u_{-3m-2} \quad (m=0, 1, \ldots, n-1)$$

of the set U lies at infinity.

6. Now we draw in the (x, y)-plane the three asymptotes

$$\Gamma', \Gamma'', \Gamma'''$$

of C, all the straight lines

$$\Gamma_1', \Gamma_1'', \ldots, \Gamma_1^{(\mu)}$$

which go through the origin $(0, 0)$ and one of the points

$$u_1', u_1'', \ldots, u_1^{(\mu)},$$

and all the straight lines

$$\Gamma_2', \Gamma_2'', \ldots, \Gamma_2^{(\nu)}$$

which go through the origin and one of the points

$$u_2', u_2'', \ldots, u_2^{(\nu)}.$$

The total number of these straight lines is not greater than

$$3+72n^3+54n^3 \leqq 130n^3.$$

But on every one of these lines Γ, Γ_1, Γ_2 there are at most

$$130n^3+1$$

lattice points which lie in the square Q defined by

$$\max(|x|, |y|) \leqslant 65n^3.$$

Hence the number of lattice points in Q, which lie on any one of the straight lines Γ, Γ_1, Γ_2, does not exceed

$$130n^3(130n^3+1),$$

and so is less than the number

$$(130n^3+1)^2$$

of all lattice points in Q. It follows that there must be a point

$$(x_1, y_1)$$

with integer coordinates satisfying the inequality

$$\max(|x_1|, |y_1|) \leqslant 65n^3,$$

which does *not* lie on any one of the straight lines

$$\Gamma', \Gamma'', \Gamma'''; \Gamma_1', \Gamma_1'', \ldots, \Gamma_1^{(\mu)}; \Gamma_2', \Gamma_2'', \ldots, \Gamma_2^{(\nu)}.$$

7. The number

$$F(x_1, y_1) = k_1$$

is an integer different from zero, since (x_1, y_1) does not lie on the asymptotes Γ_1, Γ_2, Γ_3. Hence the equation

$$F(x, y) = k_1$$

defines a cubic curve $C(k_1)$ which goes through (x_1, y_1). Denote by $k_1^{\frac{1}{3}}$ the real cube root of k_1. Then the transformation

$$x \to \frac{x}{k_1^{\frac{1}{3}}}, \quad y \to \frac{y}{k_1^{\frac{1}{3}}},$$

changes $C(k_1)$ into C and the point (x_1, y_1) on $C(k_1)$ into the point

$$u_1 = \left(\frac{x_1}{k_1^{\frac{1}{3}}}, \quad \frac{y_1}{k_1^{\frac{1}{3}}}\right)$$

on C. The three points

$$(0, 0),\ (x_1, y_1),\ u_1$$

lie on the same straight line, and so u_1 must be different from all the points

$$u_1',\ u_1'',\ \ldots,\ u_1^{(\mu)},\ u_2',\ u_2'',\ \ldots,\ u_2^{(\nu)},$$

since (x_1, y_1) does not lie on any one of the lines

$$\Gamma_1',\ \Gamma_1'',\ \ldots,\ \Gamma_1^{(\mu)},\ \Gamma_2',\ \Gamma_2'',\ \ldots,\ \Gamma_2^{(\nu)}.$$

Now construct the $2n$ points

$$u_{3m+1} = \left(\frac{x_{3m+1}}{k_1^{\frac{1}{3}}},\ \frac{y_{3m+1}}{k_1^{\frac{1}{3}}}\right),\quad u_{-3m-2} = \left(\frac{x_{-3m-2}}{k_1^{\frac{1}{3}}},\ \frac{y_{-3m-2}}{k_1^{\frac{1}{3}}}\right)\quad (m = 0, 1, \ldots, n-1)$$

of the set U belonging to u_1 on the curve C. Then we know that they are all different from one another and that none of them lies at infinity. The $2n$ corresponding points

$$(x_{3m+1},\ y_{3m+1}),\ (x_{-3m-2},\ y_{-3m-2})\quad (m = 0, 1, \ldots, n-1)$$

on $C(k_1)$ must therefore also be different from one another, and none of them can lie at infinity. Furthermore, when m_1, m_2, m_3 are three different members of the set of $2n$ indices $3m+1$, $-3m-2$ $(m = 0, 1, \ldots, n-1)$ with

$$m_1+m_2+m_3 = 0,$$

then the three points

$$(x_{m_1},\ y_{m_1}),\ (x_{m_2},\ y_{m_2}),\ (x_{m_3},\ y_{m_3})$$

on $C(k_1)$ lie on the same straight line, and when m_1 and m_2 are two different members of this set of indices with

$$2m_1+m_2 = 0,$$

then the straight line through the two points

$$(x_{m_1},\ y_{m_1}),\ (x_{m_2},\ y_{m_2})$$

is a tangent to $C(k_1)$ at $(x_{m_1},\ y_{m_1})$.

8. We make use of the abbreviations

$$F_1(x, y) = \frac{\partial}{\partial x} F(x, y),\quad F_2(x, y) = \frac{\partial}{\partial y} F(x, y).$$

$F_1(x, y)$ and $F_2(x, y)$ vanish simultaneously only at the point $x = y = 0$; for the binary form $F(x, y)$ has simple linear factors only, and its discriminant, *i.e.* the resultant of $F_1(x, y)$ and $F_2(x, y)$, cannot be zero.

Let (x''', y''') be the third point of intersection of the straight line through two points (x', y') and (x'', y'') on $C(k_1)$ with this curve. Then

$$F(x''', y''') = k_1,$$

and

$$\begin{vmatrix} x' & y' & 1 \\ x'' & y'' & 1 \\ x''' & y''' & 1 \end{vmatrix} = x'''(y'-y'')+y'''(x''-x')+(x'y''-x''y') = 0,$$

and therefore x''' and y''' must satisfy the cubic equations

$$F\{x'''(x'-x''),\ x'''(y'-y'')+(x'y''-x''y')\} = k_1(x'-x'')^3,$$

$$F\{y'''(x'-x'')-(x'y''-x''y'),\ y'''(y'-y'')\} = k_1(y'-y'')^3.$$

If we expand the left-hand sides in powers of x''' or y''', we get

$$F(x'-x'', y'-y'')x'''^3+F_2(x'-x'', y'-y'')(x'y''-x''y')x'''^2+*x''' +*1 = 0,$$

$$F(x'-x'', y'-y'')y'''^3-F_1(x'-x'', y'-y'')(x'y''-x''y')y'''^2+*y''' +*1 = 0,$$

where the asterisks denote the coefficients of x''' and 1, or of y''' and 1, which are of no importance.

Now we know two roots x' and x'', or y' and y'', of these cubic equations; their third roots therefore have the values

$$(A_1) \qquad \begin{cases} x''' = -x'-x''-\dfrac{(x'y''-x''y')\,F_2(x'-x'', y'-y'')}{F(x'-x'', y'-y'')}, \\ y''' = -y'-y''+\dfrac{(x'y''-x''y')\,F_1(x'-x'', y'-y'')}{F(x'-x'', y'-y'')}. \end{cases}$$

When we introduce homogeneous coordinates

$$x':y':1 = x^{(1)}:y^{(1)}:z^{(1)}, \quad x'':y'':1 = x^{(2)}:y^{(2)}:z^{(2)}, \quad x''':y''':1 = x^{(3)}:y^{(3)}:z^{(3)},$$

and use the abbreviations

$$(xy) = x^{(1)}y^{(2)}-x^{(2)}y^{(1)}, \quad (xz) = x^{(1)}z^{(2)}-x^{(2)}z^{(1)}, \quad (yz) = y^{(1)}z^{(2)}-y^{(2)}z^{(1)}$$

and

$$G_1(x^{(1)}y^{(1)}z^{(1)},\ x^{(2)}y^{(2)}z^{(2)})$$
$$= -(x^{(1)}z^{(2)}+x^{(2)}z^{(1)})\,F\{(xz),\ (yz)\}-z^{(1)}z^{(2)}(xy)\,F_2\{(xz),\ (yz)\},$$

$$G_2(x^{(1)}y^{(1)}z^{(1)},\ x^{(2)}y^{(2)}z^{(2)})$$
$$= -(y^{(1)}z^{(2)}+y^{(2)}z^{(1)})\,F\{(xz),\ (yz)\}-z^{(1)}z^{(2)}\,(xy)\,F_1\{(xz),\ (yz)\},$$

$$G_3(x^{(1)}y^{(1)}z^{(1)},\ x^{(2)}y^{(2)}z^{(2)}) = z^{(1)}z^{(2)}\,F\{(xz),\ (yz)\},$$

these formulae take the simple form

$$(\mathrm{A}_2)\qquad \begin{cases} x^{(3)} = G_1(x^{(1)}y^{(1)}z^{(1)},\ x^{(2)}y^{(2)}z^{(2)}), \\ y^{(3)} = G_2(x^{(1)}y^{(1)}z^{(1)},\ x^{(2)}y^{(2)}z^{(2)}), \\ z^{(3)} = G_3(x^{(1)}y^{(1)}z^{(1)},\ x^{(2)}y^{(2)}z^{(2)}). \end{cases}$$

The expressions G_1, G_2, G_3 on the right-hand side are forms in the six arguments $x^{(1)}$, $y^{(1)}$, $z^{(1)}$, $x^{(2)}$, $y^{(2)}$, $z^{(2)}$ with integer coefficients; they are of degree four in the coordinates $x^{(1)}$, $y^{(1)}$, $z^{(1)}$, and also in the coordinates $x^{(2)}$, $y^{(2)}$, $z^{(2)}$. It is obvious that, when $x^{(1)}$, $y^{(1)}$, $z^{(1)}$, $x^{(2)}$, $y^{(2)}$, $z^{(2)}$ have integral values, so also have $x^{(3)}$, $y^{(3)}$, $z^{(3)}$. If we write

$$w^{(i)} = \max(|x^{(i)}|,\ |y^{(i)}|,\ |z^{(i)}|) \quad (i = 1,\ 2,\ 3),$$

the formulae give the inequality

$$(\mathrm{A}_3)\qquad w^{(3)} \leqslant c_1(w^{(1)}w^{(2)})^4,$$

where c_1 is a positive constant depending only on the coefficients of the cubic form $F(x, y)$.

Assume, in particular, that none of the three points

$$\left(\frac{x^{(1)}}{z^{(1)}},\ \frac{y^{(1)}}{z^{(1)}}\right),\quad \left(\frac{x^{(2)}}{z^{(2)}},\ \frac{y^{(2)}}{z^{(2)}}\right),\quad \left(\frac{x^{(3)}}{z^{(3)}},\ \frac{y^{(3)}}{z^{(3)}}\right)$$

lies at infinity, that $(x^{(1)}/z^{(1)}, y^{(1)}/z^{(1)})$ and $(x^{(2)}/z^{(2)}, y^{(2)}/z^{(2)})$ do not coincide, and that $z^{(1)} \neq 0$ and $z^{(2)} \neq 0$. Then it is obvious that the straight line through these three points is not parallel to one of the asymptotes Γ', Γ'', Γ'''; hence the point

$$\left(\frac{x^{(1)}}{z^{(1)}}-\frac{x^{(2)}}{z^{(2)}},\ \frac{y^{(1)}}{z^{(1)}}-\frac{y^{(2)}}{z^{(2)}}\right)$$

does not lie on Γ' or Γ'' or Γ''', and so the number

$$F\Big((xz),\ (yz)\Big) = (z^{(1)}z^{(2)})^3\,F\left(\frac{x^{(1)}}{z^{(1)}}-\frac{x^{(2)}}{z^{(2)}},\ \frac{y^{(1)}}{z^{(1)}}-\frac{y^{(2)}}{z^{(2)}}\right)$$

does not vanish, and we also have

$$(A_4) \qquad z_3 = z^{(1)} z^{(2)} F\Big((xz), (yz)\Big) \neq 0.$$

9. By the method of § 8 we can calculate the coordinates (x''', y''') of the point of intersection of $C(k)$ with its tangent at the point (x', y'). The result is

$$(B_1) \qquad \begin{cases} x''' = -2x' - \dfrac{F_2(Y', -X')(x'X' + y'Y')}{F(Y', -X')}, \\ y''' = -2y' + \dfrac{F_1(Y', -X')(x'X' + y'Y')}{F(Y', -X')}, \\ \qquad\qquad [X' = F_1(x', y'),\ Y' = F_2(x', y')]. \end{cases}$$

We also arrive at these formulae by making (x'', y'') tend to (x', y') in the formulae (A_1), for then

$$(x' - x'') : (y' - y'') \to Y' : (-X').$$

It is useful to remark that the binary form of degree 6,

$$F(Y', -X'),$$

is divisible by the cubic form

$$x'X' + y'Y' = 3F(x', y').$$

To establish this result, we show that the first form vanishes when the second vanishes. This is trivial for $x' = y' = 0$. Therefore, let x' (or y') be different from zero; then

$$X' = -\frac{y'}{x'} Y' \quad \left(\text{or } Y' = -\frac{x'}{y'} X'\right),$$

and so

$$F(Y', -X') = F\left(Y', +\frac{y'}{x'} Y'\right) = \left(\frac{Y'}{x'}\right)^3 F(x', y') = 0$$

$$\left[\text{or } = F\left(-\frac{x'}{y'} X', -X'\right) = -\left(\frac{X'}{y'}\right)^3 F(x', y') = 0\right].$$

Hence $$F(Y', -X') = \frac{z}{a} F(x', y') F^*(x', y'),$$

where $F^*(x', y')$ is a certain cubic form with integer coefficients and a is a constant integer depending only on the coefficients of $F(x, y)$.

If we introduce homogeneous coordinates

$$x' : y' : 1 = x^{(1)} : y^{(1)} : z^{(1)}, \quad x''' : y''' : 1 = x^{(3)} : y^{(3)} : z^{(3)}$$

and use the abbreviations

$$H_1(x^{(1)}y^{(1)}z^{(1)}) = -2x^{(1)}z^{(1)3} - a\,F_2\{F_2(x^{(1)}, y^{(1)}), -F_1(x^{(1)}, y^{(1)})\},$$

$$H_2(x^{(1)}y^{(1)}z^{(1)}) = -2y^{(1)}z^{(1)3} + a\,F_1\{F_2(x^{(1)}, y^{(1)}), -F_1(x^{(1)}, y^{(1)})\},$$

$$H_3(x^{(1)}y^{(1)}z^{(1)}) = z^{(1)}F^*(x^{(1)}, y^{(1)}),$$

the formulae (B_1) now take the form

$$(B_2) \qquad \begin{cases} x^{(3)} = H_1(x^{(1)}y^{(1)}z^{(1)}), \\ y^{(3)} = H_2(x^{(1)}y^{(1)}z^{(1)}), \\ z^{(3)} = H_3(x^{(1)}y^{(1)}z^{(1)}). \end{cases}$$

The expressions H_1, H_2, H_3 on the right-hand side are forms in the three arguments $x^{(1)}$, $y^{(1)}$, $z^{(1)}$ with integer coefficients; they are of degree four in these variables. Hence the two maxima

$$w^{(i)} = \max(|x^{(i)}|, |y^{(i)}|, |z^{(i)}|) \quad (i = 1, 3)$$

are connected by the inequality

$$(B_3) \qquad w^{(3)} \leqslant c_2(w^{(1)})^4,$$

where c_2 is a positive constant depending only on the coefficients of the cubic form $F(x, y)$.

If neither of the points

$$\left(\frac{x^{(1)}}{z^{(1)}}, \frac{y^{(1)}}{z^{(1)}}\right), \quad \left(\frac{x^{(3)}}{z^{(3)}}, \frac{y^{(3)}}{z^{(3)}}\right)$$

is at infinity and if $z^{(1)}$ is not zero, then the tangent to $C(k_1)$ at the first point cannot be parallel to one of the asymptotes Γ', Γ'', Γ'''; hence

$$F(Y', -X') \neq 0 \quad \text{and also} \quad F^*(x^{(1)}, y^{(1)}) \neq 0,$$

and so we have also

$$(B_4) \qquad z^{(3)} = z^{(1)}F^*(x^{(1)}, y^{(1)}) \neq 0.$$

We add the obvious remark that, if $x^{(1)}$, $y^{(1)}$, $z^{(1)}$ are integers, so also are $x^{(3)}$, $y^{(3)}$, $z^{(3)}$.

10. Now we proceed to the application of the results (A) and (B) to the study of the $2n$ points

$$P_{3m+1} = (x_{3m+1}, y_{3m+1}), \quad P_{-3m-2} = (x_{-3m-2}, y_{-3m-2}) \quad (m = 0, 1, \ldots, n-1)$$

on the curve $C(k_1)$. We know that P_{-2} is the tangential of P_1, and P_4 the tangential of P_{-2}; that for $m = 2, 3, \ldots, n-1$ the three points

$$P_{3m+1},\ P_{-3m+1},\ P_{-2}$$

are collinear; and for $m = 1, 2, \ldots, n-1$ the three points

$$P_{-3m-2},\ P_{3m+1},\ P_1$$

are collinear. We know also that the coordinates of P_1 are integers and that none of the $2n$ points P_{3m+1}, P_{-3m-2} lies at infinity. Hence, from the formulae (A_1) and (B_1), it is obvious that the coordinates of all these points are rational numbers, say

$$x_{3m+1} = \frac{x^{(3m+1)}}{z^{(3m+1)}},\ y_{3m+1} = \frac{y^{(3m+1)}}{z^{(3m+1)}};\quad x_{-3m-2} = \frac{x^{(-3m-2)}}{z^{(-3m-2)}},\ y_{-3m-2} = \frac{y^{(-3m-2)}}{z^{(-3m-2)}},$$

where the new x's, y's, and z's are integers; in particular,

$$x^{(1)} = x_1,\ y^{(1)} = y_1,\ z^{(1)} = 1.$$

Since $z^{(1)} \neq 0$, all denominators

$$z^{(3m+1)},\ z^{(-3m-2)} \quad (m = 0, 1, \ldots, n-1)$$

are different from zero. If we write

$$w^{(3m+1)} = \max\left(|x^{(3m+1)}|, |y^{(3m+1)}|, |z^{(3m+1)}|\right),$$

$$w^{(-3m-2)} = \max\left(|x^{(-3m-2)}|, |y^{(-3m-2)}|, |z^{(-3m-2)}|\right),$$

then $$w^{(1)} = \max\left(|x_1|, |y_1|, 1\right) \leqslant 65n^3,$$

and, using (A_3) and (B_3), we get the system of inequalities

$$w^{(-2)} \leqslant c_2(w^{(1)})^4,\quad w^{(4)} \leqslant c_2(w^{(-2)})^4,$$

$$w^{(3m+1)} \leqslant c_1(w^{(-3m+1)}w^{(-2)})^4 \quad (m = 2, 3, \ldots, n-1),$$

$$w^{(-3m-2)} \leqslant c_1(w^{(3m+1)}w^{(1)})^4 \quad (m = 1, 2, \ldots, n-1),$$

and so we are able to give an upper bound for all numbers

$$w^{(3m+1)},\ w^{(-3m-2)} \quad (m = 0, 1, \ldots, n-1).$$

11. We obtain, however, a much better result in a different way. The equations

$$(6m+1)+(-3m-2)+(-3m+1) = 0,\ (-6m-5)+(3m+1)+(3m+4) = 0,$$

$$(-6m-2)+2(3m+1) = 0,\quad (6m+4)+2(-3m-2) = 0$$

show that, for every m, the three points

$$P^{(6m+1)},\ P^{(-3m-2)},\ P^{(-3m+1)} \quad (m \geqslant 1)$$

are collinear; also the three points

$$P^{(-6m-5)},\ P^{(3m+1)},\ P^{(3m+4)}$$

are collinear; and that, of the two points

$$P^{(-6m-2)},\ P^{(3m+1)},$$

or of the two points

$$P^{(6m+4)},\ P^{(-3m-2)},$$

the first is the tangential of the second. Hence we obtain the recurrence inequalities

$$w^{(6m+1)} \leqslant c_1 (w^{(-3m-2)} w^{(-3m+1)})^4, \quad w^{(6m+4)} \leqslant c_2 (w^{(-3m-2)})^4,$$

$$w^{(-6m-2)} \leqslant c_2 (w^{(3m+1)})^4, \quad w^{(-6m-5)} \leqslant c_1 (w^{(3m+1)} w^{(3m+4)})^4.$$

They assume a simpler form when the new numbers

$$c_3 = \max (c_1^{\frac{1}{7}}, c_2^{\frac{1}{3}}, 1), \quad c_3 w^{(3m+1)} = w_{3m+1}, \quad c_3 w^{(-3m-2)} = w_{-3m-2}$$

$$(m = 0, 1, \ldots, n-1)$$

are introduced; they become

$$w_{6m+1} \leqslant w_{-3m-2}^{\frac{4}{3}} w_{-3m+1}^{\frac{4}{3}}, \quad w_{6m+4} \leqslant w_{-3m-2}^{\frac{4}{3}},$$

$$w_{-6m-2} \leqslant w_{3m+1}^{\frac{4}{3}}, \quad w_{-6m-5} \leqslant w_{3m+1}^{\frac{4}{3}} w_{3m+4}^{\frac{4}{3}}.$$

Hence

$$w_{3m+1} \leqslant w_1^{f(3m+1)}, \quad w_{-3m-2} \leqslant w_1^{f(-3m-2)} \quad (m = 0, 1, \ldots, n-1),$$

where the arithmetical function $f(h)$ is defined for all $h \equiv 1 \pmod 3$ by the equations

$$\text{(C)} \begin{cases} f(6m+1) = 4f(-3m-2) + 4f(-3m+1), & f(6m+4) = 4f(-3m-2), \\ f(-6m-2) = 4f(3m+1), & f(-6m-5) = 4f(3m+1) + 4f(3m+4), \end{cases}$$

and the initial value $f(1) = 1$.

12. Although $f(h)$ is a complicated function, it is not difficult to find a simple upper bound for its values when h is large.

By applying the formulae C twice to this function, we obtain the system of equations

$$f(12m+1) = 32f(3m+1)+16f(3m-2),$$

$$f(-12m-2) = 16f(-3m-2)+16f(-3m+1),$$

$$f(12m+4) = 16f(3m+1),$$

$$f(-12m-5) = 32f(-3m-2)+16f(-3m+1),$$

$$f(12m+7) = 16f(3m+4)+32f(3m+1),$$

$$f(-12m-8) = 16f(-3m-2),$$

$$f(12m+10) = 16f(3m+4)+16f(3m+1),$$

$$f(-12m-11) = 16f(-3m-5)+32f(-3m-2).$$

Now let α be a number such that

$$4^\alpha > 48,$$

and introduce the new function

$$g(h) = f(h)|h|^{-\alpha}.$$

It is obvious then that, for every $\epsilon > 0$ and $m \geqslant m_0(\epsilon)$, we have the system of inequalities

$$g(12m+1) \leqslant (1+\epsilon)\left(\frac{32}{4^\alpha}g(3m+1)+\frac{16}{4^\alpha}g(3m-2)\right),$$

$$g(-12m-2) = (1+\epsilon)\left(\frac{16}{4^\alpha}g(-3m-2)+\frac{16}{4^\alpha}g(-3m+1)\right),$$

$$g(12m+4) \leqslant (1+\epsilon)\left(\frac{16}{4^\alpha}g(3m+1)\right),$$

$$g(-12m-5) = (1+\epsilon)\left(\frac{32}{4^\alpha}g(-3m-2)+\frac{16}{4^\alpha}g(-3m+1)\right),$$

$$g(12m+7) \leqslant (1+\epsilon)\left(\frac{16}{4^\alpha}g(3m+4)+\frac{32}{4^\alpha}g(3m+1)\right),$$

$$g(-12m-8) = (1+\epsilon)\left(\frac{16}{4^\alpha}g(-3m-2)\right),$$

$$g(12m+10) \leqslant (1+\epsilon)\left(\frac{16}{4^\alpha}g(3m+4)+\frac{16}{4^\alpha}g(3m+1)\right),$$

$$g(-12m-11) = (1+\epsilon)\left(\frac{16}{4^\alpha}g(-3m-5)+\frac{32}{4^\alpha}g(-3m-2)\right).$$

Assume that, in particular,

$$4^{\alpha} = 48(1+\epsilon),$$

and write

$$\max_{\substack{|h| \leqq 12m_0(\epsilon) \\ h \equiv 1 \bmod 3}} g(h) = c_4.$$

Then it can be deduced at once from the last two inequalities that, for all integers $h \equiv 1 \pmod 3$,

$$g(h) \leqq c_4.$$

Hence

$$f(h) \leqq c_4 |h|^{\alpha}$$

for all such values of h.

13. The results in §§ 11 and 12 show that there are on $C(k_1)$ at least $2n$ rational points

$$P_{3m+1},\ P_{-3m-2} \quad (m = 0, 1, \ldots, n-1),$$

and that the coordinates of these points are of the form

$$x_{3m+1} = \frac{x^{(3m+1)}}{z^{(3m+1)}},\ y_{3m+1} = \frac{y^{(3m+1)}}{z^{(3m+1)}};\quad x_{-3m-2} = \frac{x^{(-3m-2)}}{z^{(-3m-2)}},\ y_{-3m-2} = \frac{y^{(-3m-2)}}{z^{(-3m-2)}}$$

$$(m = 0, 1, \ldots, n-1),$$

where the x's, y's, z's are integers which satisfy the inequalities

$$\max(|x^{(h)}|,\ |y^{(h)}|,\ |z^{(h)}|) \leqq \frac{1}{c_3} w_1^{f(h)} \leqq \frac{1}{c_3} (65 c_3 n^3)^{c_4 |h|^{\alpha}},\ z^{(h)} \neq 0$$

$$\begin{pmatrix} h = 3m+1,\ -3m-2 \\ m = 0, 1, \ldots, n-1 \end{pmatrix},$$

α being a number such that

$$4^{\alpha} > 48,$$

and c_4 being a positive constant depending only on α and the coefficients of the form $F(x, y)$.

Therefore, in particular,

$$|z^{(3m+1)}| \leqq \frac{1}{c_3} (65 c_3 n^3)^{c_4 (3m+1)^{\alpha}},\quad |z^{(-3m-2)}| \leqq \frac{1}{c_3} (65 c_3 n^3)^{c_4 (3m+2)^{\alpha}}$$

$$(m = 0, 1, \ldots, n-1),$$

and the least common multiple Z of all denominators

$$z^{(3m+1)},\ z^{(-3m-2)} \quad (m = 0, 1, \ldots, n-1)$$

has a value

$$Z \leqslant \left(\frac{1}{c_3}\right)^{2n} (65c_3 n^3)^{\sum_{m=0}^{n-1} \{c_4 (3m+1)^\alpha + c_4 (3m+2)^\alpha\}} \leqslant \left(\frac{1}{c_3}\right)^{2n} (65c_3 n^3)^{2c_4 (3n)^\alpha n}.$$

14. Let us now write

$$k = Z^3 k_1$$

and

$$Zx_{3m+1} = p_{3m+1},\ Zy_{3m+1} = q_{3m+1};\ Zx_{-3m-2} = p_{-3m-2},\ Zy_{-3m-2} = q_{-3m-2}$$
$$(m = 0, 1, \ldots, n-1).$$

Then all the $2n$ points

$$(p_{3m+1}, q_{3m+1}),\ (p_{-3m-2}, q_{-3m-2}) \quad (m = 0, 1, \ldots, n-1)$$

have integer coordinates; they are different from one another and they lie on the same cubic curve $C(k)$. Now, evidently,

$$0 < |k_1| \leqslant c_5 (65n^3)^3,$$

with a positive constant c_5 depending only on the coefficients of the form $F(x, y)$, and so

$$0 < |k| \leqslant c_5 (65n^3)^3 \left\{ \left(\frac{1}{c_3}\right)^{2n} (65c_3 n^3)^{2c_4 (3n)^\alpha n} \right\}^3.$$

Since α is restricted only by the condition

$$4^\alpha > 48,$$

it may be assumed less than 3. Hence, if γ is any positive constant, we have, for $n \geqslant n_0(\gamma)$,

$$0 < |k| \leqslant e^{16\gamma n^4},$$

and, when we replace $2n$ by t, the following theorem is proved.

THEOREM 1. *If γ is any positive number, then there is a positive number $i_0(\gamma)$ such that, corresponding to every integer $t \geqslant t_0(\gamma)$, there exists an integer k with*

$$0 < |k| \leqslant e^{\gamma t^4},$$

which can be represented by the binary form $F(x, y)$ in at least t different ways,

$$k = F(p_h, q_h) \quad (h = 1, 2, \ldots, t)$$

with integers p_h, q_h.

15. We apply Theorem 1 to the special form

$$F(x, y) = x(y^2 - ax^2),$$

where $a \neq 0$ is an arbitrary integer. Then we know that, for large t, there exist integers k with

$$0 < |k| \leqslant e^{t^4},$$

such that the equation

$$p_h(q_h^2 - ap_h^2) = k$$

has at least $2t$ different integer solutions

$$(p_h, q_h) \quad (h = 1, 2, \ldots, 2t).$$

Solving with respect to q_h, we get

$$(p_h\, q_h)^2 = ap_h^4 + kp_h,$$

and hence it is obvious that to every p_h there belong at most two different q_h. Therefore the numbers

$$p_1, p_2, \cdots, p_{2t}$$

assume at least t different values, and we have proved

THEOREM 2. *If $a \neq 0$ is any integer and t is a sufficiently large positive integer, then there is an integer k with*

$$0 < |k| \leqslant e^{t^4},$$

such that the polynomial

$$f(x) = ax^4 + kx$$

is a perfect square for at least t different integer values of the argument x, and these values of x may be assumed to divide the number k.

By a similar method we can also prove

THEOREM 3. *If $a \neq 0$ is any integer and t is a sufficiently large positive integer, then there is an integer k with*

$$0 < |k| \leqslant e^{t^4},$$

such that the polynomial

$$g(x) = ax^3 + k$$

is a perfect square for at least t different integer values of the argument x, and these values of x may be assumed to divide the number k.

THEOREM 4. *If $a \neq 0$ is any integer and t is a sufficiently large positive integer, then there is an integer k with*

$$0 < |k| \leqslant e^{t^4},$$

such that the polynomial

$$g(x) = ax^3 + k$$

is a perfect cube for at least t different integers x.

II.

16. The result of Theorem 1 can be generalized by a simple change in the method of the first chapter.

We have constructed a set of $2n$ rational points

$$P_{3m+1} = (x_{3m+1}, y_{3m+1}), \quad P_{-3m-2} = (x_{-3m-2}, y_{-3m-2}) \qquad (m = 0, 1, \ldots, n-1)$$

on the curve $C(k_1)$ with the following properties:

(*a*) The corresponding points

$$u_{3m+1} = \left(\frac{x_{3m+1}}{k_1^{\frac{1}{3}}}, \frac{y_{3m+1}}{k_1^{\frac{1}{3}}}\right), \quad u_{-3m-2} = \left(\frac{x_{-3m-2}}{k_1^{\frac{1}{3}}}, \frac{y_{-3m-2}}{k_1^{\frac{1}{3}}}\right) \quad (m = 0, 1, \ldots, n-1)$$

on the curve C have arguments of the form

$$u_{3m+1} = (3m+1)u_1, \quad u_{-3m-2} = -(3m+2)u_1 \quad (m = 0, 1, \ldots, n-1).$$

(*b*) If h_1, h_2, h_3 are three different indices of the set

$$3m+1, \quad -3m-2 \quad (m = 0, 1, \ldots, n-1)$$

with

$$h_1 + h_2 + h_3 = 0,$$

then $P_{h_1}, P_{h_2}, P_{h_3}$ are collinear, and when h_1 and h_2 are two different indices with

$$h_1 + 2h_2 = 0$$

then P_{h_1} is the tangential of P_{h_2}.

(*c*) The $2n$ points

$$P_{3m+1}, \quad P_{-3m-2} \quad (m = 0, 1, \ldots, n-1)$$

are all different.

(*d*) None of these $2n$ points lies at infinity.

17. Now let us consider, also, the additional points

$$u_{3m+1} = \left(\frac{x_{3m+1}}{k_1^{\frac{1}{3}}}, \frac{y_{3m+1}}{k_1^{\frac{1}{3}}}\right), \quad u_{-3m-2} = \left(\frac{x_{-3m-2}}{k_1^{\frac{1}{3}}}, \frac{y_{-3m-2}}{k_1^{\frac{1}{3}}}\right) \quad (m = n, n+1, n+2, \ldots)$$

on C with elliptic arguments

$$u_{3m+1} = (3m+1)\,u_1, \quad u_{-3m-2} = (-3m-2)\,u_1 \quad (m = n, n+1, n+2, \ldots)$$

and their corresponding points

$$P_{3m+1} = (x_{3m+1}, y_{3m+1}), \quad P_{-3m-2} = (x_{-3m-2}, y_{-3m-2}) \quad (m = n, n+1, n+2, \ldots)$$

on $C(k_1)$; we arrange them with their indices in the order

$$1, -2, 4, -5, 7, -8, \ldots, -3m+1, 3m+1, -3m-2, \ldots.$$

It is obvious that the enlarged system of points

$$P_1, P_{-2}, P_4, P_{-5}, P_7, P_{-8}, \ldots$$

still has the property (b); but in general the two other properties (c) and (d) no longer hold.

It can easily be proved that all arguments

$$u_{3m+1}, \quad u_{-3m-2} \quad (m = 0, 1, 2, \ldots)$$

are congruent (mod ω_1, ω_2) to real numbers. For the curve C has at least one real asymptote; hence there is a real linear transformation

$$x = \alpha X + \beta Y, \quad y = \gamma X + \delta Y \quad (\alpha\delta - \beta\gamma \neq 0),$$

such that C takes the form

$$X(Y^2 - aX^2) = 1 \quad (a \neq 0)$$

with a real constant a; then, if we write

$$X = \frac{1}{\xi}, \quad Y = \frac{\eta}{\xi},$$

it becomes

$$\eta^2 = \xi^3 + a.$$

But the curve in the (ξ, η)-plane corresponding to this equation has only one real branch; hence, as is well known, all real points of the curve and, therefore, also all real points of C are obtained, if, and only if, u assumes all values congruent to a real number (mod ω_1, ω_2).

Denote by ω the real fundamental period of the two functions $\phi(u)$ and $\psi(u)$. We may assume that u_1, and so all arguments

$$u_{3m+1},\ u_{-3m-2} \quad (m = 0,\ 1,\ 2,\ \ldots),$$

are real numbers. Two real arguments u' and u'' will give the same point on C, if, and only if,

$$u' \equiv u'' \pmod{\omega},$$

i.e. if there is an integer h with

$$u' = u'' + h\omega.$$

18. It is possible that no two points of the system of points

$$P_{3m+1},\ P_{-3m-2} \quad (m = 0,\ 1,\ 2,\ \ldots)$$

on $C(k_1)$ coincide. Then, also, all points

$$u_{3m+1},\ u_{-3m-2} \quad (m = 0,\ 1,\ 2,\ \ldots)$$

on C must be different; there is no integer $h \neq 0$ with

$$3hu_1 \equiv 0 \pmod{\omega}$$

and therefore the quotient

$$\frac{u_1}{\omega}$$

is an irrational number. Hence, by a well-known theorem, the system of real numbers

$$u_1,\ u_{-2},\ u_4,\ u_{-5},\ u_7,\ u_8,\ \ldots$$

is "gleichverteilt" mod ω, and the corresponding points on C will be everywhere dense on every arc Γ of this curve. To every arc Γ there belongs a positive constant γ, such that, for sufficiently large N, at least

$$\gamma N$$

of the points

$$u_{3m+1},\ u_{-3m-2} \quad (m = 0,\ 1,\ \ldots,\ N-1)$$

on C lie on Γ. Hence, also, at least γN of the points

$$P_{3m+1},\ P_{-3m-2} \quad (m = 0,\ 1,\ \ldots,\ N-1)$$

lie on the arc $\Gamma(k_1)$ on $C(k_1)$ corresponding to Γ on C.

Since both curves C and $C(k_1)$ are cut in only one real point by any straight line through the origin, we may define the arcs Γ and $\Gamma(k_1)$ by conditions of the form

$$A \leqslant \frac{y}{x} \leqslant B,$$

or the form

$$A \leqslant \left(\frac{y}{x}\right)^{-1} \leqslant B,$$

where A and B are any two real numbers with $A < B$.

19. If at least two points of the set

$$P_{3m+1},\ P_{-3m-2} \quad (m = 0, 1, 2, \ldots)$$

coincide, then there is an index $N \geqslant n$ such that all points

$$P_{3m+1},\ P_{-3m-2} \quad (m = 0, 1, \ldots, N-1)$$

are different, while at least two of the points

$$P_{3m+1},\ P_{-3m-2} \quad (m = 0, 1, \ldots, N)$$

coincide. This means that all the numbers

$$u_{3m+1} = (3m+1)u_1,\ u_{-3m-2} = (-3m-2)u_1 \quad (m = 0, 1, \ldots, N-1),$$

but not all the numbers

$$u_{3m+1} = (3m+1)u_1,\ u_{-3m-2} = (-3m-2)u_1 \quad (m = 0, 1, \ldots, N),$$

are incongruent (mod ω). Therefore, either

$$u_{3N+1} \equiv u_{-3N+1} \pmod{\omega}, \quad i.e. \quad 6Nu_1 \equiv 0 \pmod{\omega},$$

with $$3hu_1 \not\equiv 0 \pmod{\omega} \quad \text{for} \quad h = 1, 2, \ldots, 2N-1;$$

or $$u_{-3N-2} \equiv u_{3N+1} \pmod{\omega}, \quad i.e. \quad 3(2N+1)u_1 \equiv 0 \pmod{\omega},$$

with $$3hu_1 \not\equiv 0 \pmod{\omega} \quad \text{for} \quad h = 1, 2, \ldots, 2N.$$

It follows that, in the first case,

$$u_1 = \frac{M\omega}{6N},$$

and that, in the second case,

$$u_1 = \frac{M\omega}{3(2N+1)},$$

with an integer $N \geqq n$ and a second integer M prime to $2N$ or $2N+1$ respectively. Hence in the first case the points

$$u_1, u_{-2}, u_4, u_{-5}, \ldots, u_{3N-2}, u_{-3N+1}$$

are the same as the points

$$\frac{M(3g+1)\omega}{6N} \quad (g = 0, 1, \ldots, 2N-1),$$

and in the second case the points

$$u_1, u_{-2}, u_4, u_{-5}, \ldots, u_{-3N+1}, u_{3N+1}$$

are the same as the points

$$\frac{M(3g+1)\omega}{3(2N+1)} \quad (g = 0, 1, .,\ldots, 2N),$$

when we disregard the order of the terms.

Therefore, for sufficiently large N, *i.e.* for sufficiently large n, those of the points

$$u_{3m+1}, \; u_{-3m-2} \quad (m = 0, 1, 2, \ldots),$$

on C which are different, will be everywhere dense on every arc of the curve, and similarly for the corresponding points

$$P_{3m+1}, \; P_{-3m-2} \quad (m = 0, 1, 2, \ldots)$$

on $C(k_1)$. To every such arc there belongs again a positive constant γ, such that for sufficiently large n and N at least

$$\gamma N$$

of the different points of the set

$$u_{3m+1}, \; u_{-3m-2} \quad (m = 0, 1, 2, \ldots)$$

on C, or of the points

$$P_{3m+1}, \; P_{-3m-2} \quad (m = 0, 1, 2, \ldots)$$

on $C(k_1)$, lie on that arc. As in § 18 the arc can be defined by inequalities of the form

$$A \leqslant \frac{y}{x} \leqslant B,$$

or the form

$$A \leqslant \left(\frac{y}{x}\right)^{-1} \leqslant B,$$

where the real numbers A and B satisfy the condition $A < B$.

20. The results of the last two paragraphs lead to the following lemma.

Assume G to be an angle about the origin $(0, 0)$, i.e. the part of the (x, y)-plane with

$$A \leqslant \frac{y}{x} \leqslant B \quad or \quad A \leqslant \left(\frac{y}{x}\right)^{-1} \leqslant B,$$

where A and B are two real numbers with $A < B$. Then there is a positive constant c_6 depending only on G, such that corresponding to every integer $t > 0$, there are three integers n, N, k_1, and $2N$ points

$$P_{3m+1} = (x_{3m+1}, y_{3m+1}), \quad P_{-3m-2} = (x_{-3m-2}, y_{-3m-2}) \quad (m = 0, 1, \ldots, N-1)$$

on the curve $C(k_1)$ with the following properties:

(1) *All points P_{3m+1}, P_{-3m-2} have rational coordinates; P_1 has integer coordinates with $\max(|x_1|, |y_1|) \leqslant 65n^3$.*

(2) *All points P_{3m+1}, P_{-3m-2} are different from one another; at least $t+3$ of them lie in the angle G.*

(3) *When h_1, h_2, h_3 are three different indices of the set*

$$3m+1, \quad -3m-2 \quad (m = 0, 1, \ldots, N-1)$$

with $h_1+h_2+h_3 = 0$, then P_{h_1}, P_{h_2}, P_{h_3} are collinear; when h_1 and h_2 are two of them with $h_1+2h_2 = 0$, then P_{h_1} is the tangential of P_{h_2}.

(4) *The integers n, N, and t satisfy the inequality*

$$n \leqslant N \leqslant c_6 t.$$

It is now possible that some, say j, of the $2N$ points P_{3m+1}, P_{-3m-2} lie at infinity; but $C(k_1)$ being of degree 3, there are at most three such points, and so $j = 0$, 1, 2, or 3. Call these points $P_{\kappa_1}, \ldots, P_{\kappa_j}$.

21. The coordinates of the points P_{3m+1}, P_{-3m-2} are rational numbers and can be written in the form

$$x_{3m+1} = \frac{x^{(3m+1)}}{z^{(3m+1)}}, \quad y_{3m+1} = \frac{y^{(3m+1)}}{z^{(3m+1)}}; \quad x_{-3m-2} = \frac{x^{(-3m-2)}}{z^{(-3m-2)}}, \quad y_{-3m-2} = \frac{y^{(-3m-2)}}{z^{(-3m-2)}}$$

$$(m = 0, 1, \ldots, N-1),$$

as in § 10, with integer x's, y's, and z's, which are finite and not all three zero. The denominators z_{3m+1} and z_{-3m-2} are different from zero, with the

exception of the j denominators $z_{\kappa_1}, \ldots, z_{\kappa_j}$. We write

$$w^{(3m+1)} = \max(|x^{(3m+1)}|, |y^{(3m+1)}|, |z^{(3m+1)}|)$$

$$w^{(-3m-2)} = \max(|x^{(-3m-2)}|, |y^{(-3m-2)}|, |z^{(-3m-2)}|)$$

$$(m = 0, 1, \ldots, N-1),$$

and have again

$$w^{(1)} \leqslant 65n^3.$$

Furthermore, it is obvious that

$$w^{(\kappa_i)} \leqslant c_6 \quad (i = 1, 2, \ldots, j),$$

where c_6 is a positive constant depending only on the coefficients of the form $F(x, y)$, but not on t, n, or N; and similarly

$$w^{(-2\kappa_i)} \leqslant c_7 \quad (i = 1, 2, \ldots, j),$$

where the positive constant c_7 also depends only on the coefficients of the form $F(x, y)$. We remark that, if $j \geqslant 2$, then $j = 3$, for then P_{κ_1}, P_{κ_2}, and P_{κ_3} lie on the line at infinity.

If one of these points P_{κ_i} is collinear with two different finite points P_{h_1} and P_{h_2}, three equations of the following form will be satisfied by the coordinates of these finite points:

$$x^{(h_2)} = K_1^{(i)}(x^{(h_1)}, y^{(h_1)}, z^{(h_1)}), \quad y^{(h_2)} = K_2^{(i)}(x^{(h_1)}, y^{(h_1)}, z^{(h_1)}),$$

$$z^{(h_2)} = K_3^{(i)}(x^{(h_1)}, y^{(h_1)}, z^{(h_1)});$$

here $K_1^{(i)}, K_2^{(i)}, K_3^{(i)} (i = 1, \ldots, j)$ denote ternary forms with integer coefficients, of degree e say, which depend only on the coefficients of the form $F(x, y)$. Their actual calculation by the method in §§ 8 and 9 shows that $z^{(h_i)}$ is not zero, when P_{h_1} and P_{h_2} are different and finite. Obviously these equations lead to the inequality

$$w^{(h_2)} \leqslant c_8 (w^{(h_1)})^e$$

with another constant $c_8 > 0$ depending only on the coefficients of $F(x, y)$.

22. It is possible now to obtain an upper bound for all maxima

$$w^{(3m+1)}, \; w^{(-3m-2)} \quad (m = 0, 1, \ldots, N-1)$$

by using the method of § 11. As far as these maxima correspond to finite points, they are connected by the recurrence formulae

$$w^{(6m+1)} \leqq c_1 (w^{(-3m-2)} w^{(-3m+1)})^4, \quad w^{(6m+4)} \leqq c_2 w^{(-3m-2)4},$$

$$w^{(-6m-2)} \leqq c_2 w^{(3m+1)4}, \quad w^{(-6m-5)} \leqq c_1 (w^{(3m+1)} w^{(3m+4)})^4.$$

But these formulae must be replaced by others when at least one of the points in them lies at infinity. Hence at most three of them change into

$$w^{(\kappa_i)} \leqq c_6,$$

at most three into

$$w^{(-2\kappa_i)} \leqq c_7,$$

and at most six into

$$w^{(h_2)} \leqq c_8 (w^{(h_1)})^e.$$

Therefore the results in § 12 lead to a system of inequalities

$$c_9 w^{(3m+1)} \leqq (c_9 w^{(1)})^{e^6 f(3m+1)}, \quad c_9 w^{(-3m-2)} \leqq (c_9 w^{(1)})^{e^6 f(-3m-2)}$$

$$(m = 0, 1, \ldots, N-1),$$

where c_9 is a constant depending only on the coefficients of $F(x, y)$ and where the arithmetical function $f(h)$ satisfies the inequality

$$f(h) \leqq c_4 |h|^\alpha.$$

So we have, in particular,

$$|z^{(3m+1)}| \leqq \frac{1}{c_9} (65 c_9 n^3)^{c_4 e^6 (3m+1)^\alpha}, \quad |z^{(-3m-2)}| \leqq \frac{1}{c_9} (65 c_9 n^3)^{c_4 e^6 (3m+2)^\alpha}$$

$$(m = 0, 1, \ldots, N-1).$$

Now, by § 20, at least $t+3$ of the points

$$P_{3m+1}, \; P_{-3m-2} \quad (m = 0, 1, \ldots, N-1)$$

lie in the angle G, and so there must be at least t of these points, say

$$P_{h_1}, \; P_{h_2}, \; \ldots, \; P_{h_t},$$

which lie in G and are all finite. Therefore their denominators are different from zero and satisfy the inequalities

$$|z^{(h_i)}| \leqq \frac{1}{c_9} (65 c_9 n^3)^{c_4 e^6 (3N)^\alpha} \quad (i = 1, 2, \ldots, t),$$

so that their least common multiple is not greater than

$$\left\{\frac{1}{c_9}(65c_9 n^3)^{c_4 e^{6(3N)^\alpha}}\right\}^t.$$

If Z is this least common multiple, we write

$$Zx_{h_i} = \frac{Zx^{(h_i)}}{z^{(h_i)}} = p_i, \quad Zy_{h_i} = \frac{Zy^{(h_i)}}{z^{(h_i)}} = q_i \quad (i = 1, 2, \ldots, t)$$

and

$$k = Z^3 k_1.$$

Then the coordinates of all points

$$(p_i, q_i) \quad (i = 1, 2, \ldots, t)$$

are integers; these points are finite and different and all lie in the angle G and on the curve $C(k)$. For the number k we have

$$0 < |k| \leqslant c_5(65n^3)^3 \left\{\frac{1}{c_9}(65n^3)^{c_4 e^{6(3N)^\alpha}}\right\}^{3t} \quad (N \leqslant c_6 t),$$

and so for any positive constant γ and sufficiently large t, that is, sufficiently large n and N, we have

$$0 < |k| \leqslant e^{\gamma t^4},$$

since the exponent α may be assumed less than 3.

We have thus proved the following generalisation of Theorem 1.

THEOREM 5. *Let A and B be two real numbers with $A < B$ and G be the angle*

$$A \leqslant \frac{y}{x} \leqslant B \quad \text{or} \quad A \leqslant \left(\frac{y}{x}\right)^{-1} \leqslant B$$

about the origin, and let γ be any positive number. Then there is a positive number $t_0(A, B, \gamma)$, such that to every integer $t \geqslant t_0(A, B, \gamma)$ exists an integer k with

$$0 < |k| \leqslant e^{\gamma t^4},$$

for which the conditions

$$F(x, y) = k, \quad (x, y) \text{ in } G,$$

have at least t different solutions in points

$$x = p_i, \quad y = q_i \quad (i = 1, 2, \ldots, t)$$

with finite integer coordinates.

We specialize the binary form $F(x, y)$ and the angle G in this theorem and obtain the two following results.

THEOREM 6. *There is an infinite set of positive integers* $k_1, k_2, k_3, \ldots$ *with*

$$1 \leqslant k_1 < k_2 < k_3 < \ldots,$$

such that the number of representations of k_ν *as a sum of two cubes of positive integers is greater than* $\sqrt[4]{\log k_\nu}$.

THEOREM 7. *There is an infinite set of positive integers* $k_1, k_2, k_3, \ldots$ *with*

$$1 \leqslant k_1 < k_2 < k_3 < \ldots,$$

such that the number of representations of k_ν *in the form*

$$k_\nu = pq(p+q)$$

with positive integers p, q *is greater than* $\sqrt[4]{\log k_\nu}$.

III.

23. So far we have treated only cubic curves of the special kind

$$F(x, y) = k.$$

But our method suffices for the study of much more general cubic curves. Suppose

$$f(x, y) = 0$$

to be the equation of a cubic curve of genus 1, and

$$g(x, y) = 0,$$

the equation of another curve, of degree less than or equal to 3, both $f(x, y)$ and $g(x, y)$ being polynomials with rational coefficients. Then to every point (x', y') in the (x, y)-plane with rational coordinates, which is not a point of intersection of the two curves, there belongs a rational number λ, such that the cubic curve $C(\lambda)$,

$$f(x, y)+\lambda g(x, y) = 0,$$

goes through (x', y'), and if this point lies sufficiently near to $f = 0$, but not to a point of intersection of $f = 0$, $g = 0$, then $|\lambda|$ will be very small. Now there is a uniformisation of the curve $C(\lambda)$,

$$x = \phi_\lambda(u), \quad y = \psi_\lambda(u),$$

by means of two elliptic functions

$$\phi_\lambda(u), \ \psi_\lambda(u)$$

with the same periods $\omega_\lambda^{(1)}$, $\omega_\lambda^{(2)}$ and of order 2 or 3; as functions of the parameter λ the expressions

$$\phi_\lambda(u),\ \psi_\lambda(u),\ \omega_\lambda^{(1)},\ \omega_\lambda^{(2)}$$

are analytic and even regular for sufficiently small $|\lambda|$.

If the point (x', y') on $C(\lambda)$ has the elliptic argument u, we can construct on this curve the set of points with arguments

$$(3m+1)u,\ (-3m-2)u \quad (m = 0, 1, \ldots, n-1)$$

by the same process as in § 3, for any given integer $n \geqslant 1$. All these $2n$ points will be different from one another and none lie at infinity if a certain finite number of incongruences of the type

$$u \not\equiv v_\lambda^{(i)} \pmod{\omega_\lambda^{(1)}, \omega_\lambda^{(2)}} \quad (i = 1, 2, \ldots, j)$$

are satisfied; here

$$v_\lambda^{(1)},\ v_\lambda^{(2)},\ \ldots,\ v_\lambda^{(j)}$$

denote analytical functions of λ, which are regular for sufficiently small $|\lambda|$. Now for variable and sufficiently small λ, every congruence

$$u \equiv v_\lambda^{(i)} \pmod{\omega_\lambda^{(1)}, \omega_\lambda^{(2)}} \quad (i = 1, 2, \ldots, j)$$

represents an arc of a certain analytic curve. Hence, in order that all $2n$ points on $C(\lambda)$ with elliptic arguments

$$(3m+1)u,\ (-3m-2)u \quad (m = 0, 1, \ldots, n-1)$$

are different and finite, we must choose the special point (x', y') of argument u in such a way that it lies sufficiently near to the curve $f = 0$ and not on a finite number of arcs of certain analytical curves. But here it is obvious, that in any neighbourhood of any arc of the curve $f = 0$, there is a point (x', y') with these properties and with rational coordinates. Therefore we arrive at the following result.

THEOREM 8. *Suppose that*

$$f(x, y) = 0 \quad \textit{and} \quad g(x, y) = 0$$

are the equations of two different cubic curves, of which the first has the genus one, and that these equations have rational coefficients. Let ϵ be any positive number, $t \geqslant 1$ any integer. Then there is a rational number λ with

$$0 < |\lambda| \leqslant \epsilon,$$

such that at least t different finite points with rational coordinates lie on the cubic curve

$$f(x, y) + \lambda g(x, y) = 0.$$

The method of the second chapter may be applied to prove that all these t points can also be assumed to lie in any neighbourhood of any finite arc of the curve $f = 0$.

It is also possible to obtain an upper bound

$$e^{t^{\alpha}}$$

for the general denominator of the coordinates of these t points, when t is large enough; here α denotes an absolute positive constant.

24. We now mention some applications of Theorem 8. Take

$$f(x, y) = x^3+y^3+1-3Axy$$

with any rational number $A \neq 1$, so that this curve has genus 1, and

$$g(x, y) = 3\lambda xy.$$

Then we get the result:

THEOREM 9. *For any given rational number $A \neq 1$, any given positive number ϵ, and any integer $t \geqslant 1$, there is a rational number A' with*

$$0 < |A'-A| < \epsilon,$$

such that the equation

$$x^3+y^3+z^3-3A'xyz = 0$$

has at least t different solutions in co-prime integers x, y, z.

It is trivial that this theorem remains true also for $A = 1$. For then take a rational number A'' with

$$A+\tfrac{1}{2}\epsilon \leqslant A'' \leqslant A+\tfrac{3}{4}\epsilon,$$

and apply the theorem with A'' instead of A and $\frac{1}{4}\epsilon$ instead of ϵ; then

$$|A'-A''| \leqslant \tfrac{1}{4}\epsilon \quad \text{and therefore} \quad 0 < |A'-A| \leqslant \epsilon.$$

As a second example take

$$f(x, y) = y^2-(4x^3-g_2x-g_3),$$

where g_2 and g_3 are two rational numbers with

$$g_2{}^3-27g_3{}^2 \neq 0,$$

so that the curve $f = 0$ is of genus 1, and

$$g(x, y) = -(4x^3-g_2x-g_3).$$

We have

$$f(x, y)+\lambda g(x, y) = y^2-(1+\lambda)(4x^3-g_2x-g_3),$$

and so obtain the result:

THEOREM 10. *For any two given rational numbers g_2 and g_3 with*

$$g_2^3-27g_3^2 \neq 0,$$

any given positive number ϵ and any integer $t \geqslant 1$ there exists a rational number λ with

$$0 < |\lambda| \leqslant \epsilon,$$

such that the cubic curve

$$y^2-(1+\lambda)(4x^3-g_2x-g_3) = 0$$

has at least t different points with rational coordinates.

25. The last theorem has a remarkable consequence. Evidently the curve

$$C_\lambda(1): \qquad y^2-(1+\lambda)(4x^3-g_2x-g_3) = 0$$

has the same absolute invariant

$$J = \frac{g_2^3}{g_2^3-27g_3^2}$$

as the curve

$$C: \qquad y^2-(4x^3-g_2x-g_3) = 0,$$

and the same is true for all curves

$$C_\lambda(Z): \qquad \left(\frac{y}{Z}\right)^2-(1+\lambda)\left\{4\left(\frac{x}{Z}\right)^3-g_2\frac{x}{Z}-g_2\right\} = 0,$$

where $Z \neq 0$ is any number. Now we may choose this number Z as an integer in such a way that the t rational points on $C_\lambda(1)$ change into points on $C_\lambda(Z)$ with integer coordinates. It is obvious also that, corresponding to every rational value of the absolute invariant J, two rational numbers g_2 and g_3 can be found with

$$J = \frac{g_2^3}{g_2^3-27g_3^2}.$$

Hence we have:

THEOREM 11. *Corresponding to every integer $t \geqslant 1$ and to every rational number J, there exists a cubic curve with absolute invariant J, on which lie at least*

t different points with integer coordinates, and which is defined by an equation of the special form

$$Ay^2+Bx^3+Cx+D=0$$

with integer coefficients.

26. Theorem 10 is a special case of the following more general result.

THEOREM 12. *Suppose that $f(x)$ is a polynomial of exact degree 3 or 4 in x with rational coefficients, and that $t \geqslant 1$ is an integer. Then there is an integer $k \neq 0$ such that, for at least t different rational values of x, the polynomial $kf(x)$ is the square of an integer.*

Proof. If $f(x)=0$ has a multiple root this result is rather trivial; henceforth we assume that the roots of $f(x)=0$ are all simple. Evidently it is sufficient to prove that, for a certain rational number $k_1 \neq 0$, there are at least t different rational points on the curve

$$C(k_1): \qquad y^2-k_1 f(x)=0.$$

The curve $C(1)=C$ has a uniformisation

$$x=\phi(x), \quad y=\psi(x),$$

where $\phi(x)$ and $\psi(x)$ are two elliptic functions with the periods ω_1, ω_2 say. It is cut by every parabola

$$y=Ax^2+Bx+C$$

in exactly four points, and the arguments u_1, u_2, u_3, u_4 of these points satisfy the congruence

$$u_1+u_2+u_3+u_4 \equiv c \pmod{\omega_1, \omega_2},$$

with a certain constant c, which may be assumed equal to zero without loss of generality. Of these four points of intersection, three may be given arbitrarily, and then the last one can be found by a rational construction. In the special case in which the parabola osculates C in the point u_1, we must count this point thrice, and so there is only one other point of intersection u_2, given by the congruence

$$u_2 \equiv -3u_1 \pmod{\omega_1, \omega_2}.$$

When we now construct the set of t points with arguments

$$u_1, \ -3u_1, \ +9u_1, \ \ldots, \ (-3)^{t-1}u_1$$

on C, there will be only a finite number of initial positions for u_1, for which some of these t points coincide or lie at infinity: this is shown by a method similar to that of the first chapter. Through all these exceptional positions of u_1 and through all points with $f(x)=0$, $y=0$ we draw straight lines perpendicular to the x-axis, and then take a rational point (x_1, y_1) not lying on any one of these lines or on the x-axis. Then there is exactly one curve $C(k_1)$ going through this point, with

$$k_1 = \frac{y_1^2}{f(x)}.$$

On this curve we construct the t points

$$(x_1, y_1), \quad (x_2, y_2), \quad \ldots, \quad (x_t, y_t)$$

such that, for $m=1, 2, \ldots, t-1$, the point (x_{m+1}, y_{m+1}) is the point of intersection with $C(k_1)$ of the parabola

$$y = Ax^2+Bx+C,$$

which osculates this curve in the point (x_m, y_m). All points

$$(x_1, y_1), \quad (x_2, y_2), \quad \ldots, \quad (x_t, y_t)$$

are different and none lies at infinity, and their coordinates are rational numbers; these facts may be shown by transforming $C(k_1)$ into C by replacing y by $y/\sqrt{|k_1|}$; or by actually giving a recurrence formula for the coordinates of the $(m+1)$-th point, when the m-th is known. This proves our theorem.

27. We mention two trivial consequences of the last theorem.

THEOREM 13. *Let t be an arbitrary positive integer. Then there exists a polynomial $a_0x^2+a_1$ $(a_0a_1 \neq 0)$ of exact degree 2 with integer coefficients, which is a perfect cube for at least t different integer values of the argument.*

THEOREM 14. *Let t be an arbitrary positive integer. Then there exists a polynomial $a_0x^2+a_1$ $(a_0a_1 \neq 0)$ of exact degree 2 with integer coefficients, which is a perfect fourth power for at least t different integer values of the argument.*

To prove these two theorems we need only apply Theorem 12 to the two polynomials

$$f(x) = x^3-a \quad \text{and} \quad f(x) = x^4-a,$$

where $a \neq 0$ denotes an arbitrary integer.

28. So far we have considered the lattice points only on plane curves of genus 1. Our method, however, can also be used for the study of these points on curves of genus 1 in spaces of three or more dimensions. It may be sufficient to give one result of this kind.

THEOREM 15. *Suppose that a, b, c, and A, B, C are six integers with*

$$aB-bA \neq 0, \quad aC-cA \neq 0, \quad bC-cB \neq 0,$$

and that t is an arbitrary positive integer. Then there are two integers $k \neq 0$ and $K \neq 0$ such that the system of equations

$$ax^2+by^2+cz^2 = k, \quad Ax^2+By^2+Cz^2 = K$$

has at least t different solutions in integers x, y, z.

Proof. We choose two rational numbers k_0 and K_0 such that the curve

$$C(k_0, K_0): \qquad ax^2+by^2+cz^2 = k_0, \quad Ax^2+By^2+Cz^2 = K_0$$

in three dimensions does not consist of single real points, but has real arcs and is of genus unity. Then the same is true for all curves

$$C(k_1, K_1): \qquad ax^2+by^2+cz^2 = k_1, \quad Ax^2+By^2+Cz^2 = K_1,$$

where k_1 and K_1 are two rational numbers sufficiently near to k_0 and K_0. $C(k_1, K_1)$ is not a plane curve and is of degree 4 and of genus 1. Hence it has a uniformisation

$$x = \phi_{k_1 K_1}(u), \quad y = \psi_{k_1 K_1}(u), \quad z = \chi_{k_1 K_1}(u),$$

by means of three elliptic functions

$$\phi_{k_1 K_1}(u), \quad \psi_{k_1 K_1}(u), \quad \chi_{k_1 K_1}(u),$$

with the same periods $\omega^{(1)}_{k_1 K_1}$, $\omega^{(2)}_{k_1 K_1}$, say. As functions of variable parameters k_1 and K_1, the expressions

$$\phi_{k_1 K_1}(u), \quad \psi_{k_1 K_1}(u), \quad \chi_{k_1 K_1}(u), \quad \omega^{(1)}_{k_1 K_1}, \quad \omega^{(2)}_{k_1 K_1}$$

are analytic and even regular, when

$$|k_1-k_0| \quad \text{and} \quad |K_1-K_0|$$

are sufficiently small.

An arbitrary plane cuts $C(k_1, K_1)$ in four points of elliptic arguments u_1, u_2, u_3, u_4, say; they are connected by the congruence

$$u_1+u_2+u_3+u_4 \equiv c \pmod{\omega^{(1)}_{k_1 K_1}, \omega^{(2)}_{k_1 K_1}}$$

with a certain constant c, which may be assumed equal to zero without loss of generality. Of these four points of intersection, three may be given arbitrarily, and then the last one can be found by a rational construction. In the special case when the plane osculates $C(k_1, K_1)$ in the point u_1, this point must be counted thrice, and so there is only one other point of intersection u_2, given by the congruence

$$u_2 \equiv -3u_1 \pmod{\omega^{(1)}_{k_1 K_1},\ \omega^{(2)}_{k_1 K_1}}.$$

When we now construct the t points with the elliptic arguments

$$u_1,\ -3u_1,\ +9u_1,\ \ldots,\ (-3)^{t-1}u_1,$$

they will all be different from one another and none lie at infinity, when a certain finite number of incongruences of the type

$$u \not\equiv v^{(i)}_{k_1 K_1} \pmod{\omega^{(1)}_{k_1 K_1},\ \omega^{(2)}_{k_1 K_1}} \quad (i = 1, 2, \ldots, j)$$

is satisfied; here

$$v^{(i)}_{k_1 K_1} \quad (i = 1, 2, \ldots, j)$$

denote analytical functions of k_1, K_1, which are regular for sufficiently small $|k_1 - k_0|$ and $|K_1 - K_0|$. Now for such k_1 and K_1 every congruence

$$u \equiv v^{(i)}_{k_1 K_1} \pmod{\omega^{(1)}_{k_1 K_1},\ \omega^{(2)}_{k_1 K_1}}$$

represents a piece of a certain analytical surface. Hence all t points

$$u_1,\ -3u_1,\ +9u_1,\ \ldots,\ (-3)^{t-1}u_1$$

on $C(k_1, K_1)$ will be different and finite, when the first point

$$u_1 = (x', y', z')$$

is chosen in such a way that it lies sufficiently near to the curve $C(k_0, K_0)$ and does not lie on a finite number of pieces of certain analytical surfaces. We can satisfy these conditions by rational numbers x', y', z'. The point (x', y', z') determines uniquely the numbers

$$k_1 = ax'^2 + by'^2 + cz'^2, \quad K_1 = Ax'^2 + By'^2 + Cz'^2,$$

and so also the curve $C(k_1, K_1)$; both k_1 and K_1 are rational, and the former method shows that the t points with elliptic arguments

$$u_1,\ -3u_1,\ +9u_1,\ \ldots,\ (-3)^{t-1}u_1$$

on $C(k_1, K_1)$ are different and finite and have rational coordinates. Assume Z to be the least common multiple of the denominators of these coordinates; then, on the curve

$$C(k_1 Z^2, K_1 Z^2): \quad ax^2 + by^2 + cz^2 = k_1 Z^2, \quad Ax^2 + By^2 + Cz^2 = K_1 Z^2,$$

there are at least t different and finite lattice points, which proves our theorem.

29. All the considerations of the previous pages were based on the construction of rational points on a curve of genus 1, when one such rational point was known. Evidently the method will lead to still better results when we know more rational points on the curve and when the elliptic arguments, say $u_1, u_2, \ldots, u_s$, of these points do not satisfy any congruence

$$h_1 u_1 + h_2 u_2 + \ldots + h_s u_s \equiv 0 \pmod{\omega_1, \omega_2}$$

with too small integers $h_1, h_2, \ldots, h_s$. Now we have a cubic curve through any nine given points in the plane. We may choose their coordinates as integers in such a way that the curve has no double point and hence is of genus 1, and that the elliptic arguments of the nine points are sufficiently independent in the above-mentioned sense; this will be the case when a certain finite system of inequalities is satisfied. It is very probable that in this way we may be able to prove the result:

"There are an infinity of cubic curves

$$\sum_{\substack{i=0 \\ i+j \leqslant 3}}^{3} \sum_{j=0}^{3} a_{ij} x^i y^j = 0,$$

of genus 1 and with integer coefficients, on which at least

$$(\log a)^2$$

different lattice points lie, where

$$a = \max_{i,\, j} (|a_{ij}|)".$$

I hope to attack this question in a later paper.

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SOME REMARKS ON THE COEFFICIENTS OF SCHLICHT FUNCTIONS

By V. LEVIN.

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We denote by S_k, k being throughout a positive integer, the class of functions of the form

$$(F) \qquad f_k(z) = z + a_1^{(k)} z^{k+1} + a_2^{(k)} z^{2k+1} + \ldots$$

$$= \sum_{\nu=0}^{\infty} a_\nu^{(k)} z^{\nu k+1} \qquad (a_0^{(k)} = 1),$$

regular and schlicht in $|z| < 1$.[1]

Since the functions belonging to S_k form a normal set (in the sense of Montel) there exists

$$\max_{f_k \subset S_k} |a_n^{(k)}| = \alpha_n^{(k)}.$$

For $k = 1$ it is known that[2]

$$\alpha_1^{(1)} = 2, \quad \alpha_2^{(1)} = 3,$$

and Littlewood[3] has proved that, for all $n \geqslant 1$,

$$\alpha_n^{(1)} < en.$$

[1] Throughout the whole paper we write $f_k(z)$, $g_k(z)$, etc. for functions belonging to S_k.

[2] For the first (classical) result see L. Bieberbach, "Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln", *Sitzungsber. der preuss. Ak. d. Wiss., Math. Phys. Klasse* (1916), 940–955, and L. Bieberbach, *Lehrbuch der Funktionentheorie*, 2, 71–74 (2nd ed., Berlin and Leipzig, 1931); for the second, K. Löwner, "Untersuchungen über schlichte konforme Abbildungen des Einheitskreises", *Math. Annalen*, 89 (1923), 103–121. It can be easily shown that $\alpha_3^{(1)} < 5$ and $\alpha_4^{(1)} < 8$ [for the second inequality compare G. Szegö, "Zur Theorie der schlichten Abbildungen", *Math. Annalen*, 100 (1928), 192].

[3] J. E. Littlewood, "On inequalities in the theory of functions", *Proc. London Math. Soc.* (2), 23 (1924), 481–519, particularly 498; compare also [4].

Landau[4] has proved that

$$\overline{\lim_{n\to\infty}}\ \frac{a_n^{(1)}}{n} \leqslant \left(\tfrac{1}{2}+\frac{1}{\pi}\right) e.$$

Recently Littlewood and Paley[5] have shown that

$$\text{upper bound}_{0\leqslant n<\infty}\ a_n^{(2)} = A < \infty.$$

It is of great interest that, as has been proved by Fekete and Szegö[6],

$$A > 1.$$

The last mentioned authors prove, in fact, that[7]

$$a_2^{(k)} = \frac{2}{k}\exp\left\{-\frac{2(k-1)}{k+1}\right\}+\frac{1}{k}$$

for $k \geqslant 1$, so that

$$A \geqslant a_2^{(2)} = e^{-\frac{2}{3}}+\tfrac{1}{2} = 1{\cdot}013\ldots > 1.$$

In this paper we shall first show that

(I) $$a_3^{(2)} < 1{\cdot}25,\quad a_4^{(2)} < 1{\cdot}38,\quad a_5^{(2)} < 1{\cdot}70,\quad a_6^{(2)} < 2{\cdot}05,$$
$$a_7^{(2)} < 2{\cdot}46,\quad a_8^{(2)} < 2{\cdot}95,$$

and

(II) $$A < 2^{\frac{1}{2}}3^{\frac{1}{2}}e^{\frac{1}{2}} = 3{\cdot}39\ldots.$$

The proof of (II) will be obtained by a slight modification of the method of Littlewood and Paley[5].

Then, using an idea of Landau[4], we shall prove that

(III) $$\overline{\lim_{n\to\infty}}\ a_n^{(2)} \leqslant 2^{-\frac{3}{4}}3^{\frac{1}{4}}e^{\frac{1}{4}}\left\{1+\frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{7}{8})}{\Gamma(\frac{11}{8})}\right\}^{\frac{1}{4}}\left\{1+\frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{5}{8})}{\Gamma(\frac{9}{8})}\right\}^{\frac{1}{4}}$$
$$= 3{\cdot}006\ldots.$$

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[4] E. Landau, "Über schlichte Funktionen", *Math. Zeitschrift*, 30 (1929), 635–638.

[5] J. E. Littlewood and R. E. A. C. Paley, "A proof that an odd schlicht function has bounded coefficients", *Journal London Math. Soc.*, 7 (1932), 167–169. See also E. Landau, "Über ungerade schlichte Funktionen", *Math. Zeitschrift*, 37 (1933), 33–35. In a paper which appeared in *Math. Zeitschrift*, 38 (1934), 306–311, I proved that

$$a_n^{(3)} = O(n^{-\frac{1}{2}}),\quad a_n^{(4)} = O(n^{-\frac{1}{2}}\log n),\quad a_n^{(k)} = O(n^{-\frac{1}{2}}\log^{\frac{1}{2}} n)$$

uniformly for $k \geqslant 5$. Szegö's conjecture

$$a_n^{(k)} = O(n^{-1+(2/k)})$$

is thus proved for $k=1$, 2, and 3.

[6] M. Fekete and G. Szegö, "Eine Bemerkung über ungerade schlichte Funktionen", *Journal London Math. Soc.*, 8 (1933), 85–89.

[7] From $a_1^{(1)} = 2$ it follows easily that $a_1^{(k)} = 2/k$ for $k \geqslant 1$.

Considering further the class S_k^s of schlicht functions of the form (F) which represent $|z|<1$ on a star region with respect to the origin, and putting

$$\max_{f_k \subset S_k^s} |a_n^{(k)}| = \beta_n^{(k)},$$

we have, as is known[8],

$$\beta_n^{(1)} = n, \quad \beta_n^{(2)} = 1.$$

We prove now that, generally,

(IV) $$\beta_n^{(k)} = \left| \binom{-2/k}{n} \right|,$$

a result which seems not to have been noticed.

As a corollary to (IV) it follows that

$$\gamma_n^{(k)} = \frac{1}{nk+1} \left| \binom{-2/k}{n} \right|,$$

where $$\gamma_n^{(k)} = \max_{f_k \subset S_k^c} |a_n^{(k)}|,$$

and S_k^c denotes the class of schlicht functions of the form (F) which represent $|z|<1$ on a convex region. In fact, after a remark of Alexander[9], if $f_k(z) \subset S_k^s$, $g_k(z)$, defined by

$$f_k(z) = z g_k'(z), \quad g_k(0) = 0,$$

belongs to S_k^c, and for every $g_k(z) \subset S_k^c$ there exists a corresponding $f_k(z) \subset S_k^s$.

1. The function

$$g(z) = f_2^{-1}(z^{-1}) = z + \sum_{\nu=1}^{\infty} b_\nu z^{-2\nu+1}$$

[8] For the first equality see L. Bieberbach, "Neue Forschungen im Gebiete der konformen Abbildung", *Glasnik hrv. prirod. društa*, 33 (1921), and R. Nevanlinna, "Über die konforme Abbildung von Sterngebieten", *Fiuska Vetenskaps-Soc. Foerh.*, 53 (A), (1921), No. 6; for the second, F. Privalov, "Sur les fonctions, qui donnent la représentation conforme bi-univoque", *Recueil Math. d. l. Soc. Math. d. Moscou*, 31 (1924), 350–365, where are also given the exact values of $\gamma_n^{(1)}$ and $\gamma_n^{(2)}$; compare further in text. See also E. Strohhäcker, "Beiträge zur Theorie der schlichten Funktionen", *Math. Zeitschrift*, 37 (1933), 372.

[9] I. W. Alexander, "Functions which map the interior of the unit circle upon simple regions", *Annals of Math.* (2), 17 (1915), 15, and L. Bieberbach, *Lehrbuch* (*loc. cit.* in [2]), 82.

is schlicht in $|z|>1$. It follows from a well-known theorem of Bieberbach[10] that

$$\sum_{\nu=1}^{\infty}(2\nu-1)|b_\nu|^2\leqslant 1. \tag{1.1}$$

Now we have

$$(z^{-1}+a_1^{(2)}z^{-3}+a_2^{(2)}z^{-5}+\ldots)(z+b_1z^{-1}+b_2z^{-3}+\ldots)=1,$$

and, with $a_0^{(2)}=1$,

$$a_n^{(2)}=-\sum_{\nu=1}^{n}b_\nu a_{n-\nu}^{(2)},\ ^{11}$$

$$|a_n^{(2)}|^2\leqslant\left\{\sum_{\nu=1}^{n}|b_\nu||a_{n-\nu}^{(2)}|^2\right\}^2\leqslant\left\{\sum_{\nu=1}^{n}(2\nu-1)|b_\nu|^2\right\}\left\{\sum_{\nu=1}^{n}\frac{|a_{n-\nu}^{(2)}|^2}{2\nu-1}\right\},$$

and, by (1.1),

$$|a_n^{(2)}|^2\leqslant\sum_{\nu=1}^{n}\frac{|a_{n-\nu}^{(2)}|^2}{2\nu-1}.$$

We use this formula with $|a_1^{(2)}|\leqslant 1$, $|a_2^{(2)}|\leqslant e^{-\frac{2}{3}}+\frac{1}{2}<1{\cdot}014$, and obtain successively the various results in (I).

2. Let $z=re^{i\phi}$, where $0<r<1$ and ϕ is real. We need now the following known facts.

LEMMA 1. ("Verzerrungssatz")[12].

$$|f_k(z)|\leqslant r(1-r^k)^{-2/k}.$$

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[10] "Flächensatz". See L. Bieberbach, *loc. cit.* in [2]. Also his *Lehrbuch*, 72–74.

[11] We have thus

$$a_2^{(2)}=b_1^2-b_2,\quad |a_2^{(2)}|\leqslant|b_1|^2+|b_2|.$$

But from (1.1) it follows that

$$|b_1|^2+3|b_2|^2\leqslant 1,\quad |b_1|^2\leqslant 1-3|b_2|^2,$$

and hence that $|a_2^{(2)}|\leqslant 1+|b_2|-3|b_2|^2\leqslant 1\frac{3}{12}=1{\cdot}083\ldots.$

(Compare the exact result

$$|a_2^{(2)}|\leqslant e^{-\frac{2}{3}}+\tfrac{1}{2}=1{\cdot}013\ldots.)$$

[12] See L. Bieberbach (*loc. cit.* in [2]) and his *Lehrbuch*, 76–78, where it is proved for $k=1$. But we have

$$f_k(z)=f_1^{1/k}(z^k),$$

and Lemma 1 follows.

LEMMA 2[13].

$$\frac{1}{2\pi}\int_0^{2\pi} |f_1(z)|^\lambda d\phi \leqslant \lambda \int_0^r t^{\lambda-1}(1-t)^{-2\lambda}\, dt$$

for $\lambda > 0$.

LEMMA 3[14].

$$\frac{1}{2\pi}\int_0^{2\pi} |f_k'(z)|^2 d\phi < e^{-1} r^{-1}(1-r)^{-1}(1-r^{\frac{1}{2}k})^{-4/k}.$$

Proof. We have

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} |f_k'(z)|^2 d\phi &= \sum_{\nu=0}^{\infty} (\nu k+1)^2 |a_\nu^{(k)}|^2 r^{2\nu k} \\
&\leqslant \max_{0\leqslant \nu < \infty} \{(k\nu+1) r^{k\nu+1}\} r^{-2} \sum_{\nu=0}^{\infty} (\nu k+1) |a_\nu^{(k)}|^2 (\sqrt{r})^{2\nu k+2} \\
&\leqslant e^{-1} \log^{-1}\left(\frac{1}{r}\right) r^{-2} \left\{\frac{1}{\pi} \cdot \text{area of the } f_k\text{-map of } |\zeta| \leqslant \sqrt{r}\right\} \\
&< e^{-1}(1-r)^{-1} r^{-2} \max_{|\zeta|=\sqrt{r}}{}^2 |f_k(\zeta)| \\
&\leqslant e^{-1} r^{-1}(1-r)^{-1}(1-r^{\frac{1}{2}k})^{-4/k},
\end{aligned}$$

by Lemma 1.

3. Denoting by l an integer not less than 3, we prove now

LEMMA 4.

$$\begin{aligned}
&(2n+1)|a_n^{(2)}|^2 \\
&\leqslant r^{-4n-2+(2/l)} \left\{\frac{1}{2\pi}\int_0^{2\pi} |f_1(r^2 e^{i\psi})|^{(l-1)/l} d\psi\right\} \left\{\frac{1}{2\pi}\int_0^{2\pi} |f_{2l}'(r^{1/l} e^{i\psi})|^2 d\psi\right\}.
\end{aligned}$$

Proof. We have

$$f_2(z) = f_{2l}^l(z^{1/l}),$$

$$f_2'(z) = z^{(1/l)-1} f_{2l}^{l-1}(z^{1/l}) f_{2l}'(z^{1/l}),$$

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[13] See J. E. Littlewood (*loc. cit.* in [3]) and H. Prawitz, "Über Mittelwerte analytischer Funktionen", *Arkiv för Mat.*, 20 (1927–28), No. 6, 8.

[14] See J. E. Littlewood and R. E. A. C. Paley (*loc. cit.* in [5]), where Lemma 3 is proved without determining the constant on the right.

and since $$(2n+1)a_n^{(2)} = \frac{1}{2\pi i}\oint_{|z|=r} \frac{f_2'(z)}{z^{2n+1}}\,dz,$$

we obtain

$$(2n+1)^2|a_n^{(2)}|^2 \leqslant r^{-4n}\left\{\frac{1}{2\pi}\int_0^{2\pi}|f_2'(z)|\,d\phi\right\}^2$$

$$= r^{-4n-2+(2/l)}\left\{\frac{1}{2\pi}\int_0^{2\pi}|f_{2l}(z^{1/l})|^{l-1}|f_{2l}'(z^{1/l})|\,d\phi\right\}^2$$

$$\leqslant r^{-4n-2+(2/l)}\left\{\frac{1}{2\pi}\int_0^{2\pi}|f_{2l}(z^{1/l})|^{2(l-1)}\,d\phi\right\}\left\{\frac{1}{2\pi}\int_0^{2\pi}|f_{2l}'(z^{1/l})|^2\,d\phi\right\}.$$

But we have [using again $f_k(z^{1/k}) = f_1^{1/k}(z)$]

$$\frac{1}{2\pi}\int_0^{2\pi}|f_{2l}(z^{1/l})|^{2(l-1)}\,d\phi = \frac{1}{2\pi}\int_0^{2\pi}|f_1(z^2)|^{(l-1)/l}\,d\phi$$

$$= \frac{1}{4\pi}\int_0^{4\pi}|f_1(r^2e^{i\psi})|^{(l-1)/l}\,d\psi = \frac{1}{2\pi}\int_0^{2\pi}|f_1(r^2e^{i\psi})|^{(l-1)/l}\,d\psi.$$

Further, since $f_{2l}'(z\epsilon) = f_{2l}'(z)$, if $\epsilon^{2l} = 1$, it follows that

$$\frac{1}{2\pi}\int_0^{2\pi}|f_{2l}'(z^{1/l})|^2\,d\phi = \frac{l}{2\pi}\int_0^{2\pi/l}|f_{2l}'(r^{1/l}e^{i\psi})|^2\,d\psi$$

$$= \frac{1}{2\pi}\int_0^{2\pi}|f_{2l}'(r^{1/l}e^{i\psi})|^2\,d\psi.$$

4. By Lemma 2 we have, with $\lambda = (l-1)/l$,

$$\frac{1}{2\pi}\int_0^{2\pi}|f_1(r^2e^{i\psi})|^{(l-1)/l}\,d\psi \leqslant \frac{l-1}{l}\int_0^{r^2} t^{-1/l}(1-t)^{-2+(2/l)}\,dt;$$

we shall now prove that

$$\int_0^{r^2} t^{-1/l}(1-t)^{-2+(2/l)}\,dt < \frac{l}{l-2}\,r^{2-(2/l)}(1-r^2)^{-1+(2/l)}, \tag{4.1}$$

from which it will follow that

$$\frac{1}{2\pi}\int_0^{2\pi}|f_1(r^2e^{i\psi})|^{(l-1)/l}\,d\psi < \frac{l-1}{l-2}\,r^{2-(2/l)}(1-r^2)^{-1+(2/l)}. \tag{4.2}$$

To prove (4.1) it is obviously sufficient to show that the derivate of the

left-hand side of (4.1) is less than the derivate of the right-hand side. In fact,

$$2r\,.\,r^{-2/l}(1-r^2)^{-2+(2/l)}$$

$$<\frac{2l}{l-2}\left(1-\frac{1}{l}\right)r^{1-(2/l)}(1-r^2)^{-1+(2/l)}+\frac{2l}{l-2}\left(1-\frac{2}{l}\right)r^{3-(2/l)}(1-r^2)^{-2+(2/l)},$$

because
$$1<\frac{l-1}{l-2}(1-r^2)+r^2=\frac{l-1-r^2}{l-2}.$$

From Lemma 4 it follows, with regard to (4.2) and Lemma 3 (for $k=2l$), that

$$(2n+1)^2\{a_n^{(2)}\}^2<e^{-1}\frac{l-1}{l-2}r^{-4n-(1/l)}(1-r^{1/l})^{-1}(1-r)^{-2/l}(1-r^2)^{-1+(2/l)}$$

$$<le^{-1}\frac{l-1}{l-2}r^{-4n-(1/l)}(1+r)^{-1+(2/l)}(1-r)^{-2}.$$

Substituting herein $l=4$ and $r=1-\{1/(2n+1)\}$, we obtain

$$\{a_n^{(2)}\}^2<6e^{-1}\left(1-\frac{1}{2n+1}\right)^{-4n-\frac{1}{2}}r^{\frac{1}{2}}(1+r)^{-\frac{1}{2}}. \tag{4.3}$$

Now

$$r^{\frac{1}{2}}(1+r)^{-\frac{1}{2}}<2^{-\frac{1}{2}}, \tag{4.4}$$

and, for $n\geqslant 2$,

$$\left(1-\frac{1}{2n+1}\right)^{-4n-\frac{1}{2}}<e^2. \tag{4.5}$$

To prove (4.5) we put $2n+1=m$, and have to verify that

$$\left(1-\frac{1}{m}\right)^{-2m+\frac{3}{2}}<e^2,$$

or
$$\left(1-\frac{1}{m}\right)^{-m+\frac{3}{4}}<e$$

for $m\geqslant 5$. This, however, follows from the fact that

$$\left(1-\frac{1}{m}\right)^{-m+\frac{3}{4}}<\left(1-\frac{1}{m+1}\right)^{-m-1+\frac{3}{4}} \tag{4.6}$$

for $m \geqslant 5$. To verify (4.6) we observe that it is equivalent to

$$\left(\frac{m^2-1}{m^2}\right)^{\frac{3}{4}} < \left(\frac{m^2-1}{m^2}\right)^m \frac{m+1}{m},$$

or

$$\frac{3}{4} > m - \frac{\log\{1+(1/m)\}}{\log\{1+1/(m^2-1)\}}. \tag{4.7}$$

Since

$$m - \frac{\log\{1+(1/m)\}}{\log\{1+1/(m^2-1)\}} < m-(m^2-1)\log\left(1+\frac{1}{m}\right)$$

$$< m-(m^2-1)\left(\frac{1}{m}-\frac{1}{2m^2}\right) = \frac{1}{2}+\frac{1}{m}-\frac{1}{2m^2},$$

(4.7) is obviously true for $m \geqslant 5$.

Substituting now from (4.5) and (4.4) in (4.3), we obtain

$$a_n^{(2)} < 2^{\frac{1}{4}} 3^{\frac{1}{4}} e^{\frac{1}{4}}$$

for $n \geqslant 2$, and, since $a_n^{(2)} = 1$, the whole result in (II)[7].

5. In this section we introduce the symbol $G_k(z)$ for functions belonging to S_k but regular and schlicht in $|z| \leqslant 1$ (instead of in $|z| < 1$). Then, for every $u > 1$, there exists at least one real number ω such that

$$G_k(z) \neq ue^{i\omega} = a \tag{5.1}$$

everywhere in $|z| \leqslant 1$. This follows from the fact that

$$\min_{0 \leqslant \phi < 2\pi} |G_k(e^{i\phi})| \leqslant 1.$$

We have, further,

$$G_k(z)\{1-a^{-k}G_k{}^k(z)\}^{-(1/k)} = g_k(z);$$

applying Lemma 1 to $g_k(z)$, we find

$$|G_k(z)||1-a^{-k}G_k{}^k(z)|^{-(1/k)} \leqslant r(1-r^k)^{-(2/k)} = R.$$

The functions $G_k(z)$ therefore do not assume in $|z| \leqslant r$ values w satisfying the relation

$$|w||1-a^{-k}w^k|^{-(1/k)} > R,$$

or

$$|w^{-k}-a^{-k}|^2 < R^{-2k}. \tag{5.2}$$

With $x > 0$ and real θ we put

$$w = xe^{i\omega}e^{i\theta}. \tag{5.3}$$

From (5.3) and (5.2) it follows that

$$x^{-2k}-2u^{-k}x^{-k}\cos(k\theta)+u^{-2k}<R^{-2k},$$

or

$$|\theta|>\frac{1}{k}\arccos\{\tfrac{1}{2}(x^{-k}u^{k}+x^{k}u^{-k}-x^{k}u^{k}R^{-2k})\}. \tag{5.4}$$

Suppose now that

$$R^{*}=R(1+R^{2k})^{-(1/2k)}<x<R, \tag{5.5}$$

and let

$$u=x\left\{1-\left(\frac{x}{R}\right)^{2k}\right\}^{-(1/2k)}, \tag{5.6}$$

so that, in fact, by (5.5),

$$1<u<\infty.$$

From (5.4) and (5.6) it follows that

$$|\theta|>\frac{1}{k}\arcsin\left\{\left(\frac{x}{R}\right)^{k}\right\}. \tag{5.7}$$

With regard to (5.3) and (5.7) we have thus proved that *there exists at least one real ω such that the function $G_k(z)$ does not take the values w for which*

$$|w|=x,\quad |\arg w-\arg\omega|>\frac{1}{k}\arcsin\left\{\left(\frac{x}{R}\right)^{k}\right\}.$$

Now $G_k(z)$ does not take the value $a\epsilon$, where $\epsilon^k=1$, if it does not take the value a, and, since

$$\frac{1}{k}\arcsin\left\{\left(\frac{x}{R}\right)^{k}\right\}<\frac{\pi}{2k}$$

[by the right side of (5.5)], there exist k arcs of the circle $|w|=x$, each of length greater than

$$\frac{2}{k}\arcsin\left\{\left(\frac{x}{R}\right)^{k}\right\},$$

which *do not overlap*, and which consist of points not taken by $G_k(z)$ in $|z|\leqslant r$. We have thus

LEMMA 5. *Suppose that*

$$R=r(1-r^{k})^{-(2/k)},$$

$$R^{*}=R(1+R^{2k})^{-(1/2k)}<x<R.$$

Then the circle $|w| = x$ contains some arcs which have a total length greater than

$$2 \arcsin\left\{\left(\frac{x}{R}\right)^k\right\},$$

and which consist of points corresponding to values not taken by $G_k(z)$ in $|z| \leqslant r$.

Now let $0 < \delta < 1$. Since the function $\delta^{-1} f_k(\delta z)$ is a $G_k(z)$, we obtain from Lemma 5, making $\delta \to 1$,

LEMMA 5*a*. *Lemma 5 holds also with $f_k(z)$ instead of $G_k(z)$.*

6. Let p and q be positive integers, and let us put

$$f_{2q}^{p}(\zeta) = \chi(\zeta) = \sum_{n=1}^{\infty} c_n \zeta^n. \tag{6.1}$$

We have by Lemma 1, with $|\zeta| = \bar{\rho}$,

$$|\chi(\zeta)| \leqslant \bar{\rho}^p (1-\bar{\rho}^{2q})^{-(p/q)} = \bar{P}^p;$$

now it follows from Lemma 5*a* that the area of the χ-map of $|\zeta| \leqslant \rho$ is

$$\pi \sum_{n=1}^{\infty} n |c_n|^2 \rho^{2n} < p\pi P^{2p} - \int_{x=P^{*2p}}^{P^{2p}} x^p \,.\, 2p \arcsin\left\{\left(\frac{x}{P}\right)^{2q}\right\} d(x^p),$$

where

$$P = \rho(1-\rho^{2q})^{-(1/q)} \quad \text{and} \quad P^* = P(1+P^{4q})^{-(1/4q)},$$

and, further,

$$\begin{aligned}
\sum_{n=1}^{\infty} n |c_n|^2 \rho^{2n} &< pP^{2p} - \frac{p}{\pi} P^{2p} \int_{t=(P^*/P)^{2p}}^{1} \arcsin\{t^{q/p}\}\, dt \\
&< pP^{2p}\left\{1 - \left[\frac{1}{\pi} t \arcsin(t^{q/p})\right]_{t=(1/P)^{2p}}^{1} + \frac{1}{\pi}\frac{q}{p}\int_0^1 t^{q/p}(1-t^{2(q/p)})^{-\frac{1}{2}}\, dt\right\} \\
&= \tfrac{1}{2}p\left\{1 + \frac{1}{\sqrt{\pi}}\frac{\Gamma\{\frac{1}{2}+\frac{1}{2}(p/q)\}}{\Gamma\{1+\frac{1}{2}(p/q)\}}\right\} P^{2p} + \frac{p}{\pi}\arcsin\left\{\left(\frac{1}{P}\right)^{2q}\right\},
\end{aligned}$$

We have thus

$$\begin{aligned}
&\sum_{n=1}^{\infty} n |c_n|^2 \rho^{2n} \\
&\quad < \tfrac{1}{2}p\left\{1 + \frac{1}{\sqrt{\pi}}\frac{\Gamma\{\frac{1}{2}+\frac{1}{2}(p/q)\}}{\Gamma\{1+\frac{1}{2}(p/q)\}}\right\} P^{2p} + \frac{p}{\pi}\arcsin\left\{\left(\frac{1}{P}\right)^{2q}\right\}.
\end{aligned} \tag{6.2}$$

Now we consider, for $r \geqslant \frac{1}{2}$,

$$\frac{1}{2\pi}\int_0^{2\pi} |f_1(re^{i\phi})|^{p/q}\, d\phi.$$

By (6.1) we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |f_1(re^{i\phi})|^{p/q}\, d\phi &= \frac{1}{2\pi}\int_0^{2\pi} |f_{2q}(r^{1/2q}e^{i(\phi/2q)})|^{2p}\, d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} |f_{2q}(r^{1/2q}e^{i\psi})|^{2p}\, d\psi = \frac{1}{2\pi}\int_0^{2\pi} |\chi(r^{1/2q}e^{i\psi})|^2\, d\psi \\ &= \sum_{n=1}^{\infty} |c_n|^2 \rho^{2n},\end{aligned}$$

where

$$\rho = r^{1/2q}.$$

It follows that

$$\sum_{n=1}^{\infty} |c_n|^2 \rho^{2n} = \text{const.} + \int_{t=2^{-(1/2q)}}^{\rho} \frac{2}{t} \sum_{n=1}^{\infty} n|c_n|^2 t^{2n}\, dt,$$

and hence

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} |f_1(re^{i\phi})|^{p/q}\, d\phi \\ &\quad = \text{const.} + \int_{t=2^{-(1/2q)}}^{r^{1/2q}} \frac{2}{t} \sum_{n=1}^{\infty} n|c_n|^2 t^{2n}\, dt \\ &\quad < C\left(\frac{p}{q}\right) + \tfrac{1}{2}\left\{1 + \frac{1}{\sqrt{\pi}} \frac{\Gamma\{\frac{1}{2}+\frac{1}{2}(p/q)\}}{\Gamma\{1+\frac{1}{2}(p/q)\}}\right\} \frac{p}{q} \int_{\tau=\frac{1}{2}}^{r} \tau^{(p/q)-1}(1-\tau)^{-2(p/q)}\, d\tau,\end{aligned}$$

by (6.2); $C(p/q)$ denotes a constant depending only on p/q.

From this we obtain at once

LEMMA 6. *Suppose that $p/q > \frac{1}{2}$; then we have*[15]

$$\begin{aligned}&\overline{\lim_{r\to 1}}\,(1-r)^{2(p/q)-1} \max_{f_1 \subset S_1} \left\{\frac{1}{2\pi}\int_0^{2\pi} |f_1(re^{i\phi})|^{p/q}\, d\phi\right\} \\ &\qquad \leqslant \tfrac{1}{2}\left\{1 + \frac{1}{\sqrt{\pi}} \frac{\Gamma\{\frac{1}{2}+\frac{1}{2}(p/q)\}}{\Gamma\{1+\frac{1}{2}(p/q)\}}\right\} \frac{p/q}{2(p/q)-1}.\end{aligned}$$

[15] The maximum in question exists because S_k is a normal set of functions.

We may observe in passing that Lemma 6 holds also for every real $\lambda > \frac{1}{2}$ instead of p/q, and is therefore a sharpening of a corollary of Lemma 2. In fact, from Lemma 2 it follows only that

$$\overline{\lim_{r\to 1}}\,(1-r)^{2\lambda-1} \max_{f_1 \subset S_1} \left\{\frac{1}{2\pi}\int_0^{2\pi} |f_1(re^{i\phi})|^\lambda d\phi\right\} \leqslant \frac{\lambda}{2\lambda-1}.$$

We deduce now another consequence of Lemma 5a. We have (compare the proof of Lemma 3)

$$\frac{1}{2\pi}\int_0^{2\pi} |f_k'(\rho e^{i\phi})|^2 d\phi \leqslant e^{-1}\log^{-1}\left(\frac{1}{\rho}\right).\rho^{-2}\left\{\frac{1}{\pi}\,.\,\text{area of the } f_k\text{-map of } |z| \leqslant \sqrt{\rho}\right\}.$$

But, by Lemma 5, we have, with

$$\bar{P} = \rho^{\frac{1}{2}}(1-\rho^{\frac{1}{2}k})^{-(2/k)}, \quad \bar{P}^* = \bar{P}(1+\bar{P}^{2k})^{-(1/2k)}, \tag{6.3}$$

$$\begin{aligned}
&\frac{1}{\pi}\{\text{area of the } f_k\text{-map of } |z| \leqslant \sqrt{\rho}\} \\
&< \bar{P}^2 - \frac{2}{\pi}\int_{x=\bar{P}^*}^{\bar{P}} x \arcsin\left\{\left(\frac{x}{\bar{P}}\right)^k\right\} dx = \bar{P}^2 - \frac{\bar{P}^2}{\pi}\int_{t=(\bar{P}^*/\bar{P})^2}^1 \arcsin\,(t^{\frac{1}{2}k})\,dt \\
&< \bar{P}^2\left\{1 - \left[\frac{1}{\pi}\,t \arcsin\,(t^{\frac{1}{2}k})\right]_{t=(1/\bar{P})^2}^1 + \frac{k}{2\pi}\int_0^1 t^{\frac{1}{2}k}(1-t^k)^{-\frac{1}{2}}dt\right\} \\
&= \tfrac{1}{2}\left\{1 + \frac{1}{\sqrt{\pi}}\,\frac{\Gamma\{\frac{1}{2}+(1/k)\}}{\Gamma\{1+(1/k)\}}\right\}\bar{P}^2 + \frac{1}{\pi}\arcsin\left\{\left(\frac{1}{\bar{P}}\right)^k\right\},
\end{aligned}$$

and so

$$\frac{1}{2\pi}\int_0^{2\pi} |f_k'(\rho e^{i\phi})|^2 d\phi < \tfrac{1}{2}e^{-1}\left\{1 + \frac{1}{\sqrt{\pi}}\,\frac{\Gamma\{\frac{1}{2}+(1/k)\}}{\Gamma\{1+(1/k)\}}\right\}\left(\frac{\bar{P}}{\rho}\right)^2 + O(1-\rho),$$

from which follows, by means of the first equality in (6.3),

LEMMA 7.

$$\begin{aligned}
&\overline{\lim_{\rho\to 1}}\,(1-\rho)^{1+(4/k)} \max_{f_k \subset S_k}\left\{\frac{1}{2\pi}\int_0^{2\pi} |f_k'(\rho e^{i\phi})|^2 d\phi\right\} \\
&\qquad\qquad \leqslant 2^{-1+(4/k)}e^{-1}k^{-(4/k)}\left\{1 + \frac{1}{\sqrt{\pi}}\,\frac{\Gamma\{\frac{1}{2}+(1/k)\}}{\Gamma\{1+(1/k)\}}\right\},
\end{aligned}$$

the maximum existing for the reason given in [15].

7. We now write Lemma 4 in the form

$$(1-r)^2(2n+1)^2|a_n^{(2)}|^2 \leqslant r^{-4n-2+(2/l)}\left\{(1-r)^{1-(2/l)}\frac{1}{2\pi}\int_0^{2\pi}|f_1(r^2e^{i\psi})|^{l-1/l}d\psi\right\}$$
$$\times\left\{(1-r)^{1+(2/l)}\frac{1}{2\pi}\int_0^{2\pi}|f_{2l}'(r^{1/l}e^{i\psi})|^2d\psi\right\},$$

and obtain, by Lemmas 6 and 7, with

$$p=l-1, \quad q=l, \quad k=2l, \quad r=1-\frac{1}{2n+1},$$

$$\overline{\lim_{n\to\infty}}\{a_n^{(2)}\}^2 \leqslant 2^{-3+(2/l)}el\frac{l-1}{l-2}\left\{1+\frac{1}{\sqrt{\pi}}\frac{\Gamma\{1-(1/2l)\}}{\Gamma\{\frac{3}{2}-(1/2l)\}}\right\}\left\{1+\frac{1}{\sqrt{\pi}}\frac{\Gamma\{\frac{1}{2}+(1/2l)\}}{\Gamma\{1+(1/2l)\}}\right\}.$$

Substituting herein $l=4$ we obtain the result in (III).

8. The functions belonging to S_k^s satisfy the relation

$$\mathfrak{R}\left\{\frac{zf_k'(z)}{f_k(z)}\right\} \geqslant 0 \tag{8.1}$$

in $|z|<1$, where $\mathfrak{R}w$ stands for the real part of w.

With

$$\frac{zf_k'(z)}{f_k(z)} = 1+c_1^{(k)}z^k+c_2^{(k)}z^{2k}+\ldots, \tag{8.2}$$

we find from (F) that

$$(kn+1)a_n^{(k)} = \sum_{\mu+\nu=n} c_\mu^{(k)}a_\nu^{(k)},$$

where $c_0^{(k)}=a_0^{(k)}=1$, and hence

$$|a_n^{(k)}| \leqslant \frac{1}{nk}\sum_{\nu=0}^{n-1}|c_{n-\nu}^{(k)}||a_\nu^{(k)}|. \tag{8.3}$$

But from (8.2) and (8.1) it follows that[16]

$$|c_\kappa^{(k)}| \leqslant 2, \quad \kappa \geqslant 1,$$

and we obtain, from (8.3),

$$|a_n^{(k)}| \leqslant \frac{2}{nk}\sum_{\nu=0}^{n-1}|a_\nu^{(k)}|. \tag{8.4}$$

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[16] By a classical result of Carathéodory; compare C. Carathéodory, "Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen", *Rend. di Palermo*, 32 (1911), 193–217.

In particular,

$$|a_1^{(k)}| \leqslant \frac{2}{k}. \tag{8.5}$$

Now we have, for $\nu \geqslant 2$,

$$\left|\binom{-(2/k)}{\nu}\right| = \frac{2}{\nu k}\prod_{\mu=1}^{\nu-1}\left(1+\frac{2}{k\mu}\right) = \frac{(\nu+1)\,k}{2}\left|\binom{-(2/k)}{\nu+1}\right| - \frac{\nu k}{2}\left|\binom{-(2/k)}{\nu}\right|, \tag{8.6}$$

and this holds obviously also for $\nu = 0$ and $\nu = 1$.

If the result in IV is true for $n = 1, 2, \ldots, N-1$, it follows from (8.7) and (8.6) that

$$\begin{aligned} |a_N^{(k)}| &\leqslant \frac{2}{Nk}\sum_{\nu=0}^{N-1}\left|\binom{-(2/k)}{\nu}\right| \\ &= \frac{2}{Nk}\sum_{\nu=0}^{N-1}\left\{\frac{(\nu+1)\,k}{2}\left|\binom{-(2/k)}{\nu+1}\right| - \frac{\nu k}{2}\left|\binom{-(2/k)}{\nu}\right|\right\} = \left|\binom{-(2/k)}{N}\right|. \end{aligned}$$

Since, by (8.5), the result is true for $n = 1$, it is true by induction for all $n \geqslant 1$.

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TYPES OF ELEMENTS AND CHARACTERISTIC SUBGROUPS OF ABELIAN GROUPS

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We consider (additively written) Abelian groups $\mathfrak{A}$, the elements of which are all of finite order; two elements $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathfrak{A}$ are of the same type in $\mathfrak{A}$ [*i.e.* there exists a (proper) automorphism of $\mathfrak{A}$ mapping $\mathfrak{a}$ upon $\mathfrak{b}$], if and only if their primary components are of the same type in the respective primary component of $\mathfrak{A}$. We shall show that the type of an element of a primary group is uniquely determined by its order and the exponents (="Höhenexponent" in the sense of H. Prüfer) of its different multiples provided that $\mathfrak{A}$ is a direct sum of cyclic groups and groups of the type p^∞†, where p is the prime belonging to the primary group $\mathfrak{A}$. (§ 1.)

We apply this characterization of the types of elements to enumerate the characteristic subgroups of $\mathfrak{A}$, *i.e.* those subgroups of $\mathfrak{A}$ which are mapped on themselves by all (proper) automorphisms of $\mathfrak{A}$. Since a subgroup $\mathfrak{U}$ of $\mathfrak{A}$ is a characteristic subgroup of $\mathfrak{A}$ if and only if the primary components of $\mathfrak{U}$ are characteristic subgroups of the respective primary components of $\mathfrak{A}$, it is sufficient to consider primary groups and their characteristic subgroups.

For primary groups $\mathfrak{A}$ satisfying the above-mentioned condition, each characteristic subgroup determines uniquely a set of types of elements, and the subgroup is also uniquely determined by this set of types of elements. Each of the types of this set is defined by a certain maximum-property, but the set as a whole has a minimum-property so that in general no proper subset determines the characteristic subgroup, and, in particular, it does not contain the types of all elements of the characteristic subgroup. (§ 2.)

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† A group of the type p^∞ is isomorphic with the quotient group of the additive group of all rational numbers with regard to the subgroup of all rational numbers with a denominator prime to p.

We give necessary and sufficient conditions for a set of types of elements to determine, as above, a characteristic subgroup. (§ 3, § 4.)

A characteristic subgroup may be called regular if, for any of the types of elements determining, as above, this characteristic subgroup, the sum of exponent-order and exponent is the same for all its multiples other than 0. Thus the regular characteristic subgroups are completely determined by exponent-order and exponent of the determining set. We get all regular characteristic subgroups as join groups of groups which contain all elements of $\mathfrak{A}$ of an exponent-order not greater than a certain number, and of an exponent not less than another number; these numbers are uniquely determined by the regular characteristic subgroup considered. Finally, a subgroup is a regular characteristic subgroup, if and only if it is mapped upon itself by all proper and improper automorphisms of $\mathfrak{A}$. (§ 3.)

Irregular characteristic subgroups exist only if the prime number belonging to the primary group $\mathfrak{A}$ is 2. We give necessary and sufficient conditions for the existence of irregular characteristic subgroups. (§ 4.)

Thus we have enumerated all characteristic subgroups by means of invariants. K. Shoda† has given an enumeration of all characteristic subgroups of finite Abelian groups by determining sets of elements which generate the characteristic subgroups. But these generating sets are expressed in terms of a given basis of the group considered, and may be affected by a change of basis. Moreover, he uses the "matrix-method" of Chatelet and O. Schreier which cannot be applied in the more general cases considered by us, so that it is difficult, except in a few cases, to derive Shoda's and our results from one another.

1. *Types of elements.*

Let $\mathfrak{A}$ be an (additively written) primary Abelian group, *i.e.* the order of any element of $\mathfrak{A}$ is a power of the fixed prime number p.

Two elements $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathfrak{A}$ are said to be isotypic if there exists an automorphism‡ α such that $\alpha(\mathfrak{a}) = \mathfrak{b}$. A function $f(\mathfrak{a})$ defined for all elements $\mathfrak{a}$ of $\mathfrak{A}$ is called an invariant of $\mathfrak{a}$ if it has the same value for isotypic elements; $f(\mathfrak{a})$ is called a characteristic invariant or also the type of $\mathfrak{a}$ if $f(\mathfrak{a}) = f(\mathfrak{b})$, if and only if $\mathfrak{a}$ and $\mathfrak{b}$ are isotypic.

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† *Cp.* K. Shoda, "Über die charakteristischen Untergruppen einer endlichen Abelschen Gruppe", *Math. Zeitschrift*, 31 (1930), 611–624.

‡ In the sequel an automorphism is always a proper one unless otherwise stated.

The following notation for invariants will be used throughout:

$n(\mathfrak{a}) =$ *exponent-order of* $\mathfrak{a}$

$=$ least number greater than or equal to 0 such that $p^{n(\mathfrak{a})}\mathfrak{a} = 0$;

$h(\mathfrak{a}) =$ *exponent of* $\mathfrak{a}$

$=$ greatest number such that $p^{h(\mathfrak{a})}\mathfrak{b} = \mathfrak{a}$ has a solution $\mathfrak{b}$ in $\mathfrak{A}$.

$h(\mathfrak{a})$ is always greater than or equal to 0 and may be infinite [*i.e.* $h(\mathfrak{a}) = \infty$ if the equation $p^i\mathfrak{b} = \mathfrak{a}$ has a solution in $\mathfrak{A}$ for every $i \geqslant 0$]. Since the exponent depends on the group $\mathfrak{A}$ containing the element, we shall sometimes write $h_{\mathfrak{A}}(\mathfrak{a}) =$ exponent of $\mathfrak{a}$ in $\mathfrak{A}$.

For exponent-order and exponent the following important results are true:

Suppose that $\mathfrak{a}+\mathfrak{b} = \mathfrak{c}$. *If* $n(\mathfrak{a}) < n(\mathfrak{b})$, *then* $n(\mathfrak{c}) = n(\mathfrak{b})$; *if* $n(\mathfrak{a}) = n(\mathfrak{b})$, *then* $n(\mathfrak{c}) \leqslant n(\mathfrak{b})$; *if* $h(\mathfrak{a}) < h(\mathfrak{b})$, *then* $h(\mathfrak{c}) = h(\mathfrak{a})$; *if* $h(\mathfrak{a}) = h(\mathfrak{b})$, *then* $h(\mathfrak{c}) \geqslant h(\mathfrak{a})$.

Finally we define:

$$s(\mathfrak{a}) = \begin{cases} n(\mathfrak{a})+h(\mathfrak{a}) & \text{for finite } h(\mathfrak{a}), \\ \infty & \text{for } h(\mathfrak{a}) = \infty. \end{cases}$$

If $s(\mathfrak{a})$ is finite, it is the exponent-order of a greatest cyclic subgroup of $\mathfrak{A}$ containing $\mathfrak{a}$; and $s(\mathfrak{a})$ is infinite, if for every $i \geqslant n(\mathfrak{a})$ there exist cyclic subgroups of order p^i containing $\mathfrak{a}$.

An example of a characteristic invariant is

$c(\mathfrak{a}) =$ the class of all elements of $\mathfrak{A}$ which are isotypic with $\mathfrak{a}$.

We consider the following invariant:

$$I(\mathfrak{a}) = \{s(\mathfrak{a}),\ s(p\mathfrak{a}),\ \ldots,\ s(p^{n(\mathfrak{a})-1}\mathfrak{a})\},$$

and in this paragraph we prove that $I(\mathfrak{a})$ *is a characteristic invariant*, provided that the structure of $\mathfrak{A}$ satisfies certain conditions.

Generally $I(\mathfrak{a})$ is not a characteristic invariant, since there exist groups $\mathfrak{A}$ containing elements $\mathfrak{a}$ and $\mathfrak{b}$ both of exponent-order 1 and of exponent ∞ such that $\mathfrak{a}$ is contained in a subgroup of the type p^∞†, but $\mathfrak{b}$ is not contained in any subgroup of the type p^∞ of $\mathfrak{A}$.

Since every subgroup of $\mathfrak{A}$ containing $\mathfrak{a}$ also contains $p\mathfrak{a}$, we have, from the properties of $s(\mathfrak{a})$ mentioned above:

$$s(\mathfrak{a}) \leqslant s(p\mathfrak{a}) \leqslant \ldots \leqslant s(p^{n(\mathfrak{a})-1}\mathfrak{a}),$$

† A group of the type p^∞ is generated by an enumerable set of elements $\mathfrak{a}_1, \ldots, \mathfrak{a}_i, \ldots$ satisfying the relations $0 = p\mathfrak{a}_1, \ldots, \mathfrak{a}_i = p\mathfrak{a}_{i+1}, \ldots$; *cp.* also foot-note †, p. 481.

If, in particular, $s(\mathfrak{a}) = \ldots = s(p^{n(\mathfrak{a})-1}\mathfrak{a})$, *the element* $\mathfrak{a}$ *is said to be regular; otherwise irregular. E.g.*, all elements of order p are regular.

Finally, the numbers $d(\mathfrak{a})$ and $d(l, \mathfrak{a})$, for $0 \leqslant l \leqslant d(\mathfrak{a})$ and any $\mathfrak{a} \neq 0$, may be defined in the following way:

$$-1 = d(0, \mathfrak{a})$$

$$s(p^{d(l,\mathfrak{a})+1}\mathfrak{a}) = \ldots = s(p^{d(l+1,\mathfrak{a})}\mathfrak{a}) \quad \text{for} \quad 0 \leqslant l \leqslant d(\mathfrak{a}),$$

$$s(p^{d(l,\mathfrak{a})}\mathfrak{a}) < s(p^{d(l,\mathfrak{a})+1}\mathfrak{a}) \quad \text{for} \quad 0 \leqslant l < d(\mathfrak{a}),$$

$$n(\mathfrak{a})-1 = d\{d(\mathfrak{a}), \mathfrak{a}\}.$$

For any given $\mathfrak{a} \neq 0$ in $\mathfrak{A}$, $d(l, \mathfrak{a})$ is a monotonic increasing function of l. Thus an element $\mathfrak{a} \neq 0$ is regular if and only if $d(\mathfrak{a}) = 1$.

LEMMA 1. *Let every element of* $\mathfrak{A}$ *of exponent* ∞ *be contained in a subgroup of the type* p^∞.

Then, for any $\mathfrak{a} \neq 0$ *in* $\mathfrak{A}$, *there exist* $d(\mathfrak{a})$ *regular elements* $\mathfrak{r}_1, \ldots, \mathfrak{r}_{d(\mathfrak{a})}$ *such that*

$$\mathfrak{a} = \mathfrak{r}_1 + \ldots + \mathfrak{r}_{d(\mathfrak{a})}$$

and (using the abbreviation $\infty \pm 1 = \infty$)

$$s(\mathfrak{r}_l) = h(p^{d(l,\mathfrak{a})}\mathfrak{a})+1, \quad n(\mathfrak{r}_l) = d(l, \mathfrak{a})+1.$$

Any such representation of $\mathfrak{a}$ *as a sum of regular elements may be called a* CANONICAL *representation*† *of* $\mathfrak{a}$.

We have in particular:

$$h(\mathfrak{r}_l) = h(p^{d(l,\mathfrak{a})}\mathfrak{a}) - d(l, \mathfrak{a}) = s(p^{d(l,\mathfrak{a})}\mathfrak{a}) - n(\mathfrak{a}),$$

$$n(\mathfrak{r}_1) < n(\mathfrak{r}_2) < \ldots < n(\mathfrak{r}_{d(\mathfrak{a})}),$$

and therefore

$$h(\mathfrak{r}_1) < h(\mathfrak{r}_2) < \ldots < h(\mathfrak{r}_{d(\mathfrak{a})}),$$

$$s(\mathfrak{r}_1) < s(\mathfrak{r}_2) < \ldots < s(\mathfrak{r}_{d(\mathfrak{a})}),$$

so that $s(\mathfrak{r}_l)$ and $h(\mathfrak{r}_l)$ are finite for $l < d(\mathfrak{a})$, and

$$s(\mathfrak{r}_{d(\mathfrak{a})}) = s(p^{n(\mathfrak{a})-1}\mathfrak{a}), \quad n(\mathfrak{r}_{d(\mathfrak{a})}) = n(\mathfrak{a}).$$

Note that these inequalities are also sufficient conditions for the regular elements $\mathfrak{r}_l$ to form a canonical representation of their sum.

Remark. Without the assumption for the elements of exponent ∞, our lemma is not true for elements with multiples other than 0 of exponent ∞.

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† In general many different canonical representations of $\mathfrak{a}$ exist.

Proof. Before constructing the elements $\mathfrak{r}_l$ by complete induction with regard to $d(\mathfrak{a})$, we show that:

(1) For any $\mathfrak{a} \neq 0$ there exists a regular element $\mathfrak{r}$ such that

$$s(\mathfrak{r}) = s(p^{n(\mathfrak{a})-1}\mathfrak{a}), \quad n(\mathfrak{r}) = n(\mathfrak{a}),$$

and

$$n(\mathfrak{a}-\mathfrak{r}) = d\{d(\mathfrak{a})-1, \mathfrak{a}\}+1,$$

$$s[p^l(\mathfrak{a}-\mathfrak{r})] = s(p^l\mathfrak{a})-n(\mathfrak{a})+d\{d(\mathfrak{a})-1, \mathfrak{a}\}+1 \quad \text{for} \quad 0 \leqslant l \leqslant d\{d(\mathfrak{a})-1, \mathfrak{a}\}.$$

Let $\bar{\mathfrak{a}} = p^{d\{d(\mathfrak{a})-1, \mathfrak{a}\}+1}\mathfrak{a}$; then there exists a regular element $\mathfrak{r}$ in $\mathfrak{A}$ for which

$$p^{d\{d(\mathfrak{a})-1, \mathfrak{a}\}+1}\mathfrak{r} = \bar{\mathfrak{a}};$$

this follows for finite $h(\bar{\mathfrak{a}})$ by considering an element $\mathfrak{b}$ satisfying

$$p^{h(\bar{\mathfrak{a}})}\mathfrak{b} = \bar{\mathfrak{a}},$$

and choosing

$$\mathfrak{r} = p^{n(\mathfrak{b})-n(\mathfrak{a})}\mathfrak{b},$$

and for infinite $h(\bar{\mathfrak{a}})$ by noting that $\bar{\mathfrak{a}}$ is an element of a subgroup of $\mathfrak{A}$ of the type p^∞.

Since $\bar{\mathfrak{a}}$ is regular, we have

$$s(\mathfrak{r}) = s(\bar{\mathfrak{a}}) = s(p^{n(\bar{\mathfrak{a}})-1}\bar{\mathfrak{a}}) = s(p^{n(\mathfrak{a})-1}\mathfrak{a}),$$

and $n(\mathfrak{r}) = n(\mathfrak{a})$, since $p^{d\{d(\mathfrak{a})-1, \mathfrak{a}\}+1}\mathfrak{r} = p^{d\{d(\mathfrak{a})-1, \mathfrak{a}\}+1}\mathfrak{a}$.

Since $\mathfrak{r}$ and all its multiples are regular, while $p^{d\{d(\mathfrak{a})-1, \mathfrak{a}\}}\mathfrak{a}$ is not regular, $d\{d(\mathfrak{a})-1, \mathfrak{a}\}+1$ is the smallest number x satisfying $p^x\mathfrak{r} = p^x\mathfrak{a}$, and therefore we have

$$n(\mathfrak{a}-\mathfrak{r}) = d\{d(\mathfrak{a})-1, \mathfrak{a}\}+1.$$

(For regular $\mathfrak{a}$ this statement is obvious, since $\mathfrak{a} = \mathfrak{r}$.)

If $0 \leqslant l \leqslant d\{d(\mathfrak{a})-1, \mathfrak{r}\}$, then

$$s[p^l(\mathfrak{a}-\mathfrak{r})] = h[p^l(\mathfrak{a}-\mathfrak{r})]+n[p^l(\mathfrak{a}-\mathfrak{r})],$$

and

$$n[p^l(\mathfrak{a}-\mathfrak{r})] = n(\mathfrak{a}-r)-l = d\{d(\mathfrak{a})-1, \mathfrak{a}\}+1-l.$$

Since $\mathfrak{r}$ is regular, $s(p^l\mathfrak{a}) < s(p^l\mathfrak{r})$, and since $n(\mathfrak{a}) = n(\mathfrak{r})$, *i.e.* $n(p^l\mathfrak{a}) = n(p^l\mathfrak{r})$, we have also $h(p^l\mathfrak{a}) < h(p^l\mathfrak{r})$, and therefore $h[p^l(\mathfrak{a}-\mathfrak{r})] = h(p^l\mathfrak{a})$.

Hence

$$\begin{aligned} s[p^l(\mathfrak{a}-\mathfrak{r})] &= h(p^l\mathfrak{a})+d\{d(\mathfrak{a})-1, \mathfrak{a}\}+1-l \\ &= s(p^l\mathfrak{a})-n(p^l\mathfrak{a})+d\{d(\mathfrak{a})-1, \mathfrak{a}\}+1-l \\ &= s(p^l\mathfrak{a})-n(\mathfrak{a})+d\{d(\mathfrak{a})-1, \mathfrak{a}\}+1; \end{aligned}$$

this completes the proof of (1).

Our lemma is obviously true if $\mathfrak{a}$ is regular, *i.e.* if $d(\mathfrak{a}) = 1$. Therefore we may assume that $d(\mathfrak{a}) > 1$ and that the lemma is true for all elements $\mathfrak{b}$ with $d(\mathfrak{b}) < d(\mathfrak{a})$.

Applying (1), we find a regular element $\mathfrak{r}$ with the properties mentioned; then for $\mathfrak{b} = \mathfrak{a} - \mathfrak{r}$ we have

$$n(\mathfrak{b}) = d\{d(\mathfrak{a})-1, \mathfrak{a}\}+1,$$

$$s(p^l\mathfrak{b}) = s(p^l\mathfrak{a})-n(\mathfrak{a})+d\{d(\mathfrak{a})-1, \mathfrak{a}\}+1$$

for $$0 \leqslant l \leqslant d\{d(\mathfrak{a})-1, \mathfrak{a}\} = n(\mathfrak{b})-1.$$

Therefore $d(l, \mathfrak{b}) = d(l, \mathfrak{a})$ for $0 \leqslant l \leqslant d(\mathfrak{a})-1$ and $d(\mathfrak{b}) = d(\mathfrak{a})-1$.

Hence Lemma 1 is true for $\mathfrak{b}$ and there are therefore $d(\mathfrak{b})$ regular elements $\mathfrak{r}_1, \ldots, \mathfrak{r}_{d(\mathfrak{b})}$ in $\mathfrak{A}$ such that $\mathfrak{b} = \mathfrak{r}_1+\ldots+\mathfrak{r}_{d(\mathfrak{b})}$ and $s(\mathfrak{r}_l) = h(p^{d(l,\mathfrak{b})}\mathfrak{b})+1$, $n(\mathfrak{r}_l) = d(l, \mathfrak{b})+1$. Since $d(l, \mathfrak{b}) = d(l, \mathfrak{a})$ for $0 \leqslant l \leqslant d(\mathfrak{b}) = d(\mathfrak{a})-1$, we get

$$n(\mathfrak{r}_l) = d(l, \mathfrak{a})+1,$$

and

$$\begin{aligned} s(\mathfrak{r}_l) &= s[p^{d(l,\mathfrak{a})}\mathfrak{b}]-n[p^{d(l,\mathfrak{a})}\mathfrak{b}]+1 \\ &= s[p^{d(l,\mathfrak{a})}\mathfrak{a}]-n(\mathfrak{a})+d\{d(\mathfrak{a})-1, \mathfrak{a}\}+1-d\{d(\mathfrak{a})-1, \mathfrak{a}\}-1+d(l, \mathfrak{a})+1 \\ &= s[p^{d(l,\mathfrak{a})}\mathfrak{a}]-n(\mathfrak{a})+d(l, \mathfrak{a})+1 \\ &= h[p^{d(l,\mathfrak{a})}\mathfrak{a}]+n[p^{d(l,\mathfrak{a})}\mathfrak{a}]-n(\mathfrak{a})+d(l, \mathfrak{a})+1 \\ &= h[p^{d(l,\mathfrak{a})}\mathfrak{a}]+1. \end{aligned}$$

Hence $\mathfrak{r}_1, \ldots, \mathfrak{r}_{d(\mathfrak{a})-1}$ and $\mathfrak{r} = \mathfrak{r}_{d(\mathfrak{a})}$ are suitable regular elements in $\mathfrak{A}$, since

$$s(\mathfrak{r}) = s[p^{d\{d(\mathfrak{a}),\mathfrak{a}\}}\mathfrak{a}], \quad n(\mathfrak{r}) = n(\mathfrak{a}).$$

LEMMA 2. *Let $\mathfrak{A}$ be a direct sum of cyclic subgroups and sub-groups of the type p^∞, and suppose that $\mathfrak{A}$ contains elements other than 0 of exponent ∞.*

Then (a) every element of $\mathfrak{A}$ of exponent ∞ is contained in a subgroup of the type p^∞†, and

(b) if $\mathfrak{r}_{i1}, \ldots, \mathfrak{r}_{in}$ $(i = 1, 2)$ are regular elements of $\mathfrak{A}$ such that

$$s(\mathfrak{r}_{i1}) < \ldots < s(\mathfrak{r}_{in})$$

† If $\mathfrak{A}$ is at most enumerable, this assertion (*a*) is equivalent to our assumption, but in general this equivalence does not hold; *cp.* H. Prüfer, "Untersuchungen über die Zerlegbarkeit der abzählbaren primären Abelschen Gruppen", *Math. Zeitschrift*, 17 (1923), 35–61, in particular the theorem of § 17, p. 57, and the foot-note [6] of p. 57.

and $$s(\mathfrak{r}_{1j}) = s(\mathfrak{r}_{2j}), \quad n(\mathfrak{r}_{1j}) = n(\mathfrak{r}_{2j}),$$

then there exists an automorphism α of $\mathfrak{A}$ such that

$$\alpha(\mathfrak{r}_{11}) = \mathfrak{r}_{21}, \quad \ldots, \quad \alpha(\mathfrak{r}_{1n}) = \mathfrak{r}_{2n}.$$

Proof. By hypothesis $\mathfrak{A}$ is a direct sum:

$$\mathfrak{A} = \sum_{1 \leqslant g \leqslant \infty} \sum_{1 \leqslant \nu < \mu_g} \mathfrak{Z}_{g\nu},$$

where μ_g is a finite or infinite ordinal (possibly 1) and $\mathfrak{Z}_{g\nu}$ is for finite g a cyclical group of order p^g, while for $g = \infty$ it is a group of the type p^∞.

Since $\sum_{1 \leqslant \nu < \mu_\infty} \mathfrak{Z}_{\infty\nu}$ contains all elements of exponent ∞, our assertion (a) is evidently true.

Denote by $\bar{\mathfrak{A}}$ the group of all elements of order p in $\mathfrak{A}$, by $\bar{\mathfrak{A}}_r$ $(r = 0, 1, 2, \ldots, \infty)$ the subgroup of all elements of $\bar{\mathfrak{A}}$ of exponent not less than r.

A *basis* of $\bar{\mathfrak{A}}$ is a subset $\mathfrak{B}$ of $\bar{\mathfrak{A}}$ such that $\bar{\mathfrak{A}}$ is the direct sum of all the cycles $\{\mathfrak{b}\}$ with $\mathfrak{b}$ in $\mathfrak{B}$. If, in addition, $\mathfrak{B}$ contains a basis for each $\bar{\mathfrak{A}}_r$, we may call $\mathfrak{B}$ a *normal basis* of $\bar{\mathfrak{A}}$.

(2) *If $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ are elements of $\bar{\mathfrak{A}}$ such that $h(\mathfrak{b}_1) < \ldots < h(\mathfrak{b}_n)$, then there exists a normal basis of $\bar{\mathfrak{A}}$ containing $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$.*

If $\mathfrak{z}_{g\nu}$ is any element of order p in $\mathfrak{Z}_{g\nu}$, the set $\mathfrak{Z}$ of all these elements $\mathfrak{z}_{g\nu}$ is a normal basis of $\bar{\mathfrak{A}}$. Then there exists a finite subset $\mathfrak{S}$ of $\mathfrak{Z}$ such that $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ are contained in the subgroup $\{\mathfrak{S}\}$ of $\bar{\mathfrak{A}}$ generated by $\mathfrak{S}$. Since $\{\mathfrak{S}\}$ is finite, it is easy to construct a normal basis $\mathfrak{T}$ of $\{\mathfrak{S}\}$ containing $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$. The set $\mathfrak{B}$ consisting of all elements of $\mathfrak{T}$ together with all those elements of $\mathfrak{Z}$ which are not in $\mathfrak{S}$ is a suitable normal basis of $\bar{\mathfrak{A}}$.

Now put $\bar{\mathfrak{r}}_{ij} = p^{n(\mathfrak{r}_{ij})-1}\mathfrak{r}_{ij}$; since $\mathfrak{r}_{ij}$ is regular, $h(\bar{\mathfrak{r}}_{ij}) = s(\mathfrak{r}_{ij}) - 1$, and therefore in view of (2) there exists a normal basis $\mathfrak{B}_i$ of $\bar{\mathfrak{A}}$ containing $\bar{\mathfrak{r}}_{i1}, \ldots, \bar{\mathfrak{r}}_{in}$.

If $\mathfrak{b}$ is in $\mathfrak{B}_i$ and if $h(\mathfrak{b})$ is finite, then there exists an element $\mathfrak{a}_i(\mathfrak{b})$ with $p^{h(\mathfrak{b})}\mathfrak{a}_i(\mathfrak{b}) = \mathfrak{b}$; but, if $h(\mathfrak{b}) = \infty$, in view of (a), a sequence $\mathfrak{a}_{ij}(\mathfrak{b})$ exists such that $\mathfrak{a}_{i1}(\mathfrak{b}) = \mathfrak{b}$, $p\mathfrak{a}_{i,j+1}(\mathfrak{b}) = \mathfrak{a}_{ij}(\mathfrak{b})$. It is always possible to perform this construction in such a way that

$$p^{h(\bar{\mathfrak{r}}_{ij})-n(\mathfrak{r}_{ij})+1}\mathfrak{a}_i(\bar{\mathfrak{r}}_{ij}) = \mathfrak{r}_{ij}, \text{ if } h(\bar{\mathfrak{r}}_{ij}) \text{ is finite},$$

and such that

$$\mathfrak{a}_{i,\,n(\mathfrak{r}_{ij})}(\bar{\mathfrak{r}}_{ij}) = \mathfrak{r}_{ij}, \text{ if } h(\bar{\mathfrak{r}}_{ij}) \text{ is infinite}.$$

Now it is easy to prove that

$$\mathfrak{A} = \sum_{\substack{\mathfrak{b} \text{ in } \mathfrak{B}_i \\ h(\mathfrak{b}) < \infty}} \{\mathfrak{a}_i(\mathfrak{b})\} + \sum_{\substack{\mathfrak{b} \text{ in } \mathfrak{B}_i \\ h(\mathfrak{b}) = \infty}} \{\mathfrak{a}_{i1}(\mathfrak{b}), \ldots, \mathfrak{a}_{ij}(\mathfrak{b}), \ldots\}.$$

Since $\mathfrak{B}_i$ is a normal basis, there exists a one-one representation β of $\mathfrak{B}_1$ upon $\mathfrak{B}_2$ such that

$$\beta(\mathfrak{B}_1) = \mathfrak{B}_2, \quad h[\beta(\mathfrak{b})] = h(\mathfrak{b}), \quad \beta(\bar{\mathfrak{r}}_{1j}) = \bar{\mathfrak{r}}_{2j}.$$

We find a suitable automorphism α by the following rule:

$$\alpha[\mathfrak{a}_1(\mathfrak{b})] = \mathfrak{a}_2[\beta(\mathfrak{b})] \text{ for any } \mathfrak{b} \text{ in } \mathfrak{B}_1 \text{ with } h(\mathfrak{b}) \neq \infty,$$

and $$\alpha[\mathfrak{a}_{1j}(\mathfrak{b})] = \mathfrak{a}_{2j}[\beta(\mathfrak{b})] \text{ for any } \mathfrak{b} \text{ in } \mathfrak{B}_1 \text{ with } h(\mathfrak{b}) = \infty.$$

Hence Lemma 2 is proved.

From now on, we assume that $\mathfrak{A}$ is a direct sum of cyclic subgroups and subgroups of the type p^∞; it may be mentioned that usually, but not always, we require only the two assertions (a) and (b) of Lemma 2.

THEOREM 1. *The elements $\mathfrak{a}^{(1)}$ and $\mathfrak{a}^{(2)}$ of $\mathfrak{A}$ are isotypic if, and only if, $I(\mathfrak{a}^{(1)}) = I(\mathfrak{a}^{(2)})$.*

Proof. We have only to show that $\mathfrak{a}^{(1)}$ and $\mathfrak{a}^{(2)}$ are isotypic if

$$I(\mathfrak{a}^{(1)}) = I(\mathfrak{a}^{(2)}).$$

In view of Lemma 2 (a), it follows from Lemma 1 that there exist $d(\mathfrak{a}^{(i)})$ regular elements $\mathfrak{r}_l^{(i)}$ such that

$$\mathfrak{a}^{(i)} = \mathfrak{r}_1^{(i)} + \ldots + \mathfrak{r}_{d(\mathfrak{a}^{(i)})}^{(i)},$$

$$s(\mathfrak{r}_l^{(i)}) = h(p^{d(l,\,\mathfrak{a}^{(i)})}\mathfrak{a}^{(i)}) + 1, \quad n(\mathfrak{r}_l^{(i)}) = d(l,\,\mathfrak{a}^{(i)}) + 1.$$

Since $I(\mathfrak{a}^{(1)}) = I(\mathfrak{a}^{(2)})$, we have

$$d(\mathfrak{a}^{(1)}) = d(\mathfrak{a}^{(2)}), \quad d(l,\,\mathfrak{a}^{(1)}) = d(l,\,\mathfrak{a}^{(2)}),$$

and also $$n(\mathfrak{r}_l^{(1)}) = n(\mathfrak{r}_l^{(2)}), \quad s(\mathfrak{r}_l^{(1)}) = s(\mathfrak{r}_l^{(2)}).$$

Therefore, by Lemma 2 (b), there exists an automorphism α of $\mathfrak{A}$ such that

$$\alpha(\mathfrak{r}_1^{(1)}) = \mathfrak{r}_1^{(2)}, \quad \ldots, \quad \alpha(\mathfrak{r}_{d(\mathfrak{a}^{(1)})}^{(1)}) = \mathfrak{r}_{d(\mathfrak{a}^{(2)})}^{(2)},$$

and so we have $$\alpha(\mathfrak{a}^{(1)}) = \mathfrak{a}^{(2)},$$

i.e. $\mathfrak{a}^{(1)}$ and $\mathfrak{a}^{(2)}$ are isotypic.

To find the types of all elements contained in the given group $\mathfrak{A}$, we consider sets

$$F = \{f_0, \ldots, f_t\},$$

where the f_l are either positive integers or ∞, such that

$$d(0, F) = -1,$$

$$f_{d(l,F)+1} = \ldots = f_{d(l+1,F)} \quad \text{for} \quad 0 \leqslant l \leqslant d(F),$$

$$f_{d(l,F)} < f_{d(l,F)+1} \quad \text{for} \quad 0 \leqslant l < d(F),$$

$$d\{d(F), F\} = t,$$

and we have to find conditions for the existence of elements of the type F.

THEOREM 2. *There exist elements of the type F in $\mathfrak{A}$ if and only if*

(1) $1+t \leqslant f_0$,

(2) $\mathfrak{A}$, *when expressed as a direct sum, has at least one component of the type*† $p^{f_{d(l,F)}-t+d(l,F)}$ *for* $0 < l \leqslant t$.

Proof. A. Suppose that there exists an element $\mathfrak{a}$ in $\mathfrak{A}$ such that $I(\mathfrak{a}) = F$, *i.e.* $n(\mathfrak{a})-1 = t$, $s(p^l\mathfrak{a}) = f_l$ for $0 \leqslant l \leqslant t$, and therefore $d(\mathfrak{a}) = d(F)$, $d(l, \mathfrak{a}) = d(l, F)$.

Then $f_0 = s(\mathfrak{a}) = n(\mathfrak{a})+h(\mathfrak{a}) \geqslant n(\mathfrak{a}) = t+1$, and so (1) holds.

To prove the necessity of (2), we consider a canonical representation of $\mathfrak{a}$ as a sum of regular elements $\mathfrak{r}_l$; then, by Lemma 1, $\mathfrak{r}_l$ belongs to a direct component of the type $p^{f_{d(l,F)}-t+d(l,F)}$.

B. If (1) and (2) hold, we have $1+t \leqslant f_0 \leqslant f_{d(l,F)}$ for $0 < l \leqslant d(F)$, and therefore there exist regular elements $\mathfrak{r}_1, \ldots, \mathfrak{r}_{d(F)}$ in $\mathfrak{A}$ such that

$$s(\mathfrak{r}_l) = f_{d(l,F)}-t+d(l, F), \quad n(\mathfrak{r}_l) = d(l, F)+1,$$

and $\mathfrak{a} = \mathfrak{r}_1+\ldots+\mathfrak{r}_{d(F)}$ is an element with $I(\mathfrak{a}) = F$.

2. *Invariants of characteristic subgroups.*

If $\mathfrak{a}$ is any element of $\mathfrak{A}$ other than 0, then we define

$$g(\mathfrak{a}) = \textit{genus of } \mathfrak{a} = h(p^{n(\mathfrak{a})-1}\mathfrak{a}).$$

† A group of the type p^i is always a cyclic group of order p^i, if i is finite.

Now consider a characteristic subgroup $\mathfrak{C}$ of $\mathfrak{A}$, *i.e.* a subgroup composed of complete classes of isotypic elements. If $\mathfrak{A}$ contains elements of genus l, then we write

$$[\mathfrak{C}, l] = \text{maximum exponent-order of elements of genus } l \text{ in } \mathfrak{C}.$$

It is possible that $[\mathfrak{C}, \infty] = \infty$, namely if the exponent-orders of the elements of genus ∞ in $\mathfrak{C}$ are not bounded. In any case, we have

$$0 \leqslant [\mathfrak{C}, l] \leqslant l+1.$$

Now suppose that $[\mathfrak{C}, l]$ is finite and let $\mathfrak{a}$ and $\mathfrak{b}$ be two elements of genus l and exponent-order $[\mathfrak{C}, l]$. Then we shall write

$$I(\mathfrak{a}) < I(\mathfrak{b})$$

if $s(p^j\mathfrak{a}) = s(p^j\mathfrak{b})$ for $k < j \leqslant [\mathfrak{C}, l]-1 = n(\mathfrak{a})-1 = n(\mathfrak{b})-1$, but

$$s(p^k\mathfrak{a}) < s(p^k\mathfrak{b}).$$

We have to show that *by our definition the types of the elements in* $\mathfrak{C}$ *of genus* l *and (finite) exponent-order* $[\mathfrak{C}, l]$ *become an ordered set.*

I. It is obvious that, if $\mathfrak{a}$ and $\mathfrak{b}$ are two elements of genus l and exponent-order $[\mathfrak{C}, l]$, either $I(\mathfrak{a}) < I(\mathfrak{b})$ or $I(\mathfrak{a}) = I(\mathfrak{b})$ or $I(\mathfrak{a}) > I(\mathfrak{b})$.

II. Let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ be three elements of genus l and exponent-order $[\mathfrak{C}, l]$ such that $I(\mathfrak{a}) < I(\mathfrak{b})$ and $I(\mathfrak{b}) < I(\mathfrak{c})$; then there exist two numbers a and b such that

$$s(p^j\mathfrak{a}) = s(p^j\mathfrak{b}) \quad \text{for} \quad a < j \leqslant [\mathfrak{C}, l]-1,$$

$$s(p^a\mathfrak{a}) < s(p^a\mathfrak{b}),$$

$$s(p^j\mathfrak{b}) = s(p^j\mathfrak{c}) \quad \text{for} \quad b < j \leqslant [\mathfrak{C}, l]-1,$$

$$s(p^b\mathfrak{b}) < s(p^b\mathfrak{c}),$$

and hence

$$s(p^j\mathfrak{a}) = s(p^j\mathfrak{b}) = s(p^j\mathfrak{c}) \quad \text{for} \quad \max[a, b] < j \leqslant [\mathfrak{C}, l]-1,$$

$$s(p^{\max[a, b]}\mathfrak{a}) < s(p^{\max[a, b]}\mathfrak{c}),$$

i.e. $I(\mathfrak{a}) < I(\mathfrak{c})$.

THEOREM 3. (*a*) *If* $[\mathfrak{C}, \infty] = \infty$, *then any regular element of genus infinity is contained in a subgroup of* $\mathfrak{C}$ *of the type* p^∞, *and any subgroup of* $\mathfrak{A}$ *of the type* p^∞ *is contained in* $\mathfrak{C}$.

(*b*) *If* $[\mathfrak{C}, l]$ *is finite, then there exists a* (*uniquely determined*) *greatest type in the ordered set of the types of elements in* $\mathfrak{C}$ *of genus* l *and exponent-order* $[\mathfrak{C}, l]$: *the maximum type* $\mathrm{T}(\mathfrak{C}, l)$ *of genus* l *in* $\mathfrak{C}$†.

(*c*) *If* $p \neq 2$ *the maximum type is always the type of a regular element.*

We require first the following result:

(3) If $\mathfrak{a}$ is any element of the characteristic subgroup $\mathfrak{C}$ of $\mathfrak{A}$, then there exists in $\mathfrak{C}$ a regular element $\mathfrak{r}$ with $g(\mathfrak{r}) = g(\mathfrak{a})$ and

$$n(\mathfrak{r}) = \begin{cases} n(\mathfrak{a}) & \text{for} \quad p \neq 2, \\ n(\mathfrak{a})-1 & \text{for} \quad p = 2. \end{cases}$$

By Lemma 1, § 1, there exists a canonical representation of $\mathfrak{a}$ as a sum of regular elements

$$\mathfrak{a} = \mathfrak{r}_1 + \ldots + \mathfrak{r}_{d(\mathfrak{a})},$$

with $s(\mathfrak{r}_1) < \ldots < s(\mathfrak{r}_{d(\mathfrak{a})})$ and $n(\mathfrak{r}_{d(\mathfrak{a})}) = n(\mathfrak{a})$. By Lemma 2 (*b*), § 1, there exists an automorphism α of $\mathfrak{A}$ such that

$$\alpha(\mathfrak{r}_j) = \begin{cases} \mathfrak{r}_j & \text{for} \quad j \neq d(\mathfrak{a}), \\ 2\mathfrak{r}_{d(\mathfrak{a})} & \text{for} \quad j = d(\mathfrak{a}) \quad (p \neq 2), \\ 3\mathfrak{r}_{d(\mathfrak{a})} & \text{for} \quad j = d(\mathfrak{a}) \quad (p = 2), \end{cases}$$

since $2\mathfrak{r}_{d(\mathfrak{a})}$ and $3\mathfrak{r}_{d(\mathfrak{a})}$ respectively are isotypic with $\mathfrak{r}_{d(\mathfrak{a})}$.

Hence $$\alpha(\mathfrak{a}) = \mathfrak{a} + \begin{cases} \mathfrak{r}_{d(\mathfrak{a})} & \text{for} \quad p \neq 2, \\ 2\mathfrak{r}_{d(\mathfrak{a})} & \text{for} \quad p = 2, \end{cases}$$

and, since $\mathfrak{a}$ and $\alpha(\mathfrak{a})$ are in $\mathfrak{C}$, $\alpha(\mathfrak{a}) - \mathfrak{a}$ is also in $\mathfrak{C}$ and (3) is proved.

(4) *If there are elements of genus* l *and of genus* $k < l$ *in* $\mathfrak{A}$, *then there are* REGULAR *elements of genus* l *and exponent-order* $[\mathfrak{C}, k]$ *in the characteristic subgroup* $\mathfrak{C}$ *of* $\mathfrak{A}$.

There exists in $\mathfrak{A}$ a regular element $\mathfrak{r}$ of genus l and exponent-order $[\mathfrak{C}, k]$ since $[\mathfrak{C}, k] \leqslant k+1 < l+1$; moreover, there exists an element $\mathfrak{a}$ of genus k and exponent-order $[\mathfrak{C}, k]$ in $\mathfrak{C}$, since $k < l$ is finite and therefore $[\mathfrak{C}, k]$ is finite.

We consider the element $\mathfrak{r} + \mathfrak{a} = \mathfrak{b}$.

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† Notice that $\mathrm{T}(\mathfrak{C}, l)$, like $[\mathfrak{C}, l]$, is defined only if there exist elements of genus l in $\mathfrak{A}$.

Since $\mathfrak{a}$ and $\mathfrak{r}$ are of different genus they are linearly independent, *i.e.* the only common element of the cyclic groups $\{\mathfrak{a}\}$ and $\{\mathfrak{r}\}$ is 0 and, since $n(\mathfrak{a}) = n(\mathfrak{r}) = [\mathfrak{C}, k]$, we have also $n(\mathfrak{b}) = [\mathfrak{C}, k]$.

If l is finite and $0 \leqslant i \leqslant [\mathfrak{C}, k]-1$, we have

$$\begin{aligned} h(p^i\mathfrak{r}) &= s(p^i\mathfrak{r})-n(p^i\mathfrak{r}) \\ &= s(p^{n(\mathfrak{r})-1}\mathfrak{r})-n(\mathfrak{r})+i, \text{ since } \mathfrak{r} \text{ is regular,} \\ &= g(\mathfrak{r})+1-n(\mathfrak{r})+i \\ &= l+1-[\mathfrak{C}, k]+i \\ &> k+1-[\mathfrak{C}, k]+i = g(\mathfrak{a})+1-[\mathfrak{C}, k]+i \\ &= s(p^{n(\mathfrak{a})-1}\mathfrak{a})-n(p^i\mathfrak{a}) \\ &\geqslant s(p^i\mathfrak{a})-n(p^i\mathfrak{a}) = h(p^i\mathfrak{a}); \end{aligned}$$

while, if $l = \infty$, then $h(p^i\mathfrak{r}) = \infty$ also. Thus we always have

$$h(p^i\mathfrak{r}) > h(p^i\mathfrak{a}),$$

and therefore $h(p^i\mathfrak{a}) = h(p^i\mathfrak{b})$, *i.e.* $I(\mathfrak{a}) = I(\mathfrak{b})$. Since $\mathfrak{a}$ is in $\mathfrak{C}$, it follows from Theorem 1, § 1, that $\mathfrak{b}$ is in $\mathfrak{C}$, *i.e.* $\mathfrak{r}$ is in $\mathfrak{C}$ and (4) is proved.

Proof of Theorem 3. Suppose first that there exist elements of genus ∞ in $\mathfrak{A}$ and that $[\mathfrak{C}, \infty] = \infty$. Then, for any given integer $i \geqslant 0$, there exist in $\mathfrak{C}$ elements of genus ∞ and of exponent-order $i+1$; therefore, by (3), there also exist in $\mathfrak{C}$ regular elements of genus ∞ and exponent-order i.

In particular there exists in $\mathfrak{C}$ an element $\mathfrak{a}_0$ of order p and genus ∞. By Lemma 2(a), § 1, the element $\mathfrak{a}_0$ is contained in a subgroup of $\mathfrak{A}$ of the type p^∞ generated, say, by the elements $\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_i, \ldots$ with

$$0 = p\mathfrak{a}_0, \quad \ldots, \quad \mathfrak{a}_i = p\mathfrak{a}_{i+1}, \quad \ldots .$$

Since $\mathfrak{a}_i$ is a regular element of genus ∞ and exponent-order $i+1$ and since we have shown that some elements of this type are contained in $\mathfrak{C}$, it follows from Theorem 1, § 1, that the elements $\mathfrak{a}_i$ are also contained in $\mathfrak{C}$ and so (a) is proved.

Suppose now that l and $[\mathfrak{C}, l]$ are finite; then the number of different types of elements of $\mathfrak{A}$ of genus l and exponent-order $[\mathfrak{C},l]$ is finite, in fact it is less than or equal to $(l+1)^{[\mathfrak{C}, l]}$, since $I(\mathfrak{a})$ is, from Theorem 1, § 1, a characteristic invariant; therefore the ordered set of the types of elements of $\mathfrak{C}$ of genus l and exponent-order $[\mathfrak{C}, l]$ is also finite and consequently contains a greatest type.

Now suppose that elements of genus infinity exist in $\mathfrak{A}$ and that $[\mathfrak{C}, \infty]$ is finite. Then we consider the set M of all types of elements of genus ∞ and exponent order $[\mathfrak{C}, \infty]$ in $\mathfrak{C}$. If there are regular elements† among these they would be elements of the maximum type, since any type of regular element is greater than any type of irregular element. Therefore we may assume that no regular type is contained in M.

Among the elements of a given type belonging to M, there is an $\mathfrak{a}$ such that $d\{d(\mathfrak{a})-1, \mathfrak{a}\}$ takes its least value, say m.

Therefore among the elements in $\mathfrak{C}$ of genus ∞ and exponent-order $[\mathfrak{C}, \infty]$, there is an $\mathfrak{a}$ with $s(p^{m+1}\mathfrak{a}) = \infty$; but $s(p^m\mathfrak{b})$ is finite for all $\mathfrak{b}$ in $\mathfrak{C}$ with $g(\mathfrak{b}) = \infty$, and $n(\mathfrak{b}) = [\mathfrak{C}, \infty]$. We now prove

(5) Among the numbers $s(p^m\mathfrak{a})$ with $\mathfrak{a}$ in $\mathfrak{C}$, $g(\mathfrak{a}) = \infty$, $n(\mathfrak{a}) = [\mathfrak{C}, \infty]$, and $s(p^{m+1}\mathfrak{a}) = \infty$, there is a greatest one, say v.

For suppose that the assertion is not true; then a sequence $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_i, \ldots$ exists, such that $\mathfrak{a}_i$ is an element of $\mathfrak{C}$ with $g(\mathfrak{a}_i) = \infty$, $n(\mathfrak{a}_i) = [\mathfrak{C}, \infty]$, $s(p^{m+1}\mathfrak{a}_i) = \infty$, and $h(p^m\mathfrak{a}_1) < h(p^m\mathfrak{a}_2) < \ldots < h(p^m\mathfrak{a}_i) < \ldots$.

From Lemma 1, § 1, there exists a canonical representation of $\mathfrak{a}_i$ as a sum of regular elements, say

$$\mathfrak{a}_i = \mathfrak{r}_{1i} + \ldots + \mathfrak{r}_{d(\mathfrak{a}_i)i},$$

such that, in particular,

$$n(\mathfrak{r}_{d(\mathfrak{a}_i)i}) = [\mathfrak{C}, \infty],\ g(\mathfrak{r}_{d(\mathfrak{a}_i)i}) = \infty,\ n(\mathfrak{r}_{d(\mathfrak{a}_i)-1,i}) = m+1,\ g(\mathfrak{r}_{d(\mathfrak{a}_i)-1,i}) = h(p^m\mathfrak{a}_i) < \infty.$$

Without loss of generality we may assume that $\mathfrak{r}_{ji} = \mathfrak{r}_{j^*i^*}$ if and only if $I(\mathfrak{r}_{ji}) = I(\mathfrak{r}_{j^*i^*})$; for if $\mathfrak{a}_i$ is in $\mathfrak{C}$ so also is $\alpha_i(\mathfrak{a}_i)$ for any automorphism α_i of $\mathfrak{A}$, and we may apply Lemma 2 (b), § 1, successively to $\mathfrak{a}_2$ [with regard to $\mathfrak{a}_1$], to $\mathfrak{a}_3$ [with regard to $\mathfrak{a}_1$, $\mathfrak{a}_2$ ($\mathfrak{a}_2$)], and so on. Then we have, in particular, $\mathfrak{r}_{d(\mathfrak{a}_1)1} = \mathfrak{r}_{d(\mathfrak{a}_2)2} = \ldots = \mathfrak{r}_{d(\mathfrak{a}_i)i} = \ldots$, while all the elements $\mathfrak{r}_{d(\mathfrak{a}_i)-1,i}$ are different.

Therefore the element $\mathfrak{a}_i - \mathfrak{a}_1$ $(i > 1)$ is given at once as a sum of regular elements of different types, and so

$$n(\mathfrak{a}_i - \mathfrak{a}_1) = m+1 < [\mathfrak{C}, \infty], \quad g(\mathfrak{a}_i - \mathfrak{a}_1) = h(p^m\mathfrak{a}_1).$$

Hence, by (4), $m+1 \leqslant [\mathfrak{C}, h(p^m\mathfrak{a}_i)]$ for $i > 1$.

If there are regular elements of genus $h(p^m\mathfrak{a}_i)$ and exponent order $m+1$ in $\mathfrak{C}$, then $\mathfrak{r}_{d(\mathfrak{a}_i)-1i}$ is in $\mathfrak{C}$ and $\mathfrak{b}_i = \mathfrak{a}_i - \mathfrak{r}_{d(\mathfrak{a}_i)-1i}$ has genus ∞ and exponent-

† From (3) this always occurs if $p \neq 2$, but we do not need to use it.

order $[\mathfrak{C}, \infty]$, while

$$d\{d(\mathfrak{b}_i)-1, \mathfrak{b}_i\} = d\{d(\mathfrak{a}_i)-2, \mathfrak{a}_i\} < m = d\{d(\mathfrak{a}_i)-1, \mathfrak{a}_i\},$$

which is impossible by the definition of m, since $\mathfrak{b}_i$ is in $\mathfrak{C}$. Therefore from (3) we have $[\mathfrak{C}, h(p^m\mathfrak{a}_i)]-1 < m+1 \leqq [\mathfrak{C}, h(p^m\mathfrak{a}_i)]$ for $i>1$, *i.e.*

$$m+1 = [\mathfrak{C}, h(p^m\mathfrak{a}_i)].$$

From (4), for $i>1$, there exist regular elements of genus $h(p^m\mathfrak{a}_i)$ and exponent-order $[\mathfrak{C}, h(p^m\mathfrak{a}_{i+1})]$ in $\mathfrak{C}$, because $h(p^m\mathfrak{a}_i) < h(p^m\mathfrak{a}_{i+1})$ and $[\mathfrak{C}, h(p^m\mathfrak{a}_i)] = m+1 = [\mathfrak{C}, h(p^m\mathfrak{a}_{i+1})]$; hence $\mathfrak{r}_{d(\mathfrak{a})-1i}$ would also be in $\mathfrak{C}$, which, as we have seen, is not the case. By this contradiction (5) is proved.

Since there exists only a finite number of different types of elements $\mathfrak{a}$ such that $g(\mathfrak{a})=\infty$, $n(\mathfrak{a})=[\mathfrak{C}, \infty]$, $h(p^{m+1}\mathfrak{a})=\infty$, $h(p^m\mathfrak{a})=v$, certainly not more than $(v+[\mathfrak{C}, \infty]-m)^{[\mathfrak{C}, \infty]}$, and, since no types of elements of $\mathfrak{C}$ of genus ∞ and exponent-order $[\mathfrak{C}, \infty]$ are greater than these types (occurring in $\mathfrak{C}$), elements of a maximum type of genus ∞ must exist in $\mathfrak{C}$.

Finally (*c*) is an immediate consequence of (3), and so Theorem 3 is completely proved.

THEOREM 4. *If $\mathfrak{C}^{(1)}$ and $\mathfrak{C}^{(2)}$ are two characteristic subgroups of $\mathfrak{A}$ for which*

$$[\mathfrak{C}^{(1)}, l] = [\mathfrak{C}^{(2)}, l] \text{ for all } l \text{ (such that elements of the genus } l \text{ exist in } \mathfrak{A})$$

and $$\mathrm{T}(\mathfrak{C}^{(1)}, l) = \mathrm{T}(\mathfrak{C}^{(2)}, l) \text{ if } [\mathfrak{C}^{(i)}, l] \text{ is finite and not } 0,$$

then $$\mathfrak{C}^{(1)} = \mathfrak{C}^{(2)}.$$

This means that the characteristic subgroups are completely determined by the maximum exponent-orders and the maximum types of element of the various genera.

From Theorem 1, § 1, it is sufficient to prove

THEOREM 4*a*. *Any characteristic subgroup $\mathfrak{C}$ is generated by the set $\mathfrak{G}(\mathfrak{C})$ containing all elements $\mathfrak{a}$ with either $I(\mathfrak{a}) = \mathrm{T}\{\mathfrak{C}, g(\mathfrak{a})\}$, if $[\mathfrak{C}, g(\mathfrak{a})]$ is $\neq 0$ and finite, or $g(\mathfrak{a}) = h(\mathfrak{a}) = \infty$, if $[\mathfrak{C}, g(\mathfrak{a})] = \infty$.*

Proof. When $\mathfrak{C} = \{0\}$, there is nothing to prove. Hence we may assume that $\mathfrak{C} \neq \{0\}$.

Now consider a given element $\mathfrak{a}$ not 0 in $\mathfrak{C}$ with $g(\mathfrak{a}) = l$; and choose in $\mathfrak{C}$ an element $\mathfrak{m}$ such that $g(\mathfrak{m}) = l$ and either $I(\mathfrak{m}) = \mathrm{T}(\mathfrak{C}, l)$, if $[\mathfrak{C}, l]$ is finite, or else $\mathfrak{m}$ is regular with $h(\mathfrak{m}) = \infty$, $n(\mathfrak{m}) = n(\mathfrak{a})$, if $l = \infty$ and $[\mathfrak{C}, \infty] = \infty$.

From Lemma 1, § 1, there exist canonical representations of $\mathfrak{a}$ and $\mathfrak{m}$, say

$$\mathfrak{a} = \mathfrak{a}_1 + \ldots + \mathfrak{a}_{d(\mathfrak{a})} \quad \text{with} \quad g(\mathfrak{a}_{d(\mathfrak{a})}) = l, \quad n(\mathfrak{a}_{d(\mathfrak{a})}) = n(\mathfrak{a}),$$

$$\mathfrak{m} = \mathfrak{m}_1 + \ldots + \mathfrak{m}_{d(\mathfrak{m})} \quad \text{with} \quad g(\mathfrak{m}_{d(\mathfrak{m})}) = l, \quad n(\mathfrak{m}_{d(\mathfrak{m})}) = n(\mathfrak{m}).$$

Therefore $$I(\mathfrak{a}_{d(\mathfrak{a})}) = I(p^{n(\mathfrak{m})-n(\mathfrak{a})}\mathfrak{m}).$$

From Lemma 2 (b), § 1, there exists an automorphism α of $\mathfrak{A}$ such that $\alpha(p^{n(\mathfrak{m})-n(\mathfrak{a})}\mathfrak{m}_i) = \mathfrak{a}_j$, if and only if $I(p^{n(\mathfrak{m})-n(\mathfrak{a})}\mathfrak{m}_i) = I(\mathfrak{a}_j)$.

Then we have

$$\mathfrak{a} - \alpha(p^{n(\mathfrak{m})-n(\mathfrak{a})}\mathfrak{m}) = \mathfrak{a} - p^{n(\mathfrak{m})-n(\mathfrak{a})}\alpha(\mathfrak{m}) = \sum_{i=1}^{d(\mathfrak{a})-1}{}' \mathfrak{a}_i - \sum_{j=1}^{d(\mathfrak{m})-1}{}' p^{n(\mathfrak{m})-n(\mathfrak{a})}\alpha(\mathfrak{m}_j),$$

where Σ' means that pairs of isotypic elements $\mathfrak{a}_i$, $p^{n(\mathfrak{m})-n(\mathfrak{a})}\alpha(\mathfrak{m}_j)$ are omitted.

If $\mathfrak{a} \neq \alpha(p^{n(\mathfrak{m})-n(\mathfrak{a})}\mathfrak{m})$, let k be the greatest value such that elements of genus k appear in one of the two Σ'. Then

$$l > k = g[\mathfrak{a} - \alpha(p^{n(\mathfrak{m})-n(\mathfrak{a})}\mathfrak{m})]$$

and in particular k is finite. Since $\alpha(\mathfrak{m})$ is an element of the maximum type of genus l in $\mathfrak{C}$ or an element of exponent ∞ respectively, we have proved that:

An element of genus l in $\mathfrak{C}$ is either a multiple of an element of the maximum type of genus l in $\mathfrak{C}$ or of an element of exponent ∞ in $\mathfrak{C}$ respectively, or it is the sum of such a multiple and an element of a genus less than l.

From this it follows by complete induction (with regard to the genus) that all elements of finite genus in $\mathfrak{C}$ are generated by the elements of maximum type of a finite genus in $\mathfrak{C}$ and hence that $\mathfrak{C}$ is generated by the elements of $\mathfrak{G}(\mathfrak{C})$.

We have proved, in particular, the

COROLLARY. *If $\mathfrak{C}$ is a characteristic subgroup other than $\{0\}$ of $\mathfrak{A}$, and m is the least value such that elements of genus m are contained in $\mathfrak{C}$, then any element of the genus m in $\mathfrak{C}$ is a multiple of an element of the maximum type of genus m in $\mathfrak{C}$, or (for $m = [\mathfrak{C}, m] = \infty$) is an element of exponent ∞.*

3. *Regular characteristic subgroups.*

A characteristic subgroup $\mathfrak{C}$ is said to be *regular* if, for any value l such that elements of genus l exist in $\mathfrak{A}$ and $0 < [\mathfrak{C}, l] < \infty$, the maximum type $\mathrm{T}(\mathfrak{C}, l)$ of elements of genus l in $\mathfrak{C}$ is regular.

Since the type of a regular element is completely determined by its exponent-order and its genus, from Theorem 4, § 2, there follows

THEOREM 4 (r). *If $\mathfrak{C}^{(1)}$ and $\mathfrak{C}^{(2)}$ are two regular characteristic subgroups of $\mathfrak{A}$ such that $[\mathfrak{C}^{(1)}, l] = [\mathfrak{C}^{(2)}, l]$ for all l (for which elements of genus l exist in $\mathfrak{A}$), then $\mathfrak{C}^{(1)} = \mathfrak{C}^{(2)}$.*

This means that the regular characteristic subgroups are completely determined by the maximum exponent-order of the various genera.

THEOREM 5. *Let $e(l)$ be a non-negative integer or infinity defined for all l for which elements of genus l exist in $\mathfrak{A}$.*

There exists a regular characteristic subgroup $\mathfrak{C}$ of $\mathfrak{A}$ such that $[\mathfrak{C}, l] = e(l)$ for these values of l, if and only if

$$(1) \quad e(l) \leqq l+1,$$

$$(2) \quad e(l) \leqq e(k) \leqq e(l)+k-l \quad \text{for} \quad l < k\dagger.$$

Proof. (A) Assume the existence of such a group $\mathfrak{C}$. Then (1) is true because $n(\mathfrak{a}) \leqq g(\mathfrak{a})+1$ for every $\mathfrak{a}$ in $\mathfrak{A}$.

If $l < k$, and if elements of genus l as well as of genus k exist in $\mathfrak{A}$, then $[\mathfrak{C}, l] \leqq [\mathfrak{C}, k]$ is an immediate consequence of (4), §2, and therefore $e(l) \leqq e(k)$.

Finally we consider an element $\mathfrak{a}$ of $\mathfrak{A}$ with $g(\mathfrak{a}) = l$, $n(\mathfrak{a}) = l+1$, and an element $\mathfrak{b}$ of $\mathfrak{A}$ with $g(\mathfrak{b}) = k$, $n(\mathfrak{b}) = k+1$, and suppose that k is finite. Since $n(\mathfrak{b}) > n(\mathfrak{a})$, we have

$$n(\mathfrak{a}+\mathfrak{b}) = n(\mathfrak{b}) \quad \text{and} \quad g(\mathfrak{a}+\mathfrak{b}) = g(\mathfrak{b}),$$

because

$$p^{n(\mathfrak{a}+\mathfrak{b})-1}(\mathfrak{a}+\mathfrak{b}) = p^{n(\mathfrak{b})-1}\mathfrak{b}.$$

Therefore $\mathfrak{b}$ and $\mathfrak{a}+\mathfrak{b}$ are isotypic from Theorem 1, § 1, and so there exists an automorphism α such that $\alpha(\mathfrak{b}) = \mathfrak{a}+\mathfrak{b}$.

Since $\mathfrak{r} = p^{k+1-[\mathfrak{C}, k]}\mathfrak{b}$ is a regular element of genus k and exponent-order $[\mathfrak{C}, k]$, and since $\mathfrak{C}$ is a regular characteristic subgroup, it follows that $\mathfrak{r}$ and therefore also $\alpha(\mathfrak{r}) = \mathfrak{r}+p^{k+1-[\mathfrak{C}, k]}\mathfrak{a}$ must lie in $\mathfrak{C}$.

Hence $p^{k+1-[\mathfrak{C}, k]}\mathfrak{a}$ is an element of $\mathfrak{C}$. Since $\mathfrak{a}$ and all its multiples are regular elements, and since $l+1-[\mathfrak{C}, l]$ is the least number such that

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† If $\mathfrak{A}$ is finite, by Theorem 6 and Lemma 3, this Theorem 5 is equivalent to the theorem of A. Chatelet, *Les groupes Abéliens finis et les modules de points entiers* (Lille, 1925), § 67, 165; *cp.* also K. Shoda, *loc. cit.*, 6–7, Satz 3.

$p^{l+1-[\mathfrak{C},l]}\mathfrak{a}$ is in $\mathfrak{C}$ [for $\mathfrak{C}$ is regular and $g(\mathfrak{a})=l$], we have

$$l+1-[\mathfrak{C},l]\leqslant k+1-[\mathfrak{C},k] \quad \text{or} \quad l-e(l)\leqslant k-e(k),$$

and so (2) is proved.

(B) Suppose that the numbers $e(l)$ satisfy the conditions (1) and (2). Then, for any l for which elements of genus l exist in $\mathfrak{A}$, we form the set $\mathfrak{C}(l)$ of all elements in $\mathfrak{A}$ with exponent-order less than or equal to $e(l)$ and with exponent greater than or equal to $l+1-e(l)$; we here take $\infty+1-\infty=\infty$.

Since the set of all elements of exponent greater than or equal to $l+1-e(l)$, as well as the set of all elements of exponent-order less than or equal to $e(l)$, form characteristic subgroups of $\mathfrak{A}$, the sets $\mathfrak{C}(l)$ and hence also the smallest group $\mathfrak{C}$ containing every $\mathfrak{C}(l)$, *i.e.* the join-group of all the $\mathfrak{C}(l)$, are characteristic subgroups of $\mathfrak{A}$.

Then we have $[\mathfrak{C},l]\geqslant e(l)$, since $\mathfrak{C}(l)$ and therefore $\mathfrak{C}$ contains regular elements of genus l and exponent-order $e(l)$. In order to prove that $\mathfrak{C}$ is a regular characteristic subgroup with $[\mathfrak{C},l]=e(l)$, we have, therefore, only to show that $[\mathfrak{C},l]\leqslant e(l)$.

First we prove that

(6) If $\mathfrak{a}\neq 0$ is contained in $\mathfrak{C}(k)$, but not in any $\mathfrak{C}(l)$ with $l<k$, then $n(\mathfrak{a})>e(l)$ for every $l<k$, and also $g(\mathfrak{a})\geqslant k$.

Since $\mathfrak{a}$ is in $\mathfrak{C}(k)$ we have, from condition (2),

$$h(\mathfrak{a})\geqslant k+1-e(k)\geqslant l+1-e(l) \quad \text{for} \quad l<k.$$

Therefore $\mathfrak{a}$ is contained in $\mathfrak{C}(l)$ if and only if $n(\mathfrak{a})\leqslant e(l)$, and since $\mathfrak{a}$ is not contained in $\mathfrak{C}(l)$ when $l<k$ we have in that case $n(\mathfrak{a})>e(l)$.

From this it follows that, for any $l<k$,

$$g(\mathfrak{a})+1\geqslant s(\mathfrak{a})=n(\mathfrak{a})+h(\mathfrak{a})>e(l)+l-e(l)+1=l+1,$$

i.e. $g(\mathfrak{a})>l$ for every $l<k$, so that $g(\mathfrak{a})\geqslant k$.

Now let $\mathfrak{a}$ be any element of $\mathfrak{C}$ other than 0. Then $\mathfrak{a}=\mathfrak{b}_1+\ldots+\mathfrak{b}_a$, where $\mathfrak{b}_i$ is contained in $\mathfrak{C}(k_i)$ but not in any $\mathfrak{C}(l)$ with $l<k_i$, and $k_1<\ldots<k_a$. Then we have, by (6),

$$n(\mathfrak{b}_1)\leqslant e(k_1)<n(\mathfrak{b}_2)\leqslant e(k_2)<\ldots\leqslant e(k_{a-1})<n(\mathfrak{b}_a)\leqslant e(k_a)$$

and

$$g(\mathfrak{b}_i)\geqslant k_i.$$

Therefore we have $n(\mathfrak{a})=n(\mathfrak{b}_a)$, $g(\mathfrak{a})=g(\mathfrak{b}_a)$, and $n(\mathfrak{a})\leqslant e(k_a)\leqslant e[g(\mathfrak{a})]$ from condition (2), and consequently we have $[\mathfrak{C},l]\leqslant e(l)$ for all l; this completes the proof.

From Theorem 4 (r) and the above we deduce

COROLLARY 1. *If elements of genus k in $\mathfrak{A}$ exist, and if $e \leqslant k+1$, then, denoting by $\mathfrak{C}(k, e)$ the group of all elements of $\mathfrak{A}$ with exponent-order less than or equal to e and exponent greater than or equal to $k+1-e$, we have the result:*

If $\mathfrak{C}$ is a regular characteristic subgroup of $\mathfrak{A}$, then $\mathfrak{C}$ is the join-group of all the groups $\mathfrak{C}(k, [\mathfrak{C}, k])$.

An immediate consequence of the theorem, in particular of condition (2), is

COROLLARY 2. (a) *Suppose that there are only a finite number of finite integers l such that elements of genus l exist in $\mathfrak{A}$, say $l_1 < \ldots < l_a$.*

Then, if $\mathfrak{A}$ contains no elements of exponent ∞ (except possibly 0), the number of regular characteristic subgroups of $\mathfrak{A}$ is

$$(l_1+2)(l_2-l_1+1)\ldots(l_a-l_{a-1}+1).$$

If, however, $\mathfrak{A}$ does contain such elements, then the number of regular characteristic subgroups of $\mathfrak{A}$ is $\aleph_0$.

(b) *If, on the other hand, there are an infinity of numbers l for which elements of genus l exist in $\mathfrak{A}$, then the number of regular characteristic subgroups of $\mathfrak{A}$ is $2^{\aleph_0}$.*

THEOREM 6. *A subgroup $\mathfrak{C}$ of $\mathfrak{A}$ is a regular characteristic subgroup if and only if $\alpha(\mathfrak{C}) \leqslant \mathfrak{C}$ for every proper or improper automorphism α of $\mathfrak{A}$†.*

Proof. (A) Let $\mathfrak{C}$ be a regular characteristic subgroup and α any proper or improper automorphism of $\mathfrak{A}$. Then we have, for any element $\mathfrak{a}$ of $\mathfrak{A}$,

$$n[\alpha(\mathfrak{a})] \leqslant n(\mathfrak{a}), \quad h[\alpha(\mathfrak{a})] \geqslant h(\mathfrak{a}).$$

Therefore $\alpha\big(\mathfrak{C}(k, [\mathfrak{C}, k])\big) \leqslant \mathfrak{C}(k, [\mathfrak{C}, k])$ and, from Corollary 1 of Theorem 5, we have $\alpha(\mathfrak{C}) \leqslant \mathfrak{C}$.

(B) Suppose now that $\mathfrak{C}$ is an irregular characteristic subgroup of $\mathfrak{A}$; then there exists a value l such that any element $\mathfrak{m}$ of maximum type of

† In the notation of K. Shoda, *loc. cit.*, our regular characteristic subgroups are $\mathfrak{o}$-characteristic while the irregular ones are only $\mathfrak{G}$-characteristic but not $\mathfrak{o}$-characteristic.

genus l in $\mathfrak{C}$ is irregular; we consider any canonical representation of $\mathfrak{m}$ as a sum of regular elements [*cp.* Lemma 1]:

$$m = \mathfrak{r}_1 + \ldots + \mathfrak{r}_{d(\mathfrak{m})} \quad \text{with} \quad n(\mathfrak{m}) = n(\mathfrak{r}_{d(\mathfrak{m})}), \quad g(\mathfrak{m}) = g(\mathfrak{r}_{d(\mathfrak{m})}) = l.$$

Since $\mathfrak{m}$ is irregular and $\mathfrak{r}_{d(\mathfrak{m})}$ is regular, we have $I(\mathfrak{m}) < I(\mathfrak{r}_{d(\mathfrak{m})})$, and therefore $\mathfrak{r}_{d(\mathfrak{m})}$ is not in $\mathfrak{C}$.

Finally there exists an improper automorphism α of $\mathfrak{A}$ such that

$$\alpha(\mathfrak{r}_i) = \begin{cases} 0 & \text{for} \quad 1 \leqslant i < d(\mathfrak{m}), \\ \mathfrak{r}_{d(\mathfrak{m})} & \text{for} \quad i = d(\mathfrak{m}); \end{cases}$$

and we have $\alpha(\mathfrak{m}) = \mathfrak{r}_{d(\mathfrak{m})}$, *i.e.* $\alpha(\mathfrak{C})$ is not contained in $\mathfrak{C}$.

Theorem 6 follows immediately.

Suppose that $\mathfrak{A}$ is the direct sum of $\mathrm{N}(i)$ cyclic groups of order p^i for $i = 1, 2, \ldots$ and of $\mathrm{N}(\infty)$ groups of the type p^∞; then each N is either an integer greater than or equal to 0 or a certain $\aleph$. If $\overline{\mathfrak{A}}_r$ is the subgroup of all elements of order p and of exponent greater than or equal to r, then the number of elements of $\overline{\mathfrak{A}}_i/\overline{\mathfrak{A}}_{i+1}$ is $p^{\mathrm{N}(i)}$ if $\mathrm{N}(i)$ is finite and $\mathrm{N}(i)$ if $\mathrm{N}(i)$ is infinite; and the number of elements of $\overline{\mathfrak{A}}_\infty$ is $p^{\mathrm{N}(\infty)}$ if $\mathrm{N}(\infty)$ is finite and $\mathrm{N}(\infty)$ if $\mathrm{N}(\infty)$ is infinite. So the numbers N are *characteristic invariants of* $\mathfrak{A}$.

LEMMA 3. *If $\mathfrak{C}$ is any regular characteristic subgroup of $\mathfrak{A}$, then $\mathfrak{C}$ is the direct sum of*

$$\sum_{[\mathfrak{C}, k-1]=1} \mathrm{N}(k) \quad \textit{cyclic groups of order } p,$$

$$\sum_{[\mathfrak{C}, k-1]=2} \mathrm{N}(k) \quad \textit{cyclic groups of order } p^2,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots,$$

together with $\mathrm{N}(\infty)$ subgroups of type p^∞, when $\mathrm{N}(\infty) \neq 0$ and $[\mathfrak{C}, \infty] = \infty$.

Proof. Let

$$\mathfrak{A} = \sum_{1 \leqslant g \leqslant \infty} \sum_{1 \leqslant \nu < \mu_g} \mathfrak{Z}_{g\nu}.$$

Then $\mathfrak{Z}_{g\nu}$ has a uniquely determined subgroup $\mathfrak{Z}^*_{g\nu}$ of order $p^{[\mathfrak{C}, g-1]}$ and we consider the subgroup

$$\mathfrak{C}^* = \sum_{1 \leqslant g \leqslant \infty} \sum_{1 \leqslant \nu < \mu_g} \mathfrak{Z}^*_{g\nu}.$$

Since $\mathfrak{C}^*$ has the structure mentioned in Lemma 3 it will be sufficient to

prove that $\mathfrak{C}^* = \mathfrak{C}$. Since $\mathfrak{C}$ is a regular characteristic subgroup, it contains all elements of $\mathfrak{Z}_{g\nu}^*$ and therefore we have $\mathfrak{C}^* \leqslant \mathfrak{C}$.

Consider any element $\mathfrak{a}$ of $\mathfrak{C}$. Then $\mathfrak{a} = \sum_{i=1}^{w(\mathfrak{a})} \mathfrak{z}_i$, where $w(\mathfrak{a}) \geqslant 0$ and the elements $\mathfrak{z}_i$ are contained in different groups $\mathfrak{Z}_{g\nu}$. Since, therefore, the elements $\mathfrak{z}_i$ are linearly independent, we have

$$n(\mathfrak{a}) = \max [n(\mathfrak{z}_1), \ldots, n(\mathfrak{z}_{w(\mathfrak{a})})],$$

and $g(\mathfrak{a})$ is the least genus of an element $\mathfrak{z}_i$ for which $n(\mathfrak{z}_i) = n(\mathfrak{a})$. Then the elements $\mathfrak{z}_i$ for which $n(\mathfrak{z}_i) = n(\mathfrak{a})$ and $g(\mathfrak{z}_i) = g(\mathfrak{a})$ are contained in $\mathfrak{C}^*$ since $n(\mathfrak{a}) \leqslant [\mathfrak{C}, g(\mathfrak{a})]$. And hence, by complete induction with regard to $w(\mathfrak{a})$, we have the result that any element $\mathfrak{a}$ of $\mathfrak{C}$ is contained in $\mathfrak{C}^*$, *i.e.* $\mathfrak{C} = \mathfrak{C}^*$ and so we have also proved the

COROLLARY. *If $\mathfrak{C}$ is any regular characteristic subgroup of $\mathfrak{A}$, then the quotient group $\mathfrak{A}/\mathfrak{C}$ is the direct sum of*

$$\sum_{k-[\mathfrak{C}, k-1]=1} \mathrm{N}(k) \quad \textit{cyclic groups of the order } p,$$

$$\sum_{k-[\mathfrak{C}, k-1]=2} \mathrm{N}(k) \quad \textit{cyclic groups of the order } p^2,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots,$$

together with $\mathrm{N}(\infty)$ subgroups of the type p^∞, when $\mathrm{N}(\infty) \neq 0$ and $[\mathfrak{C}, \infty] \neq \infty$.

THEOREM 7. (*a*) *Any two isomorphic regular characteristic subgroups are identical, if and only if*

(1) *$\mathrm{N}(i)$ differs from 0 only for a finite number of values of i,*

and (2) *$\mathrm{N}(i) > \mathrm{N}(l)$ whenever $i < l$, $\mathrm{N}(i) \neq 0$, and $\mathrm{N}(l)$ is infinite.*

(*b*) *Any two regular characteristic subgroups $\mathfrak{B}$ and $\mathfrak{C}$ with isomorphic quotient groups $\mathfrak{A}/\mathfrak{B}$ and $\mathfrak{A}/\mathfrak{C}$ are identical, if and only if*

(1) *any two isomorphic regular characteristic subgroups are identical,*

(2) *$\mathrm{N}(\infty) = 0$, i.e. $\mathfrak{A}$ contains no elements of exponent ∞ (except possibly 0).*

Observe that the conditions (*a*), (1) and (2), and (*b*), (1) and (2) are both satisfied if $\mathfrak{A}$ is finite.

Proof of 7 (*a*). (A) Suppose first that condition (1) is not satisfied. Then there exists an infinity of positive integers i such that $\mathrm{N}(i) > 0$ and, since $\mathrm{N}(i)$ is always finite or an $\aleph$, there exists a least integer k such that $0 < \mathrm{N}(k) \leqslant \mathrm{N}(i)$ for every finite i for which $\mathrm{N}(i) \neq 0$. Then we have

$$\sum_{k<i<\infty} \mathrm{N}(i) \geqslant \aleph_0 \mathrm{N}(k),$$

and therefore

$$\sum_{k\leqslant i\leqslant\infty} \mathrm{N}(i) = \sum_{k<i\leqslant\infty} \mathrm{N}(i). \tag{7}$$

If condition (2) is not satisfied, the identity (7) holds for any k for which there exists an l satisfying $k < l \leqslant \infty$, $0 < \mathrm{N}(k) \leqslant \mathrm{N}(l)$, and $\mathrm{N}(l)$ infinite. So we have shown that (7) is true for a certain k if at least one of the conditions (1) and (2) is not satisfied.

If (7) is true for a certain k, there exist by Theorem 5 two regular characteristic subgroups $\mathfrak{B}$ and $\mathfrak{C}$ of $\mathfrak{A}$ such that

$$[\mathfrak{B}, l] = \begin{cases} 0 & \text{for} \quad l < k \\ 1 & \text{for} \quad l \geqslant k \end{cases}, \qquad [\mathfrak{C}, l] = \begin{cases} 0 & \text{for} \quad l \leqslant k \\ 1 & \text{for} \quad l > k \end{cases}.$$

But these different regular characteristic subgroups are isomorphic by Lemma 3 and (7). Thus the conditions (1) and (2) are necessary.

(B) Suppose that the conditions (1) and (2) are both satisfied. If $\mathrm{N}(k) > 0$ we have

$$\sum_{k\leqslant i\leqslant\infty} \mathrm{N}(i) > \sum_{k<i\leqslant\infty} \mathrm{N}(i),$$

since both sums contain only a finite number of terms different from 0 and since $\mathrm{N}(k) > \mathrm{N}(i)$ for $\mathrm{N}(i) \geqslant \aleph_0$ and $i > k$.

Now let $\mathfrak{B}$ and $\mathfrak{C}$ be two isomorphic regular characteristic subgroups; then we have, from Lemma 3,

$$\sum_{[\mathfrak{B}, k-1]=i} \mathrm{N}(k) = \sum_{[\mathfrak{C}, k-1]=i} \mathrm{N}(k) \tag{8}$$

for every finite i; and $[\mathfrak{B}, \infty] = \infty$ if and only if $[\mathfrak{C}, \infty] = \infty$, when $\mathrm{N}(\infty) \neq 0$.

If $\mathfrak{B} \neq \mathfrak{C}$ there would be a least value k such that

$$[\mathfrak{B}, k-1] \neq [\mathfrak{C}, k-1]$$

and such that both are finite; without loss of generality, we may take

$$[\mathfrak{B}, k-1] < [\mathfrak{C}, k-1].$$

From (8) we have

$$\sum_{[\mathfrak{B},\,i-1]>[\mathfrak{B},\,k-1]} \mathrm{N}(i) = \sum_{[\mathfrak{C},\,i-1]>[\mathfrak{B},\,k-1]} \mathrm{N}(i),$$

which is absurd, since $\mathrm{N}(k)$ occurs on the right-hand side but not on the left-hand side while all $\mathrm{N}(i)$ occurring on the left-hand side also occur on the right-hand side. Therefore $\mathfrak{B} = \mathfrak{C}$, *i.e.* Theorem 7 ($a$) is proved.

Proof of 7 (b). From the corollary of Lemma 3 the existence of different regular characteristic subgroups $\mathfrak{B}$ and $\mathfrak{C}$ with isomorphic quotient-groups $\mathfrak{A}/\mathfrak{B}$ and $\mathfrak{A}/\mathfrak{C}$ is obvious if $\mathrm{N}(\infty) \neq 0$.

If, on the other hand, $\mathrm{N}(\infty) = 0$, any two regular characteristic subgroups $\mathfrak{B}$ and $\mathfrak{C}$ are isomorphic if, and only if, the quotient-groups $\mathfrak{A}/\mathfrak{B}$ and $\mathfrak{A}/\mathfrak{C}$ are isomorphic, and so Theorem 7 (b) is also proved.

4. *Irregular characteristic subgroups.*

We say that the value l is a *singular genus* if $\mathrm{N}(l) = 1$, *i.e.* if $\bar{\mathfrak{A}}_l/\bar{\mathfrak{A}}_{l+1}$ (or $\bar{\mathfrak{A}}_\infty$ for $l = \infty$) contains exactly p elements, where $\bar{\mathfrak{A}}_i$ is the subgroup of $\mathfrak{A}$ of all elements of order p and exponent greater than or equal to i. Then l is a singular genus if and only if there exist elements of genus l in $\mathfrak{A}$ and, for any two regular elements $\mathfrak{a}$ and $\mathfrak{b}$ for which

$$g(\mathfrak{a}) = g(\mathfrak{b}) = l \quad \text{and} \quad n(\mathfrak{a}) = n(\mathfrak{b}),$$

either $n(\mathfrak{a}-\mathfrak{b}) < n(\mathfrak{a})$ or else $g(\mathfrak{a}-\mathfrak{b}) > l$, where $p = 2$.

THEOREM 8. *Suppose that there exist elements of genus l in $\mathfrak{A}$. Let $e(l)$ be an integer or infinity such that $0 \leqslant e(l) \leqslant l+1$ and, for $0 < e(l) < \infty$, let $\mathfrak{c}(l)$ be an element of genus l and exponent-order $e(l)$.*

There exists an irregular characteristic subgroup $\mathfrak{C}$ of $\mathfrak{A}$ such that

$$\left.\begin{aligned} [\mathfrak{C}, l] &= e(l) \\ \mathrm{T}(\mathfrak{C}, l) &= I[\mathfrak{c}(l)] \end{aligned}\right\} \textit{ for all } l \textit{ for which the right-hand sides are defined,}$$

if and only if

(1) $p = 2$;

(2) *at least one of the elements $\mathfrak{c}(l)$ is irregular*;

(3) *for any irregular element* $\mathfrak{c}(l)$ *the following conditions are satisfied*:

(*a*) l *and*

$$g(j, l) = s[2^{d[j, \mathfrak{c}(l)]}\, \mathfrak{c}(l)] - n[\mathfrak{c}(l)] + d[j, \mathfrak{c}(l)] \quad \text{for} \quad 1 \leqslant j < d[\mathfrak{c}(l)]$$

are singular genera,

(*b*) *either* $e[g(j, l)] = 0$ *or* $\mathfrak{c}[g(j, l)]$ *is regular,*

(*c*) $e(g[j, l]) = d[j, \mathfrak{c}(l)]$;

and (4) *using the notation*

$$f(h) = \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\} \text{ if } \left\{ \begin{matrix} \text{either } e(h) = 0 \text{ or } = \infty \text{ or } \mathfrak{c}(h) \text{ is regular,} \\ \mathfrak{c}(h) \text{ is irregular,} \end{matrix} \right.$$

$$F(h) = \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\} \text{ if } h \left\{ \begin{matrix} \neq \\ = \end{matrix} \right\} g(i, t) \text{ for } \left\{ \begin{matrix} \text{any } i, t \\ \text{a certain pair } i, t, \end{matrix} \right.$$

the following inequality is satisfied for $l > k$:

$$e(k) + f(l) + F(k) \leqslant e(l) \leqslant e(k) + l - k - f(k) - F(l).$$

Proof. (A) Suppose that there exists an irregular characteristic subgroup $\mathfrak{C}$ such that

$$[\mathfrak{C}, l] = e(l) \quad \text{and} \quad \mathrm{T}(\mathfrak{C}, l) = I[\mathfrak{c}(l)] \quad \text{for} \quad 0 < e(l) < \infty.$$

Then $p = 2$ by Theorem 3 (*c*), § 2, and at least one of the elements $\mathfrak{c}(l)$ is irregular.

Consider one of the irregular elements $\mathfrak{c}(l)$; by Lemma 1, § 1, there exists a canonical representation of $\mathfrak{c}(l)$ as a sum of $d[\mathfrak{c}(l)] > 1$ regular elements, say

$$\mathfrak{c}(l) = \mathfrak{r}_1(l) + \ldots + \mathfrak{r}_{d[\mathfrak{c}(l)]}(l).$$

(9) $2\mathfrak{r}_j(l)$ is contained in $\mathfrak{C}$.

For, since $3\mathfrak{r}_j(l)$ and $\mathfrak{r}_j(l)$ are isotypic regular elements, by Lemma 2 (*b*), § 1, there exists an automorphism α of $\mathfrak{A}$ such that

$$\alpha[\mathfrak{r}_i(l)] = \begin{cases} \mathfrak{r}_i(l) & \text{for } i \neq j, \\ 3\mathfrak{r}_j(l) & \text{for } i = j, \end{cases}$$

and we have $\alpha[\mathfrak{c}(l)] = \mathfrak{c}(l) + 2\mathfrak{r}_j(l)$, *i.e.* $2\mathfrak{r}_j(l)$ is in $\mathfrak{C}$.

(10) If $\mathfrak{r}_1+\ldots+\mathfrak{r}_{d(\mathfrak{a})}=\mathfrak{a}$ is a canonical representation of an element $\mathfrak{a}$ of $\mathfrak{C}$ for which $g(\mathfrak{a})<l$, then

$$I(\mathfrak{r}_{d(\mathfrak{a})}) \neq I\{\mathfrak{r}_j(l)\} \quad \text{for} \quad j \neq d[\mathfrak{c}(l)].$$

For suppose not. Then there exists an automorphism α such that $\alpha(\mathfrak{r}_i)=\mathfrak{r}_l(l)$ if and only if $I(\mathfrak{r}_i)=I[\mathfrak{r}_l(l)]$.

Since $\alpha[\mathfrak{r}_{d(\mathfrak{a})}]=\mathfrak{r}_j(l)$ for a certain j, the element $\mathfrak{c}(l)-\alpha(\mathfrak{a})$ has a canonical representation of the form

$$\mathfrak{r}_1'+\ldots+\mathfrak{r}_q'+\mathfrak{r}_{j+1}(l)+\ldots+\mathfrak{r}_{d[\mathfrak{c}(l)]}(l) \quad \text{with} \quad g(\mathfrak{r}_i')<g[\mathfrak{r}_j(l)].$$

Therefore $\mathfrak{c}(l)-\alpha(\mathfrak{a})$ is an element of genus l and exponent-order $[\mathfrak{C}, l]$, exactly like $\mathfrak{r}_{d[\mathfrak{c}(l)]}(l)$, but

$$I[\mathfrak{c}(l)-\alpha(\mathfrak{a})]>I[\mathfrak{c}(l)];$$

this is impossible since $\mathfrak{a}$, $\alpha(\mathfrak{a})$, and $\mathfrak{c}(l)-\alpha(\mathfrak{a})$ are in $\mathfrak{C}$.

An immediate consequence of (9) and (10) together is

(11) For $j \neq d[\mathfrak{c}(l)]$ we have either

$$e\Big[g\big(\mathfrak{r}_j(l)\big)\Big]=0 \quad \text{or} \quad I[2\mathfrak{r}_j(l)]=I\Big[\mathfrak{c}\big(g[\mathfrak{r}_j(l)]\big)\Big].$$

Since, from Lemma 1, § 1,

$$n[\mathfrak{r}_j(l)]=d[j, \mathfrak{c}(l)]+1,$$

and

$$g[\mathfrak{r}_j(l)]=s[2^{d[j,\,\mathfrak{c}(l)]}\mathfrak{c}(l)]-n[\mathfrak{c}(l)]+d[j, \mathfrak{c}(l)],$$

we have proved (3) (*b*) and (3) (*c*).

(12) If $I(\mathfrak{r})=I(\mathfrak{r}+\mathfrak{w})$ and if $\mathfrak{r}$ is a term of a canonical representation of an element of $\mathfrak{C}$, then $\mathfrak{w}$ is contained in $\mathfrak{C}$.

For then there exists an element $\mathfrak{a}$ of $\mathfrak{C}$ with a canonical representation

$$\mathfrak{a}=\mathfrak{r}_1+\ldots+\mathfrak{r}_{d(\mathfrak{a})}$$

such that $\mathfrak{r}=\mathfrak{r}_u$ for some u satisfying $1 \leqslant u \leqslant d(\mathfrak{a})$, and there exists also an automorphism α of $\mathfrak{A}$ such that

$$\alpha(\mathfrak{r}_i)=\begin{cases}\mathfrak{r}_i & \text{for} \quad i \neq u,\\ \mathfrak{r}+\mathfrak{w} & \text{for} \quad i=u.\end{cases}$$

Therefore $\alpha(\mathfrak{a})=\mathfrak{a}+\mathfrak{w}$ and $\mathfrak{w}$ is contained in $\mathfrak{C}$.

(13) $g[\mathfrak{r}_j(l)]$ is a singular genus for $1 \leqslant j \leqslant d[\mathfrak{c}(l)]$.

For otherwise there would exist an element $\mathfrak{w}$ such that

$$I[\mathfrak{r}_j(l)] = I(\mathfrak{w}) = I[\mathfrak{r}_j(l)+\mathfrak{w}],$$

and from (12) $\mathfrak{w}$ and therefore $\mathfrak{r}_j(l)$ would be contained in $\mathfrak{C}$. But this is impossible by (11) for $j \neq d[\mathfrak{c}(l)]$, and by the regularity of $\mathfrak{r}_{d[\mathfrak{c}(l)]}(l)$ and the irregularity of $\mathfrak{c}(l)$ for $j = d[\mathfrak{c}(l)]$.

From the value of $g[\mathfrak{r}_j(l)]$ mentioned above, the condition (3)(a) follows from (11) and (13).

(14) Suppose that there exist elements of genus k in $\mathfrak{A}$, that $k < l \leqslant m$, and that regular elements of genus l arise in a canonical representation of $\mathfrak{c}(m)$. Then

$$k+1-[\mathfrak{C}, k]\begin{cases} \leqslant l+1\begin{cases} -[\mathfrak{C}, l] & \text{for } l = m \\ -[\mathfrak{C}, l]-1 & \text{for } l < m \end{cases} & \text{if } [\mathfrak{C}, k] = 0 \text{ or if } \mathfrak{c}(k) \text{ is regular,} \\ < l+1\begin{cases} -[\mathfrak{C}, l] & \text{for } l = m \\ -[\mathfrak{C}, l]-1 & \text{for } l < m \end{cases} & \text{if } \mathfrak{c}(k) \text{ is irregular.} \end{cases}$$

We consider a regular element $\mathfrak{r}$ which is a term of a canonical representation of $\mathfrak{c}(m)$ such that $g(\mathfrak{r}) = l < \infty$. Then there exists a (regular) element $\mathfrak{a}$ of $\mathfrak{A}$ such that

$$2^{l+1-n(\mathfrak{r})}\mathfrak{a} = \mathfrak{r}.$$

If $\mathfrak{b}$ is any regular element of $\mathfrak{A}$ for which $g(\mathfrak{b}) = k < l$ and $n(\mathfrak{b}) = k+1$, then $I(\mathfrak{a}) = I(\mathfrak{a}+\mathfrak{b})$ and $2^{l+1-n(\mathfrak{r})}\mathfrak{b}$ is in $\mathfrak{C}$, by (12). Therefore we have

$$l+1-n(\mathfrak{r})\begin{Bmatrix} \geqslant \\ > \end{Bmatrix} k+1-[\mathfrak{C}, k] \quad \begin{cases} \text{if either } [\mathfrak{C}, k] = 0 \text{ or } \mathfrak{c}(k) \text{ is regular,} \\ \text{if } \mathfrak{c}(k) \text{ is irregular in view of the condition (3) proved above.} \end{cases}$$

Since, finally,

$$n(\mathfrak{r}) = \begin{cases} n[\mathfrak{c}(m)] = [\mathfrak{C}, m] & \text{for } l = m, \\ [\mathfrak{C}, l]+1 & \text{for } l < m, \end{cases}$$

again in view of the condition (3) proved above, we have proved (14).

(15) Suppose that there exist elements of genus k in $\mathfrak{A}$, that $l < k$, $l \leqslant m$, and that regular elements of genus l occur in a canonical representation of

$\mathfrak{c}(m)$. Then

$$[\mathfrak{C}, k]\begin{cases} \geqq \begin{cases} [\mathfrak{C}, l] & \text{for } l = m \\ [\mathfrak{C}, l]+1 & \text{for } l < m \end{cases} & \text{if } [\mathfrak{C}, k] = 0 \text{ or if } \mathfrak{c}(k) \text{ is regular,} \\ > \begin{cases} [\mathfrak{C}, l] & \text{for } l = m \\ [\mathfrak{C}, l]+1 & \text{for } l < m \end{cases} & \text{if } \mathfrak{c}(k) \text{ is irregular.} \end{cases}$$

We consider a regular element $\mathfrak{x}$ which is a term of a canonical representation of $\mathfrak{c}(m)$ and such that $g(\mathfrak{x}) = l$. Then there exists a regular element $\mathfrak{a}$ of $\mathfrak{A}$ for which $g(\mathfrak{a}) = k > l$ and $n(\mathfrak{a}) = n(\mathfrak{x})$. Hence $I(\mathfrak{x}) = I(\mathfrak{a}+\mathfrak{x})$, and $\mathfrak{a}$ is in $\mathfrak{C}$, by (12). Therefore

$$n(\mathfrak{x}) = n(\mathfrak{a})\begin{cases} \leqq [\mathfrak{C}, k], \text{ if either } [\mathfrak{C}, k] = 0 \text{ or } \mathfrak{c}(k) \text{ is regular,} \\ < [\mathfrak{C}, k], \text{ if } \mathfrak{c}(k) \text{ is irregular,} \end{cases}$$

by condition (3) established above; and the proof of (15) may be completed just as the proof of (14).

Condition (4) is an immediate consequence of (14) and (15).

B. Suppose that the conditions (1)–(4) are satisfied.

(16) There exist regular characteristic subgroups $\mathfrak{C}_*$ and $\mathfrak{C}^*$ such that, in the notation of Theorem 8, (4), we have

$$[\mathfrak{C}_*, l] = e(l)-f(l) \quad \text{and} \quad [\mathfrak{C}^*, l] = e(l)+F(l).$$

(16) is an immediate consequence of Theorem 5, § 3, since, from condition (4), the values $e(l)-f(l)$ and $e(l)+F(l)$ satisfy (1) and (2) of Theorem 5, § 3.

(17) If $\mathfrak{a}$ and $\mathfrak{b}$ are regular elements such that $g(\mathfrak{a}) = g(\mathfrak{b}) = l$ and

$$n(\mathfrak{a}) = n(\mathfrak{b}) = [\mathfrak{C}_*, l]+1,$$

and if $\mathfrak{a}$ or $\mathfrak{b}$ arises in a canonical representation of an irregular element $\mathfrak{c}(k)$ for which $k \geqq l$, then $\mathfrak{a}-\mathfrak{b}$ is contained in $\mathfrak{C}_*$.

From condition (2)(α), l and k are singular genera; therefore either

$$n(\mathfrak{a}-\mathfrak{b}) < n(\mathfrak{a}) = n(\mathfrak{b}) = [\mathfrak{C}_*, l]+1$$

or

$$g(\mathfrak{a}-\mathfrak{b}) > l.$$

If $\mathfrak{a}-\mathfrak{b} \neq 0$, there exists a canonical representation of $\mathfrak{a}-\mathfrak{b}$, say

$$\mathfrak{a}-\mathfrak{b} = \mathfrak{r}_1+\ldots+\mathfrak{r}_v.$$

Since $n(\mathfrak{r}_v) = n(\mathfrak{a}-\mathfrak{b})$ and $g(\mathfrak{r}_v) = g(\mathfrak{a}-\mathfrak{b})$, either

$$n(\mathfrak{r}_v) < n(\mathfrak{a}) = n(\mathfrak{b}) = [\mathfrak{C}_*, l]+1$$

or

$$g(\mathfrak{r}_v) > l.$$

In any case $\mathfrak{a}$ and $\mathfrak{a}-\mathfrak{r}_v$ are isotypic and therefore, by complete induction with regard to $v = d(\mathfrak{a}-\mathfrak{b})$, it follows that, for $1 \leqslant j \leqslant v$, either

$$n(\mathfrak{r}_j) < n(\mathfrak{a}) = n(\mathfrak{b}) = [\mathfrak{C}_*, l]+1$$

or

$$g(\mathfrak{r}_j) > l.$$

Therefore we may assume, without loss of generality, that

$$\mathfrak{a} = \mathfrak{b}+\mathfrak{r},$$

where $\mathfrak{r}$ is a regular element and not 0 and either

$$n(\mathfrak{r}) < n(\mathfrak{a}) = n(\mathfrak{b}) = [\mathfrak{C}_*, l]+1$$

or

$$g(\mathfrak{r}) > l.$$

If $l^* = g(\mathfrak{r}) > l$, we have $[\mathfrak{C}_*, l] < [\mathfrak{C}_*, l^*]$, because of condition (4) and the assumptions concerning $\mathfrak{a}$, and therefore we have

$$n(\mathfrak{r}) \leqslant n(\mathfrak{a}) = [\mathfrak{C}_*, l]+1 \leqslant [\mathfrak{C}_*, l^*] = [\mathfrak{C}_*, g(\mathfrak{r})],$$

i.e. $\mathfrak{r}$ is contained in $\mathfrak{C}_*$.

If $l^* \leqslant l$, we have $n(\mathfrak{r}) < n(\mathfrak{a})$, and therefore $n(\mathfrak{r}) \leqslant [\mathfrak{C}_*, l]$. Since $h(\mathfrak{a}) = h(\mathfrak{b}) = l+1-[\mathfrak{C}_*, l]-1 = l-[\mathfrak{C}_*, l]$, we have $h(\mathfrak{r}) \geqslant l-[\mathfrak{C}_*, l]$. Since $[\mathfrak{C}_*, l]$ is finite, $\mathfrak{r}$ is contained in $\mathfrak{C}(l, [\mathfrak{C}_*, l])$ if l is infinite or $h(\mathfrak{r}) > l-[\mathfrak{C}_*, l]$, and $\mathfrak{r}$ is then, by Corollary 1 of Theorem 5, § 3, an element of $\mathfrak{C}_*$.

If l is finite and $h(\mathfrak{r}) = l-[\mathfrak{C}_*, l]$, then we have

$$l^*+1 = n(\mathfrak{r})+h(\mathfrak{r}) \leqslant [\mathfrak{C}_*, l]+l-[\mathfrak{C}_*, l] = l,$$

i.e. $l^* < l$, and, from condition (4) and the assumption concerning $\mathfrak{a}$, we find that

$$l-[\mathfrak{C}_*, l] > l^*-[\mathfrak{C}_*, l^*],$$

and therefore

$$n(\mathfrak{x}) = l^* + 1 - l + [\mathfrak{C}_*, l] < l^* + 1 - l^* + [\mathfrak{C}_*, l^*],$$

i.e.

$$n(\mathfrak{x}) \leqq [\mathfrak{C}_*, l^*] = [\mathfrak{C}_*, g(\mathfrak{x})];$$

and we have proved that $\mathfrak{x}$, and therefore $\mathfrak{a}-\mathfrak{b}$, is in every case an element of $\mathfrak{C}_*$.

(18) Let $\mathfrak{C}$ be the smallest subgroup of $\mathfrak{A}$ containing $\mathfrak{C}_*$ and each (irregular) element $\mathfrak{c}(l)$. Then

(a) $\mathfrak{C}$ is a characteristic subgroup of $\mathfrak{A}$.

(b) $\mathfrak{C}_* \leqq \mathfrak{C} \leqq \mathfrak{C}^*$.

(c) Any element of $\mathfrak{C}$ has the form

$$\mathfrak{a} + \sum_{j=1}^{t} \mathfrak{c}(l_j),$$

where $\mathfrak{a}$ is an element of $\mathfrak{C}_*$, $t \geqq 0$, $l_1 < \ldots < l_t$, and the $\mathfrak{c}(l_j)$ are irregular.

(d) $[\mathfrak{C}, l] = e(l)$

(e) $I[\mathfrak{c}(l)] = \mathrm{T}(\mathfrak{C}, l)$

} for all l for which $e(l)$ or $\mathfrak{c}(l)$ is defined, respectively.

(a) and (b) are immediate consequences of (17); (c) is a consequence of (17) and of the fact that, in consequence of the conditions (1) and (3) and the definition of $\mathfrak{C}_*$, $2\mathfrak{c}(l)$ is contained in $\mathfrak{C}_*$.

(d) follows from

$$[\mathfrak{C}, l] \leqq e(l),$$

since $[\mathfrak{C}_*, l] = e(l)$, if $e(l) = 0$ or ∞, and since $n[\mathfrak{c}(l)] = e(l)$. To establish this inequality, we have to prove that:

(d^*) For any element $\mathfrak{b}$ of $\mathfrak{C}$ we have $n(\mathfrak{b}) \leqq e[g(\mathfrak{b})]$.

Since we have already proved (18) (c), we may write

$$\mathfrak{b} = \mathfrak{a} + \sum_{j=1}^{t} \mathfrak{c}(l_j),$$

where $\mathfrak{a}$ is in $\mathfrak{C}_*$, $t \geqq 0$, $l_1 < \ldots < l_t$, and the $\mathfrak{c}(l_j)$ are irregular.

(d^*) is true for $t=0$, since then $\mathfrak{b}$ is in $\mathfrak{C}_*$ and also $[\mathfrak{C}_*,\ l]\leqslant e(l)$ by definition of $\mathfrak{C}_*$; if $t\neq 0$, we have

$$g\Big[\sum_{j=1}^{t}\mathfrak{c}(l_j)\Big]=l_t \quad\text{and}\quad n\Big[\sum_{j=1}^{t}\mathfrak{c}(l_j)\Big]=e(l_t),$$

since $l_1<\ldots<l_t$ and therefore $e(l_1)<\ldots<e(l_t)$ from condition (4).

Case 1. $n(\mathfrak{a})\leqslant e(l_t)$.

Then we have $n(\mathfrak{b})\leqslant e(l_t)=e[g(\mathfrak{b})]$, since $\mathfrak{a}$ is in $\mathfrak{C}_*$, and therefore $g(\mathfrak{a})>l_t$ for $n(\mathfrak{a})=e(l_t)$ from condition (4).

Case 2. $n(\mathfrak{a})>e(l_t)$.

Then we have $g(\mathfrak{b})=g(\mathfrak{a})$ and

$$n(\mathfrak{b})=n(\mathfrak{a})\leqslant[\mathfrak{C}_*,\ g(\mathfrak{a})]=[\mathfrak{C}_*,\ g(\mathfrak{b})]\leqslant e[g(\mathfrak{b})],$$

and (d) is completely proved.

Since regular elements have always a higher type than any irregular element of the same genus and exponent-order, (e) is true in view of (d) for the regular elements $\mathfrak{c}(l)$.

Now suppose that $\mathfrak{c}(l)$ is irregular and let $\mathfrak{w}$ be an element of maximum type of genus l in $\mathfrak{C}$. Canonical representations of $\mathfrak{w}$ and $\mathfrak{c}(l)$ are $\mathfrak{w}=\mathfrak{r}_1+\ldots+\mathfrak{r}_{d(\mathfrak{w})}$ and $\mathfrak{c}(l)=\mathfrak{r}_1(l)+\ldots+\mathfrak{r}_{d[\mathfrak{c}(l)]}(l)$, say. Since

$$n(\mathfrak{w})=n[\mathfrak{c}(l)]=[\mathfrak{C},\ l] \quad\text{and}\quad g(\mathfrak{w})=g[\mathfrak{c}(l)]=l,$$

we have $I(\mathfrak{r}_{d(\mathfrak{w})})=I[\mathfrak{r}_{d[\mathfrak{c}](l)}(l)]$.

From (d), condition (3), and Lemma 1, § 1, we find that

$$n[\mathfrak{r}_j(l)]=\Big[\mathfrak{C},\ g[\mathfrak{r}_j(l)]\Big]+1 \quad\text{for}\quad 1\leqslant j<d[\mathfrak{c}(l)],$$

and, since $\mathfrak{C}$ is a characteristic subgroup and $\mathfrak{w}$ an element of maximum type of genus l of $\mathfrak{C}$, we infer from (11) that

$$n(\mathfrak{r}_j)=[\mathfrak{C},\ g(\mathfrak{r}_j)]+1 \quad\text{for}\quad 1\leqslant j<d(\mathfrak{w}).$$

From Lemma 2, § 1, there exists an automorphism α of $\mathfrak{A}$ such that $\alpha(\mathfrak{r}_j)=\mathfrak{r}_{j'}(l)$ if and only if $I(\mathfrak{r}_j)=I[\mathfrak{r}_{j'}(l)]$. Then $\alpha(\mathfrak{w})=\mathfrak{c}(l)$, since otherwise we should have $n[\alpha(\mathfrak{w})-\mathfrak{c}(l)]=\Big[\mathfrak{C},\ g(\alpha[\mathfrak{w}]-\mathfrak{c}[l])\Big]+1$, from Lemma 1, § 1, where $g(\alpha[\mathfrak{w}]-\mathfrak{c}[l])$ is the maximum of the genera of those elements $\mathfrak{r}_j$ and $\mathfrak{r}_j(l)$ for which

$$\left.\begin{aligned}&\alpha(\mathfrak{r}_j)\neq\mathfrak{r}_t(l)\\ \text{and}\quad&\mathfrak{r}_j(l)\neq\alpha(\mathfrak{r}_t)\end{aligned}\right\}\quad\text{for any } t,$$

and this is impossible by (d).

Thus (18), and therefore also our theorem, is completely proved and we have also proved the

COROLLARY. (*a*) *If* $\mathfrak{C}$ *is an irregular characteristic subgroup of* $\mathfrak{A}$, *then there exists a uniquely determined* $\left\{\begin{matrix}\text{largest}\\ \text{smallest}\end{matrix}\right\}$ *regular characteristic subgroup*

$$\left\{\begin{matrix}\mathfrak{C}_*\\ \mathfrak{C}^*\end{matrix}\right\} \text{ of } \mathfrak{A} \left\{\begin{matrix}\text{contained in}\\ \text{containing}\end{matrix}\right\} \mathfrak{C},$$

and these subgroups satisfy

$$\left.\begin{aligned}[\mathfrak{C}_*, l] &= [\mathfrak{C}, l]-f(l)\\ [\mathfrak{C}^*, l] &= [\mathfrak{C}, l]+F(l)\end{aligned}\right\} \quad [\text{in the notation of Theorem 8 (4)}].$$

(*b*) *If* $\mathfrak{c}(l)$ *is a certain representative of* $\mathrm{T}(\mathfrak{C}, l)$ *for irregular* $\mathrm{T}(\mathfrak{C}, l)$, *then every element of* $\mathfrak{C}$ *has the form*

$$\mathfrak{a}+\sum_{j=1}^{t}\mathfrak{c}(l_j),$$

where $\mathfrak{a}$ *is in* $\mathfrak{C}_*$, $0 \leqslant t$, *and* $l_1 < \ldots < l_t$.

THEOREM 9. *There exist irregular characteristic subgroups of* $\mathfrak{A}$, *if and only if*

(1) $p = 2$ *and*

(2) *there are two singular genera* l *and* k *such that* $l \geqslant k+2$†.

Remark. It is, therefore, sufficient for the existence of irregular characteristic subgroups for ∞ and at least one finite genus to be singular.

Proof. A. The necessity of condition (1) is an immediate consequence of condition (1) of Theorem 8. To prove the necessity of condition (2) we consider an irregular characteristic subgroup $\mathfrak{C}$ of $\mathfrak{A}$. Then the maximum type of at least one genus is irregular, say $\mathrm{T}(\mathfrak{C}, l)$, and there exists an element $\mathfrak{c}$ in $\mathfrak{C}$ with $I(\mathfrak{c}) = \mathrm{T}(\mathfrak{C}, l)$. By condition (3) (*a*) of Theorem 8, l and

$$k = s[2^{d(1,\,\mathfrak{c})}\mathfrak{c}]-n(\mathfrak{c})+d(1, \mathfrak{c})$$

are singular, and so

$$k < l+1-n(\mathfrak{c})+n(\mathfrak{c})-2,$$

† From Theorem 6, § 3, this theorem is equivalent to Shoda's Theorem 5, *loc. cit.*, provided that $\mathfrak{A}$ is finite.

since $\mathfrak{c}$ is irregular and $d(1, \mathfrak{c}) < d[d(\mathfrak{c}), \mathfrak{c}] = n(\mathfrak{c}) - 1$, *i.e.*

$$k \leqslant l-2.$$

B. Suppose that $p = 2$ and that k and l are singular genera of $\mathfrak{A}$ for which $k \leqslant l-2$.

If there exist elements of genus j in $\mathfrak{A}$, then we write

$$e(j) = \begin{cases} 0 & \text{for} \quad j \leqslant k, \\ 1 & \text{for} \quad k < j < l, \\ 2 & \text{for} \quad l \leqslant j, \quad \text{and} \end{cases}$$

$$\mathfrak{c}(j) = \begin{cases} \text{a regular element of genus } j \text{ and exponent order } e(l) \text{ for } k < j \neq l, \\ \mathfrak{r}_1 + \mathfrak{r}_2 \text{ for } j = l, \text{ where } \mathfrak{r}_1 \text{ and } \mathfrak{r}_2 \text{ are regular elements such that} \\ \qquad n(\mathfrak{r}_1) = 1, \quad g(\mathfrak{r}_1) = k, \\ \qquad n(\mathfrak{r}_2) = 2, \quad g(\mathfrak{r}_2) = l. \end{cases}$$

Since $l \geqslant k+2 \geqslant 2$, such elements $\mathfrak{c}(j)$ exist in $\mathfrak{A}$.

The conditions (1) and (2) of Theorem 8 are satisfied since $\mathfrak{c}(l)$ is irregular.

To verify condition (3) we have to discuss $\mathfrak{c}(l)$; clearly $n[\mathfrak{c}(l)] = 2$ and

$$I[\mathfrak{c}(l)] = \{k+2, l+1\},$$

i.e. $$d[\mathfrak{c}(l)] = 2, \quad d[2, \mathfrak{c}(l)] = 1, \quad d[1, \mathfrak{c}(l)] = 0, \quad \text{and}$$

$$g(1, l) = s[2^{d[1, \mathfrak{c}(l)]}\, \mathfrak{c}(l)] - h[\mathfrak{c}(l)] + d[1, \mathfrak{c}(l)] = k+2-2 = k,$$

and therefore (3) (*a*) is satisfied. Since $e(k) = 0$, (3) (*b*) and (3) (*c*) are also satisfied. Finally condition (4) of Theorem 8 is easily verified, and therefore there exists an irregular characteristic subgroup $\mathfrak{C}$ of $\mathfrak{A}$ such that

$$[\mathfrak{C}, l] = e(l) \quad \text{and} \quad T(\mathfrak{C}, l) = I[\mathfrak{c}(l)].$$

An analogue of Theorem 7, § 3, does not exist, since, as in Lemma 3, § 3, it is easy to verify the following statement.

If $\mathfrak{C}$ is an irregular characteristic subgroup of $\mathfrak{A}$, there exists a regular characteristic subgroup $\mathfrak{R}(\mathfrak{C})$ of $\mathfrak{A}$ isomorphic with $\mathfrak{C}$ such that†

$$[\mathfrak{C}, l] = [\mathfrak{R}(\mathfrak{C}), l]$$

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† From Theorem 6, § 3, this assertion is equivalent to Shoda's Theorem 6, *loc. cit.*, p. 619, provided that $\mathfrak{A}$ is finite.

Supplement: A Galois theory for the subgroups of an Abelian group.

We denote by A the group of all proper automorphisms of $\mathfrak{A}$ and adopt the following definitions:

If $\begin{Bmatrix}\mathfrak{U}\\ \mathfrak{Y}\end{Bmatrix}$ *is a subgroup of* $\begin{Bmatrix}\mathfrak{A}\\ \mathfrak{A}\end{Bmatrix}$ *then we denote by* $\begin{Bmatrix}\mathrm{A}(\mathfrak{U})\\ \mathfrak{A}(\mathfrak{Y})\end{Bmatrix}$ *the set of all elements* $\begin{Bmatrix}\alpha \text{ of } \mathrm{A}\\ \mathfrak{a} \text{ of } \mathfrak{A}\end{Bmatrix}$ *such that* $\begin{Bmatrix}\alpha(\mathfrak{u})=\mathfrak{u}\\ \nu(\mathfrak{a})=\mathfrak{a}\end{Bmatrix}$ *for all* $\begin{Bmatrix}\mathfrak{u} \text{ in } \mathfrak{U}\\ \nu \text{ in } \mathfrak{Y}\end{Bmatrix}$ †.

Obviously $\mathrm{A}(\mathfrak{U})$ and $\mathfrak{A}(\mathfrak{Y})$ are groups and $\mathfrak{A}[\mathrm{A}(\mathfrak{U})] \geqq \mathfrak{U}$, $\mathrm{A}[\mathfrak{A}(\mathfrak{Y})] \geqq \mathfrak{Y}$.

In particular, $\mathrm{A}(\{0\}) = \mathrm{A}$, $\mathfrak{A}(\{1\}) = \mathfrak{A}$, and $\mathrm{A}(\mathfrak{A}) = \{1\}$; but we have

THEOREM 10. *(a)* $\mathfrak{A}(\mathrm{A})$ *differs from* $\{0\}$ *if and only if*

(1) $p = 2$;

and (2) *there exists a singular genus m such that* $\mathfrak{A}$ *contains no elements of genus greater than m.*

(b) If $\mathfrak{A}(\mathrm{A})$ *differs from* $\{0\}$, *then* $\mathfrak{A}(\mathrm{A}) = \{\mathfrak{a}\}$, *where* $\mathfrak{a}$ *is the uniquely determined element of* $\mathfrak{A}$ *for which* $n(\mathfrak{a}) = 1$, $g(\mathfrak{a}) = m$.

Proof. Since $\rho(\mathfrak{x}) = -\mathfrak{x}$ is an automorphism of $\mathfrak{A}$, all elements of $\mathfrak{A}(\mathrm{A})$ satisfy $2\mathfrak{a} = 0$, and therefore $\mathfrak{A}(\mathrm{A}) = \{0\}$ for $p \neq 2$.

If $p = 2$, $n(\mathfrak{a}) = 1$, and if there exists an element $\mathfrak{b}$ in $\mathfrak{A}$ such that $n(\mathfrak{b}) = 1$ and $g(\mathfrak{b}) > g(\mathfrak{a})$, then $I(\mathfrak{a}+\mathfrak{b}) = I(\mathfrak{a})$ and $\mathfrak{a}+\mathfrak{b} \neq \mathfrak{a}$, *i.e.* $\mathfrak{a}$ is not contained in $\mathfrak{A}(\mathrm{A})$; if there exists an element $\mathfrak{b}$ in $\mathfrak{A}$ with $n(\mathfrak{a}) = n(\mathfrak{b})$ and $g(\mathfrak{a}) = g(\mathfrak{b})$, but $\mathfrak{a} \neq \mathfrak{b}$, then $I(\mathfrak{a}) = I(\mathfrak{b})$, and $\mathfrak{a}$ is not contained in $\mathfrak{A}(\mathrm{A})$.

But if $\mathfrak{a}$ is such that $n(\mathfrak{a}) = 1$ and no element $\mathfrak{b} \neq \mathfrak{a}$ exists in $\mathfrak{A}$ for which $n(\mathfrak{a}) = n(\mathfrak{b})$, $g(\mathfrak{a}) \leqq g(\mathfrak{b})$, then obviously no element $\mathfrak{b} \neq \mathfrak{a}$ exists in $\mathfrak{A}$ for which $I(\mathfrak{a}) = I(\mathfrak{b})$, and this (uniquely determined) element $\mathfrak{a}$ is contained in $\mathfrak{A}(\mathrm{A})$.

THEOREM 11. *If* $\mathfrak{C}$ *is a characteristic subgroup of* $\mathfrak{A}$ *different from* $\{0\}$, *then*

$$\mathfrak{A}[\mathrm{A}(\mathfrak{C})] = \mathfrak{C}.$$

Proof. If $\mathfrak{a}$ is an element of $\mathfrak{A}$, but not of $\mathfrak{C}$, and $\mathfrak{a} = \mathfrak{r}_1 + \ldots + \mathfrak{r}_{d(\mathfrak{a})}$ is a canonical representation of $\mathfrak{a}$, then at least one of the regular elements $\mathfrak{r}_j$, say $\mathfrak{r}_w$, is not contained in $\mathfrak{C}$.

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† *Cp.* K. Shoda, "Über den Automorphismenring bezw. die Automorphismengruppe einer endlichen Abelschen Gruppe", *Proc. Imp. Acad. Japan*, 6 (1930), 10.

Case 1. $\mathfrak{r}_w$ does not occur in any canonical representation of any element of $\mathfrak{C}$.

Now choose a number t such that $t \not\equiv 1 \bmod p^{n(\mathfrak{r}_w)}$, $t \equiv 1 \bmod p^{n(\mathfrak{r}_w)-1}$, and t is prime to p.

Such a number exists except when $p = 2$ and $n(\mathfrak{r}_w) = 1$. Using a suitable basis of $\mathfrak{A}$, we may construct an automorphism α of $\mathrm{A}(\mathfrak{C})$ such that

$$\alpha(\mathfrak{r}_j) = \begin{cases} \mathfrak{r}_j & \text{for } j \neq w, \\ t\mathfrak{r}_w & \text{for } j = w. \end{cases}$$

We have $\alpha(\mathfrak{a}) \neq \mathfrak{a}$, and therefore $\mathfrak{a}$ is not contained in $\mathfrak{A}[\mathrm{A}(\mathfrak{C})]$.

If $p = 2$ and $n(\mathfrak{r}_w) = 1$, there exists an element $\mathfrak{u} \neq \mathfrak{r}_w$ in $\mathfrak{A}$ such that $n(\mathfrak{u}) = 1$ and $g(\mathfrak{u}) = g(\mathfrak{r}_w)$. This follows from Theorem 10, because $\mathfrak{C} \neq \{0\}$ and $\mathfrak{r}_w$ is not in $\mathfrak{C}$, and from Theorems 5 and 8. Then we may construct an automorphism α of $\mathrm{A}(\mathfrak{C})$ such that

$$\alpha(\mathfrak{r}_j) = \begin{cases} \mathfrak{r}_j & \text{for } j \neq w, \\ \mathfrak{r}_w + \mathfrak{u} & \text{for } j = w, \end{cases}$$

and so $\alpha(\mathfrak{a}) \neq \mathfrak{a}$, *i.e.* $\mathfrak{a}$ is not in $\mathfrak{A}[\mathrm{A}(\mathfrak{C})]$.

Case 2. $\mathfrak{r}_w$ occurs in a canonical representation of a certain element of $\mathfrak{C}$.

Since $\mathfrak{r}_w$ is not in $\mathfrak{C}$, it follows from Theorem 5, § 3, that $\mathfrak{C}$ is irregular, and from Theorem 8, § 4, that $p = 2$ and that $\mathfrak{r}_w$ occurs in a canonical representation of a suitable irregular element of the maximum type of its genus, say in

$$\mathfrak{c}(l) = \mathfrak{r}_1(l) + \ldots + \mathfrak{r}_{d[\mathfrak{c}(0)]}(l).$$

Since $\mathfrak{a}$ is not contained in $\mathfrak{C}$, there exists a number u such that $\mathfrak{r}_u(l) \neq \mathfrak{r}_j$ for any j, and therefore in particular $\mathfrak{r}_u(l) \neq \mathfrak{r}_w = \mathfrak{r}_x(l)$. A consequence of $\mathfrak{r}_w$ not being in $\mathfrak{C}$ is the existence of a regular element $\mathfrak{b}$ such that either

$$n(\mathfrak{b}) < n(\mathfrak{r}_w) \quad \text{or} \quad g(\mathfrak{b}) > g(\mathfrak{r}_w)$$

and either $$n(\mathfrak{b}) < n\big(\mathfrak{r}_u(l)\big) \quad \text{or} \quad g(\mathfrak{b}) > g\big(\mathfrak{r}_u(l)\big).$$

We may now construct an automorphism α of $\mathrm{A}(\mathfrak{C})$ such that

$$\alpha(\mathfrak{r}_j) = \begin{cases} \mathfrak{r}_j & \text{for } j \neq w, \\ \mathfrak{r}_w + \mathfrak{b} & \text{for } j = w, \end{cases} \qquad \alpha[\mathfrak{r}_j(l)] = \begin{cases} \mathfrak{r}_j(l) & \text{for } j \neq x,\ j \neq u, \\ \mathfrak{r}_x(l) + \mathfrak{b} & \text{for } j = x, \\ \mathfrak{r}_u(l) - \mathfrak{b} & \text{for } j = u, \end{cases}$$

and since once again $\alpha(\mathfrak{a}) \neq \mathfrak{a}$, the element $\mathfrak{a}$ is not contained in $\mathfrak{A}[A(\mathfrak{C})]$. Hence we have proved that $\mathfrak{A}[A(\mathfrak{C})] = \mathfrak{C}$.

Since, for $p \neq 2$, the subgroup of A generated by the automorphism $\rho(\mathfrak{a}) = -\mathfrak{a}$ is an invariant subgroup of A and since $A[\mathfrak{A}(\{\rho\})] = A$, a similar theorem for the invariant subgroups of A is not true.

THEOREM 12. (*a*) *If* Γ *is an invariant subgroup of* A, *then* $\mathfrak{A}(\Gamma)$ *is a characteristic subgroup of* $\mathfrak{A}$.

(*b*) *If* $\mathfrak{C}$ *is a characteristic subgroup of* $\mathfrak{A}$, *then* $A(\mathfrak{C})$ *is an invariant subgroup of* A.

Proof. (*a*) If $\mathfrak{a}$ is in $\mathfrak{A}(\Gamma)$, γ in Γ, and α in A, we have

$$\gamma[\alpha(\mathfrak{a})] = \alpha[\gamma'(\mathfrak{a})], \quad \text{where } \gamma' \text{ is in } \Gamma \text{ and therefore } = \alpha(\mathfrak{a}),$$

i.e. if $\mathfrak{a}$ is contained in $\mathfrak{A}(\Gamma)$, so is $\alpha(\mathfrak{a})$, and therefore $\mathfrak{A}(\Gamma)$ is a characteristic subgroup of $\mathfrak{A}$.

(*b*) If $\mathfrak{c}$ is in $\mathfrak{C}$, α in A, and γ in $A(\mathfrak{C})$, then

$$\alpha\left[\gamma(\alpha^{-1}[\mathfrak{c}])\right] = \alpha[\gamma(\mathfrak{c}')] \quad (\text{where } \mathfrak{c}' \text{ is in } \mathfrak{C})$$

$$= \alpha(\mathfrak{c}') = \alpha[\alpha^{-1}(\mathfrak{c})] = \mathfrak{c},$$

i.e. $\alpha^{-1}\gamma\alpha$ is contained in $A(\mathfrak{C})$ for every α of A, *i.e.* $A(\mathfrak{C})$ is an invariant subgroup of A.

[*Added* 22 *June*, 1935. In the meantime there has appeared a paper by G. Birkhoff, *Proc. London Math. Soc.* (2), 38 (1934), 385–401, to which the same remarks apply as were made in the introduction about Shoda's paper. In particular Birkhoff has proved our Theorem 1, § 1, in the case when $\mathfrak{A}$ is finite. But his proof, and in particular his Corollary 9.2, cannot be generalised to infinite $\mathfrak{A}$.]

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SUR L'ÉQUATION FONCTIONNELLE DE CHAPMAN ET SUR LE PROBLÈME DES PROBABILITÉS EN CHAÎNE

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Introduction.

En étudiant le phénomène de la diffusion, S. Chapman a été conduit à introduire* l'équation fonctionnelle

$$(1) \qquad \Phi(\rho, r+r_0, r, t) = \int \Phi(\rho', r, r, t)\,\Phi(\rho-\rho', r_0, r+\rho', t+r)\,d\rho'.$$

B. Hostinsky a étendu la signification et la validité de cette équation hors du problème spécial de la diffusion. Il a fait observer qu'elle généralisait l'équation fonctionnelle obtenue précédemment par Smoluchowski pour ce même problème.

B. Hostinsky a, en outre, mis l'équation de Chapman sous une forme plus symétrique†

$$(2) \qquad \Phi(x_1, x_3, t_1, t_3) = \int_a^b \Phi(x_1, x_2, t_1, t_2)\,\Phi(x_2, x_3, t_2, t_3)\,dx_2$$

équivalente à celle de Chapman dans le cas où l'espace où s'effectue la diffusion n'a qu'une dimension.

De plus, le même auteur a découvert des solutions très générales de l'équation de Chapman en utilisant un ingénieux artifice basé sur la méthode "d'intégration des substitutions" due à V. Volterra. Plusieurs

* C'est l'équation 10, p. 39, de son mémoire "On the Brownian displacements and thermal diffusion of grains suspended in a non-uniform fluid", *Proc. Royal Soc.* (A), 119 (1928), 34–54..

† Voir l'équation (71), p. 59, de l'ouvrage *Methodes générales du calcul des probabilités* par B. Hostinsky, chez Gauthier-Villars, 1931.

des résultats encore inédits de B. Hostinsky ont été indiqués par lui dans deux récentes conférences à Paris.

D'autre part, A. Kolmogoroff a considéré directement une équation fonctionnelle très générale, valable pour un domaine à un nombre quelconque de dimensions, se réduisant à l'équation (2) dans le cas d'une dimension et qui peut être considérée comme une forme très générale de l'une des conditions qui se présente dans le problème des probabilités en chaîne étudié par Markoff.

Enfin, A. Kolmogoroff avait aussi donné des solutions particulières de l'équation (2) et montré que des classes très étendues de ses solutions vérifient des équations aux dérivées partielles linéaires et du second ordre du type parabolique*.

Nous nous proposons d'indiquer ici des solutions très générales, de formes différentes des formes données par les auteurs précédents, de l'équation de Chapman.

Nous y sommes arrivé par deux méthodes que nous avions antérieurement fait prévoir. Nous ne donnerons ici que la première méthode. (Les deux méthodes fournissent deux formes distinctes et dont l'équivalence n'est ni apparente, ni certaine†.)

Nous examinerons le cas général de ν dimensions dans la Section I. Puis nous étudierons, en détail, dans la Section II, un exemple effectif d'une solution très générale dans le cas d'une seule dimension.

Première Méthode.

I.—*Cas général de ν dimensions.*

Notre première méthode est fondée sur l'emploi d'un procédé très fréquemment usité en Physique Mathématique. Ce procédé consiste à chercher d'abord une solution particulière sous la forme d'un produit de deux fonctions G, H ne dépendant pas des mêmes variables. Les variables intervenant dans le problème sont réparties en deux groupes, la fonction G dépendant d'un groupe, la fonction H de l'autre. Alors une solution

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* Voir les équations (125), p. 447, et (133), p. 450, de son mémoire: "Über die analytischen Methoden in der Warscheinlichkeitsrechnung", *Math. Annalen*, 104 (1931), 415–458.

† La seconde méthode a été résumée dans une note aux *Comptes rendus*, 200 (1935), 369, et sera développée dans un mémoire en cours d'impression aux *Annali Sc. Norm. Sup.* (Pisa, 1935).

plus générale est obtenue comme somme d'un nombre fini ou infini de ces produits.

Appliquons la méthode à l'équation de Chapman. Pour cela, nous l'étendrons d'abord à un domaine V de l'espace à ν dimensions ($\nu = 1, 2, 3, \ldots$) où se déplacent les points M, P, Q. Et nous écrirons l'équation de Chapman sous la forme:

$$(C) \qquad f(M, s;\ P, t) = \int_V f(M, s;\ Q, u) f(Q, u;\ P, t)\, dQ,$$

où dQ est un élément (entourant Q) de la région V et où l'intégrale est ν-uple*.

Dans le problème des probabilités en chaîne, $f(M, s;\ P, t)\, dP$ est la probabilité élémentaire pour que l'état d'un système physique, déterminé par ν paramètres, et représenté par M à l'instant s, devienne à l'instant t l'un des états d'un certain ensemble élémentaire dP d'états. Bien entendu, dans cette interprétation, la fonction f devra être assujettie à des conditions particulières telles que $f \geqslant 0$ et on devra supposer $s \leqslant u \leqslant t$.

Mais pour le moment, nous ne nous occupons pas de ces conditions supplémentaires et nous cherchons seulement à vérifier l'équation de Chapman.

Premières solutions comme sommes d'un nombre fini de produits. Ceci observé, le changement que nous avons intentionnellement fait apparaitre dans l'ordre des variables qui figurent dans l'équation de B. Hostinsky, suggère de lui-même l'application de la méthode générale indiquée plus haut. Nous allons donc chercher s'il existe des solutions de l'équation (C) de la forme

$$(3) \qquad f(M, s;\ P, t) = \sum_{i=1}^{n} A_i(M, s)\, B_i(P, t).$$

Pour une telle fonction, si elle n'est pas identiquement nulle, il est toujours possible de supposer que les fonctions $A_1(M, s), \ldots, A_n(M, s)$ sont linéairement indépendantes pour au moins une valeur de s quand M se déplace sur v; et de même pour les B_i. Sans quoi, au moyen de transformations linéaires classiques de ces fonctions, on arriverait à une expression similaire à la précédente, mais où le nombre n a pu être réduit sans être nul.

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* Pour les raisons exposées ci-dessus, il serait peut-être plus indiqué d'appeler cette équation, l'équation de Chapman-Kolmogoroff.

En supposant ces manipulations faites d'avance, il suffit de substituer l'expression (3) dans l'équation (C) pour s'assurer que *la condition nécessaire et suffisante pour qu'une fonction de la forme*

$$(3) \qquad f(M, s;\ P, t) = \sum_{i-1}^{n} A_i(M, s)\, B_i(P, t)$$

vérifie l'équation fonctionnelle

$$(C) \qquad f(M, s;\ P, t) = \int_V f(M, s;\ Q, u) f(Q, u;\ P, t)\, dQ$$

est que, pour les valeurs de u considérées, les fonctions $A_i(Q, u)$, $B_j(Q, u)$ *forment un système biorthonomé sur* V, c'est à dire qu'on ait

$$(4) \qquad \begin{cases} \int_V A_i(Q, u)\, B_j(Q, u)\, dQ = \delta_{ij}, & \text{avec} \\ \delta_{ii} = 1, \quad \text{et} \quad \delta_{ij} = 0 \quad \text{pour} \quad i \neq j. \end{cases}$$

Exemple. Par exemple, considérons le cas où, ν étant égal à 1, le domaine V se réduit à l'intervalle $(0, 2\pi)$. En remplaçant, suivant les notations habituelles, M, P, Q par x, y, z, on voit que, parmi les solutions de l'équation fonctionnelle

$$(5) \qquad f(x, s;\ y, t) = \int_0^{2\pi} f(x, s;\ z, u) f(z, u;\ y, t)\, dz$$

figurent des sommes de la forme

$$(6) \qquad \frac{1}{\pi}\left\{\tfrac{1}{2}\theta_0(s, t) + \sum_{p=1}^{n} [\theta_p(s, t) \cos px \cos py + \chi_p(s, t) \sin px \sin py]\right\},$$

avec

$$(6') \qquad \theta_p(s, t) = \begin{cases} 0 \text{ ou} \\ \dfrac{a_p(s)}{a_p(t)} \end{cases} \quad (p = 0, 1, \ldots, n),$$

$$\chi_p(s, t) = \begin{cases} 0 \text{ ou} \\ \dfrac{b_p(s)}{b_p(t)} \end{cases} \quad (p = 1, 2, \ldots, n),$$

quel que soit l'entier n et quelles que soient les fonctions (non nulles) $a_p(t)$ et $b_p(t)$.

Nous ne nous préoccupons pas pour l'instant de savoir si de telles sommes d'un nombre fini de termes pourraient vérifier toutes les conditions

qui se présentent dans le problème des probabilités en chaîne. Mais ce sont certainement des solutions de l'équation fonctionnelle (5).

Un cas où les solutions (3) *vérifiant les conditions* (4) *sont les plus générales.* Quand nous voudrons faire intervenir les conditions propres au problème de Markoff, nous verrons que les solutions (3) ne peuvent convenir et qu'il faut faire intervenir des séries au lieu de sommes d'un nombre fini de termes. Il est néanmoins intéressant de signaler un cas où les solutions que nous venons de trouver sont les plus générales. Il suffit pour le voir de recourir à l'artifice que nous avons utilisé dans une note aux *Comptes Rendus de l'Académie des Sciences de Paris**. Il consiste à transformer l'équation de Chapman de façon à pouvoir lui appliquer un des résultats de la théorie de Fredholm. A cet effet, effectuons d'abord sur s, u, t une même transformation $S=\lambda(s)$, $U=\lambda(u)$, $T=\lambda(t)$, où $\lambda(s)$ est une fonction continue et croîssante de s et qui varie de 0 à 1 quand s varie de $-\infty$ à $+\infty$. Alors $f(M, s;\ P, t)$ devient une fonction de la forme $F(M, S;\ P, T)$ et l'équation de Chapman devient

$$(7) \qquad F(M, S;\ P, T)=\int_V F(M, S;\ Q, U)\,F(Q, U;\ P, T)\,dQ.$$

Cherchons maintenant celles des solutions de l'équation (C) qui sont vérifiées quel que soit l'ordre de grandeur des valeurs de s, u, t. [Dans le problème des probabilités en chaîne, on n'impose à $f(M, s;\ P, t)$ la condition de vérifier (C) que si $s<u<t$, de sorte que les solutions qu'on va chercher ne sont peut-être pas toutes celles qui résolvent ce problème.]

Dans l'hypothèse faite, la variable U qui figure dans (7) peut varier de 0 à 1 quand S, T sont fixes et on a, en intégrant dans ces limites par rapport à U,

$$F(M, S;\ P, T)=\int_0^1\left[\int_V F(M, S;\ Q, U)\,F(Q, U;\ P, T)\,dQ\right]dU.$$

On peut écrire cette équation en appelant α, β, γ les couples $M, S;\ P, T;\ Q, U$, en désignant par W la région de l'espace à $\nu+1$ dimensions décrite par γ quand Q décrit V et U varie de 0 à 1, et en posant $K_1(\alpha, \beta)=F(M, S;\ P, T)$. On a alors

$$K_1(\alpha, \beta)=\int_W K_1(\alpha, \gamma)\,K_1(\gamma, \beta)\,d\gamma.$$

* *Comptes rendus*, 198 (1934), 2053.

En considérant $K_1(\alpha, \beta)$ comme un noyau de Fredholm, le second membre n'est autre que le premier itéré K_2 de K_1 sur W.

Ainsi lorsque $f(M, s; P, t)$ vérifie l'équation de Chapman quel que soit u (le raisonnement subsiste si cette équation cesse d'être vérifiée pour des valeurs isolées de u, par exemple pour $u = s$ et $u = t$, ce qui a son intérêt par la suite), le noyau correspondant $K_1(\alpha, \beta)$ est égal à son premier itéré sur W et par suite aussi à tous ses itérés sur W.

Quand le domaine W et la fonction K_1 sont tels que la théorie de Fredholm s'applique, on sait que l'égalité de K_1 et de K_2 ne peut avoir lieu que si $K_1(\alpha, \beta)$ est de rang fini, c'est à dire si K_1 est de la forme d'une somme d'un nombre fini de termes de la forme

$$K_1(\alpha, \beta)_1 = \sum_i \psi_i(\alpha)\, \theta_i(\beta).$$

Dans ce cas, $f(M, s; P, t)$ est nécessairement de la forme (3) et nous avons vu qu'alors les conditions (4) sont nécessairement vérifiées et suffisent pour que (3) soit une solution.

On ne connait pas de conditions simples à la fois nécessaires et suffisantes pour que les théorèmes fondamentaux de Fredholm s'appliquent. Mais on connait des cas très généraux où ils sont applicables. C'est par exemple ce qui a lieu si, W étant borné, K_1 est borné sur W ou si plus généralement, l'intégrale multiple de son carré, étendue à W $\int_W \int_W [K_1(\alpha, \beta)]^2 d\alpha\, d\beta$ est finie. C'est aussi ce qui a lieu quand W étant borné et $K_1(\alpha, \beta)$ étant borné quand la distance $\alpha\beta$ est supérieure à un nombre positif, l'ordre de K_1 est inférieur à celui de $1/(\alpha\beta)^{\nu+1}$. D'après M. Giraud*, il en est encore ainsi quand W étant borné et fermé et $K_1(\alpha, \beta)$ continu pour $\alpha \neq \beta$, on a $|K_1(\alpha, \beta)| < g(\alpha\beta)/(\alpha\beta)^{\nu+1}$ où $g(t)/t^{\nu+1}$ est une fonction décroissante et où $\int_0 \frac{g(t)}{t}\, dt$ est une intégrale convergente.

En retournant à V et à f, on voit que : *la forme* (3) *obtenue est, moyennant les conditions* (4), *la forme la plus générale de celles des solutions de l'équation de Chapman vérifiée quels que soient s, u, t dans le cas où le domaine V étant borné, on se limite aux solutions f de* (C), *qui sont assujetties aux conditions très générales suivantes* :

(I) $f(M, s; P, t)$ reste borné quand M, P varient sur V et s et t prennent toutes les valeurs possibles ; ou plus généralement

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* *Bull. Soc. Math. France, C.R. des séances,* 61 (1933), 28.

(II) L'intégrale multiple

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} ds\,dt \int_V\int_V f^2(M, s;\ P, t)\,dM\,dP$$

est finie, ou encore

(III) $f(M, s;\ P, t)$ restant bornée et continue quand la distance MP ou la quantité $|t-s|$ restant supérieure à un nombre positif arbitraire, on a

$$|f(M, s;\ P, t)| \leqslant g(MP+|t-s|)/[MP+(t-s)]^{\nu+1}$$

où $g(\lambda)/\lambda^{\nu+1}$ est une fonction décroissante et $\int_0 \frac{g(\lambda)}{\lambda}\,d\lambda$ une intégrale convergente, ou enfin plus généralement

(IV) Toutes les fois que V sera tel et que la fonction cherchée f sera supposée telle, que le noyau $K_1(\alpha, \beta)$ soit justiciable des théories de Fredholm sur le domaine W.

Retour au problème des probabilités en chaîne. Si l'on avait été conduit par une voie purement mathématique au problème de la résolution de l'équation de Chapman, il est probable qu'on se serait tenu comme très satisfait du résultat précédent où les conditions imposées à f auraient paru assez lâches pour se préter à toutes les applications.

Cependant, comme nous allons le voir, aucune d'elle ne convient au problème des probabilités en chaîne et aucune des solutions sous la forme (3) de la somme d'un nombre fini de produits, ne sont alors acceptables.

Et, tout d'abord, l'hypothèse de l'existence d'une probabilité élémentaire $f(M, s;\ P, t)\,dP$ ne peut être acceptée entièrement. Une exception devra être formulée pour le cas où l'on a à la fois $s = t$, $M = P$. En effet, faisons d'abord seulement l'hypothèse plus large, qu'il y a une probabilité déterminée pour passer de l'état M à l'époque s à l'un quelconque des états P d'un certain ensemble v d'états à l'époque t. (En se limitant au cas où v est un ensemble mesurable dans l'espace à ν dimensions, et où bien entendu v appartient à V.) Cette probabilité pourra être représentée sous la forme $\varpi(M, s;\ v, t)$. Il est bien clair que, pour $s = t$, cette probabilité sera égale à l'unité quand M appartient à v, et à zéro dans le cas contraire. Une hypothèse différente reviendrait à admettre la possibilité

de sauts instantanés (qui, comme les chocs et percussions, ne peuvent être que des conceptions approchées des phénomènes réels).

Ceci étant, si l'on suppose que $\varpi(M, s;\ v, s)$ peut se représenter sous la forme d'une intégrale de Lebesgue,

$$\int_v f(M, s;\ P, s)\,dP,$$

on voit que si $P_0 \neq M$ et si v, entourant P_0, est assez petit pour ne pas contenir M, on aura

$$\int_v f(M, s;\ P, s)\,dP = 0.$$

Si f est continue en P, près de P_0, et si v tend à se réduire à P_0, cette égalité (transformée par la formule de la moyenne en divisant par la mesure de v) montre que

$$f(M, s;\ P_0, s) = 0, \quad \text{pour} \quad M \neq P_0. \tag{8}$$

On en conclut *qu'il est impossible* de représenter ϖ pour $s = t$ sous la forme

$$\varpi(M, s;\ v, s) = \int_v f(M, s;\ P, s)\,dP$$

quand M appartient à v. En effet dans ce cas, le premier membre est égal à l'unité et le second membre $f(M, s;\ P, s)$ est nul quand P varie sur tout le domaine d'intégration v, sauf peut être quand P vient en M. Même si l'on attribue à $f(M, s;\ M, s)$ une valeur infinie, l'intégrale du second membre serait nulle et l'on arriverait à l'égalité $1 = 0$. Dès lors nous voyons bien qu'il est possible de faire les hypothèses simplificatrices ; que

$$\varpi(M, s;\ v, t) = \int_v f(M, s;\ P, t)\,dP \tag{8 bis}$$

et que $f(M, s;\ P, t)$ est par rapport à P une fonction continue, mais à *condition d'introduire une exception* pour la possibilité de la représentation (8 bis) de $\varpi(M, s;\ v, t)$ quand M appartenant à v, s est égal à t, et pour $f(M, s;\ P, t)$ quand, à la fois, $M \equiv P$ et $s = t$, *cas où l'existence même de f doit être déniée*, sans qu'il suffise de supposer f discontinu ou même infini pour cet ensemble d'égalités.

Il en résulte en particulier qu'il n'y a pas non plus lieu de supposer l'équation de Chapman vérifiée quand, à la fois, $M = P$ et $s = t$, ni même de supposer qu'elle ait dans ce cas une signification.

Montrons maintenant que la solution (3) doit être rejetée quand on envisage le cas des probabilités en chaîne. On peut en donner diverses raisons. Par exemple observons, qu'en vertu des conditions (4), on a

$$\int_V A_j(P, s)\left[\sum_{i=1}^{n} A_i(M, s)\, B_i(P, s)\right] dP = \sum_{i=1}^{n} \delta_{ji} A_i(M, s) = A_j(M, s).$$

Or, en vertu de (8), le crochet est nul quand P est distinct de M, si ce crochet représente $f(M, s;\ P, s)$. La quantité sous le signe $\int$ étant nulle sur V, sauf peut-être quand P vient en M, l'intégrale est nulle et par suite $A_j(M, s)$ serait nul quel que soit s, quel que soit M sur V et pour toute valeur entière de j depuis 1 jusqu'à n. Par suite, contrairement à l'hypothèse, les fonctions $A_j(M, s)$ ne seraient pas linéairement indépendantes.

Ainsi la représentation de $f(M, s;\ P, t)$ par une somme de la forme (3) composée d'un nombre *fini* de termes (la question d'une représentation quelconque étant d'ailleurs exclue d'avance pour $M \equiv P$ et $s = t$ simultanément) est *impossible* dans le cas des probabilités en chaîne.

On en vient naturellement à se demander si dans ce même cas, on ne pourrait trouver des solutions de la forme précédente, mais d'un nombre infini de termes.

Solutions sous formes de séries de produits $A_i(M, s)\, B_i(P, t)$. Cherchons donc s'il existe des solutions de l'équation de Chapman qui seraient de la forme*.

$$f(M, s;\ P, t) = \sum_{i=1}^{+\infty} A_i(M, s)\, B_i(P, t), \tag{9}$$

l'égalité n'étant pas supposée avoir lieu quand les couples M, s et P, t coïncident, cas où f n'est pas définie.

Les termes du second membre seront encore définis dans ce cas, mais la convergence pourra cesser sans inconvénient quand les couples M, s, et P, t coïncident. Puisque dans le cas où $f(M, s;\ P, t)\, dP$ est une probabilité élémentaire, on suppose $s \leqslant t$, la convergence de la série du second membre pourra aussi cesser sans inconvénient pour $s > t$.

Nous serons enfin naturellement conduits à nous limiter au cas où,

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* M. Hostinsky avait déjà considéré le cas de solutions de la forme $\sum_i \Phi_i(M)\, \Psi_i(P)\, f_i(s, t)$ où les A_i, B_i forment deux systemes orthogonaux et où $f_i(s, t) = f_i(s, a) f_i(a, t)$. Comme j'ai démontré ailleurs que cette dernière condition suppose que f_i est de la forme $f_i(s, t) = a_i(s)/a_i(t)$, on voit que la forme de M. Hostinsky est un cas particulier de la nôtre, celui où $A_i(M, s) = \Phi_i(M)\, a_i(s)$, $B_i(P, t) = \Psi_i(P)/a_i(t)$.

comme précédemment, les fonctions A_i, B_j forment encore un système bi-orthonomé sur V, c'est à dire vérifient les conditions (4).

Mais alors on pourrait recommencer le raisonnement pour prouver que les A_j sont tous nuls si l'on supposait qu'il y eut convergence uniforme de $\sum_i A_i(M, s) B_i(P, s)$ pour chaque valeur finie de s.

En sorte que nous sommes conduits à nous limiter aux solutions de la forme (3) vérifiant les conditions (4) et pour lesquelles la série

$$\sum_i A_i(M, s) B_i(P, t)$$

est supposée converger uniformément vers $f(M, s;\ P, t)$ quand M et P varient indépendamment sur V pour chaque couple de valeurs fixes de s et t *telles que* $s < t$. On aura alors d'après (8 bis) pour $s < t$,

$$\varpi(M, s;\ v, t) = \sum_i A_i(M, s)\beta_i(v, t),$$

avec

$$\beta_i(v, t) = \int_v B_i(P, t)\, dP$$

et pour $s = t$ on posera

$$\varpi(M, s;\ v, s) = \delta(M, v), \tag{10}$$

où $\delta(M, v) = 1$ si M appartient à v et $\delta(M, v) = 0$ dans le cas contraire.

Autres conditions à réaliser. Dans le problème des probabilités en chaîne, on doit encore assujettir la probabilité à deux autres conditions évidentes

$$\varpi(M, s;\ V, t) = 1, \tag{T}$$

$$\varpi(M, s;\ v, t) \geqslant 0. \tag{P}$$

[De l'ensemble de ces deux conditions et de l'égalité évidente

$$\varpi(M, s;\ v, t) + \varpi(M, s;\ V - v, t) = \varpi(M, s;\ V, t)$$

on déduit immédiatement qu'il en résultera aussi :

$$\varpi(M, s;\ v, t) \leqslant 1.]$$

La première condition, réalisée d'avance pour $s = t$, devient pour $s < t$,

$$\sum A_i(M, s)\beta_i(V, t) = 1.$$

En multipliant par $B_j(M, s)\,dM$ et intégrant sur V, on en conclut, en vertu de (4),

$$\beta_j(V, t) = \int_V B_j(M, s)\,dM,$$

pour $s < t$.

Il en résulte que la valeur commune des deux membres est indépendante de t; car soient t_1, t_2 deux valeurs quelconques de t; prenons s inférieur à la fois à t_1 et à t_2; $\beta_j(V, t_1)$ et $\beta_j(V, t_2)$ seront égaux à la même intégrale. Ainsi $\beta_j(V, t)$ est une constante β_j indépendante de t et on a

$$1 = \sum_i \beta_i A_i(M, s). \tag{11}$$

Autrement dit pour que la condition (5) soit réalisée, il faut et il suffit que: Parmi les fonctions représentables comme une combinaison linéaire illimitée des $A_i(M, s)$ uniformément convergente quand M varie sur V, doit figurer la fonction identique à l'unité. Il n'y a d'ailleurs pas dans ce cas deux combinaisons linéaires de cette espèce susceptibles de représenter l'unité, puisque

$$\beta_j = \int_V B_j(M, s)\,dM.$$

En ce qui concerne la condition (P), on pourra en général s'assurer sur chaque exemple effectif si elle est ou non réalisée.

Calcul des $A_j(M, s)$, $B_j(P, t)$. *Si* l'on sait que $f(M, s;\ P, t)$ est de la forme (9), est-il possible de déterminer les fonctions A_j, B_j connaissant f?

Dans cette hypothèse [les A_j, B_j étant supposées biorthonormées sur V et la série (9) étant uniformément convergente sur V pour s, t fixes tels que $s < t$], on voit que *chaque fonction* $A_j(M, s)$ *est une des solutions de l'équation fonctionnelle*

$$X(M, s) = \int_V f(M, s;\ Q, u)\,X(Q, u)\,dQ \tag{H}$$

quel que soit le nombre $u > s$. De même, *chaque fonction* $B_j(P, t)$ *est une des solutions de l'équation fonctionnelle associée*

$$Y(P, t) = \int_V f(Q, u;\ P, t)\,Y(Q, u)\,dQ, \tag{H'}$$

où $u < t$.

Inversement, supposons seulement que $f(M, s;\ P, t)$ soit une solution non identiquement nulle de l'équation de Chapman pour $s < u < t$, *sans savoir* si f est de la forme (9). Cela suffit pour assurer l'existence de

solutions non identiquement nulles de (H) et de (H'). En effet, puisque f n'est pas identiquement nulle, il existe M_0, s_0; P_0, t_0 avec $s_0 < t_0$, tels que $f(M_0, s_0;\ P_0, t_0) \neq 0$: on aura alors *la solution particulière, non identiquement nulle, de* (H),

$$X(M, s) = f(M, s;\ P_0, t_0)$$

au moins pour $s < u < t_0$, et, de même, *on aura la solution particulière, non identiquement nulle, de* (H'),

$$Y(P, t) = f(M_0, s_0;\ P, t)$$

au moins pour $t > u > s_0$.

Connaissant une solution $X(M, s)$ de (H), on en déduit une infinité d'autres en la multipliant par un nombre indépendant de M et de s. Et même, si l'on connait plusieurs solutions (en nombre fini ou infini) de (H),

$$X_0(M, s),\quad X_1(M, s),\quad \ldots,$$

toute combinaison linéaire de ces solutions

$$a_0 X_0(M, s) + a_1 X_1(M, s) + \ldots$$

(supposée uniformément convergente sur V quand leur nombre est infini) à coefficients $a_0, a_1, \ldots$ indépendants de M et s, *sera aussi solution de* (H). [Nous énonçons ici une propriété évidente, qui appartient aussi aux solutions d'une équation homogène de Fredholm; mais on notera que l'équation (H) n'est pas de ce type bien qu'elle lui ressemble beaucoup.]

En faisant varier P_0, t_0, la fonction $f(M, s;\ P_0, t_0)$ fournit une infinité de solutions. On observera que ces solutions ne peuvent se réduire à une seule par multiplication d'une constante. Et même *si l'on suppose que* $f(M, s;\ P, t)$ *est une densité de probabilité, solution non identiquement nulle de l'équation de Chapman, on peut en déduire l'existence et la forme d'une infinité de solutions de* (H) *qui sont linéairement indépendantes.* Nous savons que pour chaque choix du couple P, t, quand celui-ci est fixé la fonction $X(M, s) = f(M, s;\ P, t)$ est solution de l'équation (H). Il nous suffit de montrer qu'il n'est pas possible de déterminer des couples $P_0, t_0;\ P_1, t_1, \ldots, P_n, t_n$, en nombre fini quelconque n, tels que pour tout couple P, t on puisse déterminer des nombres $b_0, b_1, \ldots, b_n$ indépendants de M, s pour lesquels

$$f(M, s;\ P, t) = b_0 f(M, s;\ P_0, t_0) + \ldots + b_n f(M, s;\ P_n, t_n)$$

pour s inférieur à $t, t_0, \ldots, t_n$. En effet, dans ce cas $b_0, \ldots, b_n$ seront, en

général, des fonctions de P, t et f serait une solution de l'équation de Chapman de la forme

$$f(M, s;\ P, t) = a_0(M, s)\,b_0(P, t) + \ldots + a_n(M, s)\,b_n(P, t).$$

Or nous avons vu (p. 523) qu'il n'existe pas de solution de cette forme *si f est densité de probabilité.*

Il sera utile, dans un instant, d'avoir fait l'observation suivante: *il n'existe aucune solution* $X(M, s)$ *de* (H) *qui soit orthogonale sur* V *à toutes les solutions de* (H') *sans être identiquement nulle.* En effet, si, par exemple, $X(M_0, s_0) \neq 0$, comme $f(M_0, s_0;\ P, t)$ est une solution de (H') et que

$$X(M_0, s_0) = \int_V f(M_0, s_0, P, t)\,X(P, t)\,dP,$$

le second membre serait nul et le premier ne serait pas nul. De même, il n'existe aucune solution $Y(P, t)$ de (H') qui soit orthogonale à toutes les solutions de (H).

Revenons maintenant au cas où, *de plus*, f est de la forme (9). Alors, parmi les solutions linéairement indépendantes, en nombre infini de (H), on pourra en choisir une suite formant un système *complet* de solutions de (H), c'est à dire un système de solutions de (H) en nombre infini $X_1(M, s)$, $X_2(M, s)$, ... telles (1°) qu'aucune combinaison linéaire d'un nombre fini ou infini (et alors supposée uniformément convergente sur V) ne soit identiquement nulle sans avoir tous ses coefficients nuls (2°); que toute solution non identiquement nulle de (H) soit une combinaison linéaire des X_n.

En effet, il suffit de prendre pour les $X_n(M, s)$, les $A_n(M, s)$:

1°. Si l'on avait

$$a_1 A_1(M, s) + \ldots + a_n A_n(M, s) + \ldots = 0$$

et si la série était uniformément convergente sur V, alors en multipliant par $B_j(M, s)\,dM$ et intégrant sur V, on obtiendrait $a_j = 0$ quel que soit j en raison de (4).

2°. Si $X(M, s)$ est solution de (H), on a dans l'hypothèse actuelle,

$$\text{(11 bis)} \qquad X(M, s) = \int_V \Big[\sum_j A_j(M, s)\,B_j(Q, u)\Big]\,X(Q, u)\,dQ$$

$$= \sum_j a_j A_j(M, s)$$

avec

$$a_j = \int_V B_j(Q, u)\,X(Q, u)\,dQ.$$

D'autre part, en multipliant (11 bis) par $B_j(M, s)dM$ et intégrant, on a

$$a_j = \int_V B_j(M, s)\, X(M, s)\, dM.$$

Donc a_j ne dépend pas de u; alors on pourrait prendre pour u dans la relation ci-dessus un nombre variable ($> s$), et même tendant vers $+\infty$ de sorte qu'on peut supposer les a_j indépendants de u, comme de s.

De même l'équation (H') possédera au moins un système complet de solutions

$$Y_1(P, t), \quad Y_2(P, t), \quad \ldots, \quad Y_n(P, t), \quad \ldots.$$

On a montré qu'aucune des solutions $X(M, s)$, non identiquement nulles, de (H) ne pourra être orthogonale à toutes les solutions de (H'); et de même en permutant (H) et (H'). En vertu d'une propriété connue, il en résulte alors que connaissant les X_n, Y_n, *on peut former*, par un procédé régulier connu, *des combinaisons linéaires de ceux-ci constituant également des systèmes complets de solutions de* (H) *et de* (H') *mais qui seront biorthonormés sur* V, soient

$$U_1(M, s), \quad U_2(M, s), \quad \ldots,$$

$$V_1(M, s), \quad V_2(M, s), \quad \ldots.$$

Ceci étant, puisque pour P et t fixes, $f(M, s;\ P, t)$ est une solution de (H) et que les U_j forment un système complet de solutions de (H), on aura

$$f(M, s;\ P, t) = \sum_k b_k U_k(M, s).$$

Quand P et t varient, les b_k varieront. On a donc

$$f(M, s;\ P, t) = \sum_k U_k(M, s)\, b_k(P, t).$$

Or $V_j(P, t)$ est une solution de (H'); on a donc

$$V_j(P, t) = \int_V \left[\sum_k U_k(Q, u)\, b_k(P, t)\right] V_j(Q, u)\, dQ$$

$$= \sum_k b_k(P, t)\, \delta_{jk} = b_j(P, t).$$

De sorte que

$$f(M, s;\ P, t) = \sum_k U_k(M, s)\, V_k(P, t).$$

En résumé,

1°. *Si une densité de probabilité $f(M, s; P, t)$ satisfait à l'équation de Chapman, chacune des équations fonctionnelles* (H), (H′) *admet une infinité de solutions linéairement indépendantes et aucune solution non $\equiv 0$ de l'une n'est orthogonale à toutes les solutions de l'autre.*

2°. *Pour qu'une densité de probabilité $f(M, s; P, t)$ satisfaisant à l'equation de Chapman puisse être mise, au moins d'une façon, sous la forme*

$$f(M, s; P, t) = \sum_{j=1}^{+\infty} A_j(M, s)\, B_j(P, t) \tag{9}$$

où la convergence est uniforme pour $s < t$ et où les A_j, B_j forment un système biorthonormé sur V, il faut et il suffit que parmi les systèmes de solutions de (H) *il en existe un qui soit complet; et de même pour* (H′). *Si cette condition est réalisée, on saura par un procédé régulier connu déduire de ces deux systèmes, deux systèmes biorthonormés grâce auxquels on pourra mettre f sous la forme* (9).

Il reste d'ailleurs à savoir s'il existe des solutions f ne satisfaisant pas à la condition ci-dessus.

Calcul des solutions de (H) *et de* (H′). Pour appliquer la condition précédente, il faut pouvoir déterminer *toutes* les solutions de (H) et de (H′). [Nous connaissons déjà les solutions $X(M, s) = f(M, s; P_0, t_0)$, $Y(P, t) = f(M_0, s_0, P, t)$.]

On peut déterminer toutes les solutions de (H) et de (H′) valables pour s et t dans un intervalle déterminé (α, β), de la manière suivante.

Toute solution $X(M, s)$ de (H) dans cet intervalle sera telle que

$$X(M, s) = \int_V f(M, s; P, \beta)\, X(P, \beta)\, dP.$$

Inversement, prenons pour $X(P, \beta)$ une fonction arbitraire de P, soit $\phi(P)$ et considérons la fonction

$$X(M, s) = \int_V f(M, s; P, \beta)\, \phi(P)\, dP.$$

Elle définit pour $s < \beta$ une fonction bien déterminée de M et de s. Cette

fonction vérifie l'équation (H) pour $s < u < \beta$. Car on a

$$X(M, s) - \int_V f(M, s;\ Q, u)\, X(Q, u)\, dQ$$

$$= \int_V f(M, s;\ P, \beta)\, \phi(P)\, dP - \int_V f(M, s;\ Q, u) \left[\int_\Gamma f(Q, u;\ P, \beta)\, \phi(P)\, dP \right] dQ$$

$$= \int_V \left\{ f(M, s;\ P, \beta) - \int_V f(M, s;\ Q, u) f(Q, u;\ P, \beta)\, dQ \right\} \phi(P)\, dP$$

$$= 0.$$

De même, *on obtiendra la solution la plus générale de* (H') *pour* $t > \alpha$, *sous la forme*

$$Y(P, t) = \int_\Gamma f(M, \alpha;\ P, t)\, \psi(M)\, dM,$$

où $\psi(M)$ *est une fonction arbitraire.*

II.—*Un exemple de solution dans le cas d'une dimension.*

Nous allons donner un exemple où sont réalisées toutes les conditions du problème des probabilités en chaîne. Ce sera, un exemple obtenu en prolongeant indéfiniment une expression du type (6) rencontré plus haut. C'est à dire que ce sera une fonction de la forme

$$\text{(6 bis)} \quad f(x, s;\ y, t) = \frac{1}{\pi} \left\{ \tfrac{1}{2}\theta_0(s, t) + \sum_{p=1}^{+\infty} [\theta_p(s, t) \cos px \cos py + \chi_p(s, t) \sin px \sin py] \right\}$$

avec

$$\theta_p(s, t) = \begin{Bmatrix} 0 & \text{ou} \\ a_p(s)/a_p(t) \end{Bmatrix}, \quad \chi_p(s, t) = \begin{Bmatrix} 0 & \text{ou} \\ b_p(s)/b_p(t) \end{Bmatrix} \quad (p = 0, 1, 2, \ldots).$$

M. Hostinsky avait déjà donné* une solution de cette forme pour l'équation de Smoluchovski, cas particulier de l'équation de Chapman où $f(M, s;\ P, t)$ ne dépend de s et t que par l'intermédiaire de $t-s$. La solution de M. Hostinsky était donc un cas particulier de la solution ci-dessus: le cas où $\theta_p(s, t)$ est une fonction de $t-s$.

Avant de préciser le choix que nous ferons des $a_p(s)$, $b_p(s)$, faisons quelques remarques générales.

* Voir chapitre III de son mémoire, *Annales de l'Institut Henri Poincaré* (1931).

Les fonctions $a_p(s)$, $b(s)$ étant supposées toujours différentes de 0, si on veut se borner au cas simple où elles sont continues, ces fonctions seront chacune d'un signe constant. Et comme la série ne change pas en remplaçant l'un des a_p(ou b_p) par $-a_p$(ou $-b_p$), *on pourra supposer que tous les a_p et tous les b_p sont constamment positifs.* D'autre part, si la série f est uniformément convergente pour $s<t$, on pourra intégrer et pour réaliser la condition (T) écrire

$$1=\varpi(x,\ s;\ V,\ t)=\int_0^{2\pi} f(x,s;\ y,\ t)\,dy=\theta_0(s,\ t).$$

De sorte que si la condition (T) est réalisée, la fonction f sera de la forme

$$(13)\quad f(x,\ s;\ y,\ t)=\frac{1}{\pi}\left\{\tfrac{1}{2}+\sum_{n=1}^{\infty}[\theta_n\cos nx\cos ny+\chi_n\sin nx\sin ny]\right\}$$

avec $$\theta_n=\begin{Bmatrix}0 & \text{ou}\\ a_n(s)/a_n(t)\end{Bmatrix},\quad \chi_n=\begin{Bmatrix}0 & \text{ou}\\ b_n(s)/b_n(t)\end{Bmatrix}.$$

On pourrait chercher à quelles conditions doivent satisfaire les fonctions $a_n(s)$, $b_n(s)$ pour que cette série converge uniformément vers une somme $\geqslant 0$ pour $s\leqslant t$. Nous nous contenterons d'en donner un exemple effectif en prenant $\theta_n(s,\ t)=[\theta(s,\ t)]^n$, avec $\theta(s,\ t)=a(s)/a(t)$; $\chi_n(s,\ t)=0$ quel que soit n, et en supposant les fonctions $a(s)$ positives et constamment croissantes. Pour éviter quelques complications secondaires, nous restreindrons l'intervalle $(0,\ 2\pi)$ à l'intervalle $(0,\ \pi)$ ce qui nous amènera à doubler f.

Exemple. Soit donc, pour $s<t$ et $0\leqslant\begin{Bmatrix}x\\ y\end{Bmatrix}\leqslant\pi$, la fonction

$$(14)\qquad f(x,\ s;\ y,\ t)=\frac{1}{\pi}\left\{1+2\sum_{n=1}^{+\infty}\theta^n\cos nx\cos ny\right\}$$

avec $\theta=a(s)/a(t)$ où $a(s)$ est une fonction positive constamment croissante. Comme on a $\theta\geqslant 0$ pour $s\geqslant t$ et $\theta<1$ pour $s<t$, cette série ne converge pas pour $s\geqslant t$ mais elle converge pour $s<t$ et même elle converge uniformément quand x, y varient de façon quelconque et en particulier quand x, y varient sur l'intervalle $(0,\ \pi)$ que nous prenons pour domaine V d'intégration.

D'après la façon dont nous avons obtenu cette fonction, elle satisfait à l'équation de Chapman,

$$f(x,\ s;\ y,\ t)=\int_0^{\pi} f(x,\ s;\ z,\ u)f(z,\ u;\ y,\ t)\,dz$$

pour $s<u<t$, et à la condition

$$\text{(T)} \qquad \int_0^\pi f(x, s;\ y, t)\,dy = 1.$$

Mais on peut s'assurer de ces deux faits par une facile vérification directe.

Pour se rendre compte qu'elle vérifie aussi la condition

$$\text{(P)} \qquad f(x, s;\ y, t) > 0 \quad \text{pour} \quad s<t,$$

il suffit d'écrire f, sous la forme, obtenue par un calcul facile :

$$(15) \quad f(x, s;\ y, t) = \frac{1-\theta^2}{2\pi}\left\{\frac{1}{1-2\theta\cos(x+y)+\theta^2}+\frac{1}{1-2\theta\cos(x-y)+\theta^2}\right\}.$$

Il est clair que pour $s<t$ et par suite $0<\theta<1$, les numérateurs et dénominateurs des différentes fractions sont positifs.

Enfin pour $s=t$, nous n'avons pas à définir $f(x, s;\ x, s)$ mais nous pouvons nous contenter pour $x\neq y$ de poser $f(x, s;\ y, s)=0$. Observons que l'on peut aussi laisser subsister l'égalité (15), écrite pour $s<t$, dans le cas où l'on a simultanément $x\neq y$, $s=t$. Car alors $1-\theta^2=0$ et les deux dénominateurs sont encore positifs. [On a à utiliser l'observation que $x+y$ ne peut être un multiple entier (ou nul) de 2π quand $0\leqslant\left\{\begin{matrix}x\\y\end{matrix}\right\}\leqslant\pi$ que si $x-y$ est aussi un multiple de 2π et par suite si $x=y$.]

Pour $s=t$ la série (14) est divergente, mais on pourrait encore laisser subsister pour $s=t$, $x\neq y$, l'égalité (14), écrite pour $s<t$ si l'on y sommait la série par le procédé de la moyenne arithmétique.

Ordres de grandeur. Puisque $f(x, s;\ y, t)$ est nul, quand x étant différent de y, $s=t$, et même puisque si $x\neq y$, $f(x, s;\ y, t)$ tend vers 0 avec $t-s$ (quand $t-s$ tend par valeurs jamais négatives), on peut se demander quel est l'ordre de grandeur de f, en considérant $t-s$ comme infiniment petit principal.

Or on a :

$$\frac{1}{t-s}f(x, s;\ y, t) = \frac{f(x, s;\ y, t)}{\theta-1}\,\frac{\theta-1}{t-s} = LM$$

avec

$$L = \frac{-(1+\theta)}{2\pi}\left\{\frac{1}{1-2\theta\cos(x+y)+\theta^2}+\frac{1}{1-2\theta\cos(x-y)+\theta^2}\right\},$$

$$M = \frac{-1}{a(t)}\times\frac{a(t)-a(s)}{t-s}.$$

En supposant toujours $x \neq y$, L a certainement une limite,

$$\lim_{t\to s} L = \lim_{\theta\to 1} L = \frac{-1}{2\pi}\left\{\frac{1}{1-\cos(x+y)} + \frac{1}{1-\cos(x-y)}\right\} = \frac{-1}{\pi}\,\frac{1-\cos x\cos y}{(\cos x-\cos y)^2}.$$

Une condition nécessaire et suffisante pour que M ait une limite est que la fonction $a(s)$ soit dérivable à droite. On a alors

$$\lim_{t\to s} M = \frac{-a'(s)}{a(s)},$$

d'où

$$\lim_{t\to s} \frac{1}{t-s} f(x, s;\ y, t) = \frac{a'(s)}{\pi a(s)}\,\frac{1-\cos x\cos y}{(\cos x-\cos y)^2}, \tag{16}$$

pour $x \neq y$. Quand $x = y$, la forme (15) de f invite à distinguer deux cas suivant que $\cos(x+y)$ est aussi égal à 1, ou non. Supposons d'abord $0 < x = y < \pi$, ce qui est le second cas. Alors

$$f(x, s;\ x, t) = \frac{1}{2\pi}\left\{\frac{(1-\theta^2)}{1-2\theta\cos 2x+\theta^2} + \frac{1+\theta}{1-\theta}\right\}.$$

Dans le second cas $x = y = \left\{\begin{matrix} 0 \text{ ou} \\ \pi \end{matrix}\right\}$, on aura

$$f(x, s;\ x, t) = \frac{1}{\pi}\,\frac{1+\theta}{1-\theta}.$$

Et par suite dans les deux cas on a:

$$\lim_{s\to t} f(x, s;\ x, t) = +\infty. \tag{17}$$

Pour trouver l'ordre de grandeur de $f(x, s;\ x, t)$ on formera

$$(t-s)f(x, s;\ x, t) = L_1/M$$

avec

$$L_1 = (\theta-1)f(x, s;\ x, t).$$

Si la dérivée à droite $a'(s)$ de la fonction croissante $a(s)$ existe, elle est partout $\geqslant 0$. En tous points où $a'(s) \neq 0$, on aura

$$\lim_{t\to s} \frac{1}{M} = \frac{-a(s)}{a'(s)}.$$

Quand à L_1 on a

$$L_1 = \frac{1}{2\pi}\left\{\frac{(1-\theta^2)(\theta-1)}{1-2\theta\cos 2x+\theta^2} - (1+\theta)\right\},$$

et en particulier

$$L_1 = -\frac{1}{\pi}(1+\theta) \quad \text{si} \quad x = y = \begin{cases} 0 \text{ ou} \\ \pi \end{cases}.$$

D'où

$$\lim_{t \to s} L_1 = \begin{cases} -\dfrac{1}{\pi} & \text{si} \quad 0 < x = y < \pi, \\ -\dfrac{2}{\pi} & \text{si} \quad x = y = \begin{cases} 0 \text{ ou} \\ \pi \end{cases}, \end{cases}$$

et finalement

$$\lim_{t \to s} (t-s) f(x, s; x, t) = \begin{cases} \dfrac{1}{\pi} \dfrac{a(s)}{a'(s)} & \text{si} \quad 0 < x = y < \pi, \\ \dfrac{2}{\pi} \dfrac{a(s)}{a'(s)} & \text{si} \quad x = y = \begin{cases} 0 \text{ ou} \\ \pi \end{cases}. \end{cases}$$

Calcul de $\varpi(x, s; v, t)$. Pour $s \neq t$, la probabilité ϖ est donnée par

$$\varpi(x, s; v, t) = \int_v f(x, s; y, t)\, dy.$$

D'où en utilisant la série uniformément convergente (14),

$$\varpi(x, s; v, t) = \sum_{n=0}^{\infty} K_n(v)[\theta(s, t)]^n \cos nx,$$

et ϖ sera donné par une série trigonomètrique uniformément convergente quand x varie sur 0, avec

$$\theta(s, t) = \frac{a(s)}{a(t)},$$

$$K_0(v) = \frac{1}{\pi} \int_v dy;$$

$$K_n(v) = \frac{2}{\pi} \int_v \cos ny\, dy \quad (n = 1, 2, \ldots).$$

En particulier si v est l'intervalle J $(\alpha \leqslant y \leqslant \beta)$, on a

$$K_0(J) = \frac{\beta - \alpha}{\pi};$$

$$K_n(J) = \frac{2}{n\pi} (\sin n\beta - \sin n\alpha)$$

d'où

$$\varpi(x, s;\ J, t) = R(x, s;\ \beta, t) - R(x, s;\ \alpha, t)$$

ou

$$R(x, s;\ J, t) = \sum_{n=0}^{+\infty} k_n(\alpha)\,\theta^n \cos nx$$

avec

$$k_0(\alpha) = \frac{\alpha}{\pi},$$

$$k_n(\alpha) = \frac{2}{n\pi} \sin n\alpha.$$

On peut aussi calculer $\varpi(x, s;\ J, t)$ sous la forme,

$$\varpi(x, s;\ J, t) = \int_\alpha^\beta f(x, s;\ z, t)\,dz$$

$$= \frac{1-\theta^2}{2\pi} \left\{ \int_\alpha^\beta \frac{dz}{1-2\theta\cos(z+x)+\theta^2} + \int_\alpha^\beta \frac{dz}{1-2\theta\cos(z-x)+\theta^2} \right\}.$$

En posant dans la première intégrale $z+x=\omega$ dans la seconde $z-x=\omega$, on a

$$(19)\quad \varpi(x, s;\ J, t) = \frac{1-\theta^2}{2\pi} \left\{ \int_{\alpha+x}^{\beta+x} \frac{d\omega}{1-2\theta\cos\omega+\theta^2} + \int_{\alpha-x}^{\beta-x} \frac{d\omega}{1-2\theta\cos\omega+\theta^2} \right\}$$

$$= \frac{1-\theta^2}{2\pi} \{[g(\beta+x, \theta) - g(\alpha+x, \theta)] + [g(\beta-x, \theta) - g(\alpha-x, \theta)]\}$$

avec

$$g(y) = \int_0^y \frac{d\omega}{1-2\theta\cos\omega+\theta^2} = \frac{2}{1-\theta^2} \operatorname{arc\,tg} \frac{1+\theta}{1-\theta} u,$$

et $u = \operatorname{tg} \frac{1}{2}y$, en prenant pour $\operatorname{arc\,tg} y$ la détermination principale entre $-\frac{1}{2}\pi$ et $+\frac{1}{2}\pi$. La valeur obtenue pour l'intégrale suppose que $\operatorname{tg} \frac{1}{2}\omega$ varie de façon continue de 0 à $\operatorname{tg} \frac{1}{2}y$ c'est à dire suppose que $-\pi \leqslant y \leqslant +\pi$. Si, par exemple, $\pi < y < 2\pi$ on aura

$$g(y) = \int_0^\pi + \int_{2\pi-y}^\pi = 2\int_0^\pi + \int_{2\pi-y}^0 = \frac{2}{1-\theta^2} \left[\pi - \operatorname{arc\,tg} \frac{1+\theta}{1-\theta} u \right].$$

D'où en particulier $g(2\pi) = 2\pi/(1-\theta^2)$.

On peut donc former l'expression complète de $\varpi(x, s;\ J, t)$. On a

$$-\pi \leqslant \alpha - x < \beta - x \leqslant +\pi;\quad 0 \leqslant \alpha + x < \beta + x \leqslant 2\pi.$$

Pour appliquer les formules précédentes, on aura donc plusieurs cas à distinguer suivant la position de π par rapport à $\alpha+x$ et $\beta+x$. Supposons par exemple $0 \leqslant \alpha + x < \beta + x \leqslant \pi$. Alors

$$(20)\quad \varpi(x, s;\ J, t) = \frac{1}{\pi}\left\{\left[\operatorname{arc\,tg}\left(\frac{1+\theta}{1-\theta}\operatorname{tg}\frac{\beta+x}{2}\right) - \operatorname{arc\,tg}\left(\frac{1+\theta}{1-\theta}\operatorname{tg}\frac{\alpha+x}{2}\right)\right]\right.$$
$$\left. + \left[\operatorname{arc\,tg}\left(\frac{1+\theta}{1-\theta}\operatorname{tg}\frac{\beta-x}{2}\right) - \operatorname{arc\,tg}\frac{1+\theta}{1-\theta}\operatorname{tg}\frac{\alpha-x}{2}\right]\right\}.$$

Limite de $\varpi(x, s;\ J, t)$ quand $t \to s$. Il est bien entendu que ce calcul suppose $s < t$. Mais il peut être interessant d'examiner ce qui se passe quand $t \to s$, c'est à dire quand $\theta \to 1$. On peut utiliser la forme intégrale (19) de ϖ, qui décompose ϖ en deux parties $\varpi = I_1 + I_2$. Or, en vertu de la formule des accroissements finis, on a

$$I_1 = \frac{1-\theta^2}{2\pi}\,\frac{\beta-\alpha}{1-2\theta\cos(\gamma+x)+\theta^2}$$

avec $$\alpha < \gamma < \beta.$$

D'ailleurs $0 \leqslant \alpha + x < \beta + x \leqslant 2\pi$. Dans le cas général on a

$$0 < \alpha + x < \beta + x < 2\pi.$$

Si on appelle ϵ le plus petit des deux nombres positifs $\alpha+x$, $2\pi-\beta-x$ on aura $\cos(\gamma+x) < \cos\epsilon$, d'où

$$0 < I_1 < \frac{1-\theta^2}{2\pi}\,\frac{\beta-\alpha}{1-2\theta\cos\epsilon+\theta^2},$$

et par suite $$\lim_{\theta\to 1} I_1 = 0.$$

Nous avons laissé de côté le cas où l'un des nombres $\alpha+x$, $2\pi-\beta-x$ serait nul, ce qui ne peut se présenter que si $x = \alpha = 0$ ou si $x = \beta = \pi$, c'est à dire quand x coïncide avec l'une des extrémités de J, celle-ci coïncidant en même temps avec l'une des extrémités de l'intervalle de variation $(0, \pi)$ de x.

Passons à I_2. Comme on a $-\pi \leqslant \alpha - x < \beta - x \leqslant +\pi$, on peut écrire

$$I_2 = \frac{1}{\pi}\operatorname{arc\,tg}\left(\frac{1+\theta}{1-\theta}\operatorname{tg}\frac{\beta-x}{2}\right) - \frac{1}{\pi}\operatorname{arc\,tg}\left(\frac{1+\theta}{1-\theta}\operatorname{tg}\frac{\alpha-x}{2}\right).$$

Supposons d'abord que x n'appartienne pas à l'intervalle J, c'est à dire que $\alpha-x$ et $\beta-x$ soient tous deux de mêmes signes. S'ils sont tous deux positifs les deux termes de I_2 tendront vers $+\frac{1}{2}$. Si $\alpha-x$ et $\beta-x$ sont négatifs, les deux termes de I_2 tendront vers $-\frac{1}{2}$. Ainsi I_2 tendra vers 0. Or dans l'hypothèse actuelle, I_1 tend aussi vers 0.

Finalement, $\varpi(x, s;\ J, t)$ *tend vers* 0 *lorsque* x *n'appartenant pas à* J, t *tend vers* s.

Supposons au contraire que x soit intérieur à J; alors $\alpha-x<0<\beta-x$, le premier terme de I_2 tend vers $+\frac{1}{2}$, le second vers $-\frac{1}{2}$ et leur différence I_2 tend vers 1.

Dès lors, $\varpi\ (x, s;\ J, t)$ *tend vers l'unité lorsque* x *étant intérieur à* J, t *tend vers* s.

Il n'y avait pas nécessité à imposer ces résultats qui sont indépendants des conditions relatives à $s=t$. Mais ils prouvent que *dans l'exemple actuel*

$$\lim_{t\to s}\varpi(x, s;\ J, t)=\varpi(x, s;\ J, s)$$

quand x ne coïncide pas avec une extrémité de J.

Limite quand le temps s'écoule. Etudions sur le même exemple (14) ce qui se passe lorsque t croît indéfiniment. $a(t)$ étant positive et croissante, on a deux cas à distinguer.

(1) $a(t)$ tend vers l'infini avec t; alors $\theta(s, t)=a(s)/a(t)$ tend vers 0. Or

$$\frac{1}{(1+\theta)^2}\leqslant\frac{1}{1-2\theta\cos\omega+\theta^2}\leqslant\frac{1}{(1-\theta)^2},$$

d'où

$$\frac{1-\theta^2}{2\pi}\left\{\frac{2(\beta-\alpha)}{(1+\theta)^2}\right\}\leqslant\varpi(x, s;\ J, t)\leqslant\frac{(1-\theta^2)}{2\pi}\left\{\frac{2(\beta-\alpha)}{(1-\theta)^2}\right\}.$$

Quand x, s, J, restant invariables, t croît indéfiniment, on aura donc

$$\lim_{t\to+\infty}\varpi(x, s;\ J, t)=\frac{\beta-\alpha}{\pi},\tag{23}$$

et cette limite, si elle dépend de J, sera indépendante de l'état initial caractérisé par x à l'instant s. On sera dans *le cas régulier* en étendant à l'équation de Chapman une dénomination proposée par M. Hadamard, dans le cas particulier de l'équation de Markoff.

(2) Mais quand t croît, $a(t)$ peut aussi tendre vers une limite finie (positive), l, et c'est la seule alternative. Dans ce cas $\theta(s, t)$ tendra en

croissant vers la quantité $a_1(s) = a(s)/l$ qui sera > 0 et < 1. La fonction

$$\left|\frac{1}{1-2\theta\cos\omega+\theta^2}-\frac{1}{1-2a_1\cos\omega+a_1^2}\right|=\left|\frac{(a_1-\theta)(a_1+\theta-2\cos\omega)}{(\ldots)(\ldots)}\leqslant\frac{4(a_1-\theta)}{(1-a_1)^4}\right|$$

converge vers 0 uniformément quand ω variant arbitrairement de 0, π, θ tend vers a_1.

De sorte que

$$\lim_{\theta\to a_1}\int_{y_1}^{y_2}\frac{d\omega}{1-2\theta\cos\omega+\theta^2}=\int_{y_1}^{y_2}\frac{d\omega}{1-2a_1\cos\omega+a_1^2},$$

et par suite

$$(24)\quad \lim_{t\to+\infty}\varpi(x,s;J,t)$$
$$=\frac{1-a_1^2}{2\pi}\left\{\int_{\alpha+x}^{\beta+x}\frac{d\omega}{1-2a_1\cos\omega+a_1^2}+\int_{\alpha+x}^{\beta-x}\frac{d\omega}{1-2a_1\cos\omega+a_1^2}\right\}.$$

Autrement dit, $\varpi(x, s; J, t)$ tend vers la valeur qu'on aurait obtenue en remplaçant dans l'expression (19) de ϖ, $\theta(s, t)$ par l'expression indépendante de t: $a_1(s)$. On voit que dans ce cas [où pourtant $f(x, s; y, t)$ est positif, et non nul, avec $t-s$], $\varpi(x, s; J, t)$ a bien encore une limite quand $t\to+\infty$, mais cette limite dépend à la fois de x, de s, et de l'intervalle J. C'est la généralisation de ce que nous avons appelé dans le même cas de Markoff, le cas *semi-régulier**. Observons que $a(s)$ est encore une fonction positive continue de s, mais tendant vers 1 quand s tend vers $+\infty$ et que l'on peut écrire $\theta(s, t) = a_1(s)/a_1(t)$. De sorte qu'on aurait pu d'avance remplacer a par a_1 dans l'expression de $\theta(s, t)$ et par suite se borner à deux cas pour $a(s)$:

$$(25)\qquad \text{ou bien } \lim_{t\to\infty}a(s)=+\infty;\quad \text{ou bien } \lim_{t\to+\infty}a(s)=+1.$$

Remarque. Dans la formule (16), la fonction $a(s)$ n'intervient que par $a'(s)/a(s)$. Ceci suggère l'idée, puisqu'en outre $a(s)$ reste positif, de poser $a(s) = e^S$, S sera une variable croîssant avec s. Alors on aura aussi, en posant $a(t) = e^T, \theta = e^{S-T}$. La fonction $f(x, s; y, t)$ deviendra $F(x, S; y, T)$ et l'équation de Chapman sera encore vérifiée quand on repère les époques

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* M. Fréchet, "Compléments á la théorie des probabilités discontinues 'en chaîne'", *Annali Sc. Norm. Sup. Pisa*, 2 (1933), 131–164.

s, t par les nombres S et T. On aura

$$(26) \qquad F(x, S; y, T) = \frac{1}{\pi}\left\{1+2\sum_{n=1}^{+\infty} e^{n(S-T)}\cos nx \cos ny\right\},$$

$$(27) \qquad \lim_{T\to S}\frac{F(x, S; y, T)}{T-S} = \frac{1}{\pi}\,\frac{1-\cos x\cos y}{(\cos x-\cos y)^2} \quad \text{pour} \quad x \neq y,$$

$$(28) \qquad \lim_{T\to S}(T-S)\,F(x, S; x, T) = \begin{cases} \dfrac{1}{\pi} & \text{si} \quad 0 < x = y < \pi, \\ \dfrac{2}{\pi} & \text{si} \quad x = y = \begin{cases} 0 \text{ ou} \\ \pi \end{cases}. \end{cases}$$

Seulement, dans ce cas, quand $t \to +\infty$, on aura deux cas à distinguer : ou bien $T \to +\infty$, ou bien T (restant négatif, ainsi que S) tend vers 0. Inversement, supposons qu'on ait obtenu une solution quelconque de l'équation de Chapman, soit $\phi(M, s; P, t)$. Soit maintenant $\lambda(s)$ une fonction continue constamment croissante quand s croit de $-\infty$ à $+\infty$. *La fonction*

$$\Phi(M, s; P, t) = \phi\{M, \lambda(s); P, \lambda(t)\}$$

sera évidemment une solution de l'équation de Chapman.

Remarque. A. Kolmogoroff* a montré qu'en assujettissant les solutions de l'équation de Chapman à certaines conditions très générales (concernant les ordres de grandeurs des moments), on obtient une classe de solutions de l'équation de Chapman qui satisfont à une même équation aux dérivées partielles de la forme

$$(30) \qquad \frac{\partial}{\partial s} f(x, s; y, t) + A(x, s)\frac{\partial}{\partial x} f(x, s; y, t) + B^2(x, s)\frac{\partial^2}{\partial x^2} f(x, s; y, t) = 0.$$

Nous allons montrer que : l'exemple de solution de l'équation de Chapman *que nous avons donné sous la forme* (14), *n'appartient pas à la classe de solutions considérée par A. Kolmogoroff.*

En effet s'il y appartenait il y existerait deux fonctions $A(x, s)$, $B(x, s)$ telles que notre solution vérifie une équation du type (30). Or en dérivant

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* *Math. Annalen*, 104 (1931), 415–458.

l'expression (14) de f, on a

$$\frac{\partial f}{\partial s} = \frac{2}{\pi}\left[\sum_{n\geqslant 1} n\theta^{n-1}\cos nx\cos ny\right]\frac{\partial\theta}{\partial s},$$

$$\frac{\partial f}{\partial x} = -\frac{2}{\pi}\left[\sum_{n\geqslant 1} n\theta^{n}\sin nx\cos ny\right],$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{2}{\pi}\left[\sum_{n\geqslant 1} n^2\theta^{n}\cos nx\cos ny\right],$$

car les séries obtenues en dérivant terme à terme la série (14) sont uniformément convergentes quand $0 \leqslant \left\{\begin{matrix} x \\ y \end{matrix}\right\} < \pi$ et quand, t restant fixe, s reste dans un intervalle borné. En substituant dans l'équation (30), on aurait

$$\frac{2}{\pi}\sum_{n\geqslant 1}\cos ny\left\{\cos nx\frac{\partial\theta}{\partial s} - \theta[\sin nx\, A(x,s) + n\cos nx\, B^2(x,s)]\right\} n\theta^{n-1} = 0.$$

Cette relation ne peut avoir lieu pour toute valeur de y entre 0 et π que si chacune des accolades est nulle, c'est à dire, en divisant par cos nx, si

$$\frac{1}{\theta}\frac{\partial\theta}{\partial s} - nB^2(x,s) = A(x,s)\operatorname{tg} nx \quad \text{pour} \quad n = 1, 2, \ldots.$$

Comme $\partial\theta/\theta\partial s$ ne dépend ni de n, ni de x, il est visible que cette relation ne peut avoir lieu que si $A(x, s)$ et $B(x, s)$ sont nuls. Et alors $\theta = a(s)/a(t)$ serait indépendant de s, ce qui est contraire à l'hypothèse que $a(s)$ est une fonction croîssante.

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THE CESÀRO-PERRON SCALE OF INTEGRATION

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1. *Introduction.*

In a recent paper† I introduced a generalization of the derivative of a function of a real variable and investigated the Perron integral associated with it. This generalized derivative was formed by taking the numerator of the incrementary ratio to be the arithmetic mean (that is, the first Cesàro mean) of the increment of the function.

Let us take, instead of the first Cesàro mean, the mean of any order and we have a further generalization. In this paper the extension is made to a C_r-derivative and a $C_r P$-integral for any positive integral value of r. The work proceeds by induction, so that, in constructing the $C_r P$-integral, we shall take as established the properties of the $C_{r-1} P$-integral. The integrals which figure in the scheme as the $C_0 P$ and $C_1 P$ integrals are respectively the special Denjoy integral and the integral called the CP-integral in the paper quoted.

2. *C_r-continuity and C_r-derivates.*

For positive r, we write $C_r(f, x, x+h)$, the r-th Cesàro mean of $f(x)$ in $(x, x+h)$, for

$$\frac{r}{h^r}\int_x^{x+h}(x+h-\xi)^{r-1}f(\xi)\,d\xi,$$

† "The Cesàro-Perron integral", *Proc. London Math. Soc.* (2), 34 (1932), 314–322. This paper will be referred to as CP.

where h may be positive or negative. The sense in which it is appropriate to understand the integral will be considered later. The mean of order zero, $C_0(f, x, x+h)$, is to be interpreted as meaning $f(x+h)$.

We say that $f(x)$ is C_r-continuous at x_0 if $C_r(f, x_0, x_0+h)$ tends to $f(x_0)$ as h tends to 0.

We define $C_r D^* f(x)$, the (two-sided) Cesàro derivate of $f(x)$ for a given value of x, to be

$$\overline{\lim_{h\to 0}} \frac{C_r(f, x, x+h)-f(x)}{h/(r+1)}.$$

The lower C_r-derivate, $C_r D_* f(x)$, is the lower limit of the same ratio. If $C_r D^* f = C_r D_* f$, then f has a C_r-derivative $C_r Df$.

We shall now prove that, if $f(x)$ is an exact C_{r-1}-derivative, in which case, according to the theory that we shall develop, it is integrable in the $C_{r-1}P$ sense, then, as h tends to 0,

$$\frac{r}{h^r}\int_x^{x+h}(x+h-\xi)^{r-1}f(\xi)\,d\xi \to f(x),$$

the integral being taken in the $C_{r-1}P$ sense.

If $r=1$, the $C_0 P$-integral is the special Denjoy integral, and the result is obvious.

Suppose that $r>1$ and that $f(x)$ is the C_{r-1}-derivative of a function $F(x)$. Then, by the definition,

$$\frac{r-1}{h^{r-1}}\int_x^{x+h}(x+h-\xi)^{r-2}F(\xi)\,d\xi - F(x) = \frac{h}{r}f(x)+o(h).$$

Now, using the theorem of integration by parts for the $C_{r-1}P$-integral (Theorem 5), we have

$$\frac{r}{h^r}\int_x^{x+h}(x+h-\xi)^{r-1}f(\xi)\,d\xi$$

$$= \frac{r}{h^r}\left\{-h^{r-1}F(x)+(r-1)\int_x^{x+h}(x+h-\xi)^{r-2}F(\xi)\,d\xi\right\}$$

(where the latter integral is a $C_{r-2}P$-integral)

$$= \frac{r}{h^r}\left\{\frac{h^r}{r}f(x)+o(h^r)\right\} = f(x)+o(1).$$

Hence it appears that an appropriate sense in which to take the integral expressing the r-th Cesàro mean is the $C_{r-1}P$ sense.

Before going further we must establish the consistency of our scale of continuity and differentiation. It must be proved that, if, for $x = x_0$, $f(x)$ is C_{r-1}-continuous, then it is C_r-continuous, and that, if $C_{r-1}Df(x_0)$ exists, then $C_rDf(x_0)$ exists and has the same value. It will be sufficient if we give the proof of the following theorem, of which the last assertion is a corollary.

THEOREM 2.1. $C_rD^*f(x_0) \leqslant C_{r-1}D^*f(x_0)$.

The proof is easy if $r = 1$; suppose that $r > 1$.

If $C_{r-1}D^*f$ is defined, then $f(x)$ is integrable $C_{r-2}P$.

For brevity, suppose that $x_0 = 0$, $f(0) = 0$, and consider only positive values of h. Let $C_{r-1}D^*f(0) = A$ (supposed finite). Then, if A_1 is any number greater than A, we have, for all $h \leqslant h_0(A_1)$,

$$\int_0^h (h-\xi)^{r-2} f(\xi)\, d\xi \leqslant \frac{A_1 h^r}{(r-1)r}, \tag{2.1}$$

the integral being a $C_{r-2}P$-integral.

We shall prove that

$$\int_0^h (h-\xi)^{r-1} f(\xi)\, d\xi = (r-1)\int_0^h d\xi \int_0^\xi (\xi-\eta)^{r-2} f(\eta)\, d\eta. \tag{2.2}$$

The integral on the left-hand side is, by repeated integration by parts, equal to

$$(r-1)\int_0^h (h-\xi)^{r-2} f_1(\xi)\, d\xi = \ldots = (r-1)!\int_0^h f_{r-1}(\xi)\, d\xi,$$

where $f_s(\xi)$ denotes

$$\int_0^\xi d\xi_1 \int_0^{\xi_1} d\xi_2 \ldots \int_0^{\xi_{s-1}} f(\xi_s)\, d\xi_s, \tag{2.3}$$

the innermost integral being a $C_{r-2}P$-integral, the next a $C_{r-3}P$-integral, and so on, the outermost being a $C_{r-s-1}P$-integral.

Similarly, the inner integral on the right-hand side of (2.2) is equal to

$$(r-2)!\int_0^\xi f_{r-2}(\eta)\, d\eta,$$

so that the right-hand side of (2.2) is equal to

$$(r-1)!\int_0^h f_{r-1}(\xi)\,d\xi.$$

Substituting from (2.1) in the right-hand side of (2.2), we have

$$\int_0^h (h-\xi)^{r-1}f(\xi)\,d\xi \leqslant (r-1)\int_0^h \frac{A_1\xi^r}{(r-1)\,r}\,d\xi = \frac{A_1 h^{r+1}}{r(r+1)}.$$

Therefore $C_r D^* f(0) \leqslant A$ and the theorem is proved.

We wish to prove now that a function with positive C-derivates is increasing.

THEOREM 2.2. *If, for $a \leqslant x \leqslant b$, $f(x)$ is C_r-continuous, $C_r D_* f(x) \geqslant 0$, then $f(b) \geqslant f(a)$.*

Integrating by parts the integral in the numerator of $C_r D_* f(x)$, as in Theorem 2.1, and taking the lower limits of integration in $f_s(\xi)$ to be x instead of 0, we find

$$C_r D_* f(x) = \underline{\lim_{h\to 0}} \frac{f_r(x+h)-(h_r/r!)\,f(x)}{h^{r+1}/(r+1)!},$$

where $f_r(x)$ is a continuous function.

It is now convenient to alter the notation and to devote a separate section to investigations including this theorem, which will appear as Theorem 3.4.

3. *Direct definition of the n-th derivative.*

Let $F(x)$ be continuous and $1 \leqslant r \leqslant n-1$.

Suppose that, for a given x, as h tends to 0, there exist finite numbers $F_1(x)$, ..., $F_{r-1}(x)$ such that

$$\frac{F(x+h)-F(x)-hF_1(x)-(h^2/2!)\,F_2(x)-\ldots-\{h^{r-1}/(r-1)!\}\,F_{r-1}(x)}{h^r/r!}$$

tends to a finite limit. Call this limit $F_r(x)$

Suppose that $F_1(x)$, ..., $F_{n-1}(x)$ have been thus inductively defined.

Denote the upper and lower limits, as $h \to \pm 0$, of

$$\frac{F(x+h)-F(x)-hF_1(x)-(h^2/2!)\,F_2(x)-\ldots-\{h^{n-1}/(n-1)!\}\,F_{n-1}(x)}{h^n/n!}$$

by F_n^+, F_{n+}, F_n^-, F_{n-}, specifying the argument x if necessary.

Write $$F_n^* = \max(F_n^+, F_n^-),$$

$$F_{n*} = \min(F_{n+}, F_{n-}).$$

Then $F_n(x)$ is a modification of the n-th derivative of $F(x)$. The n-th derivative as ordinarily defined we shall write $d^n F/dx^n$. Clearly $F_1(x) = dF/dx$. If $d^n F/dx^n$ exists, it is easy to prove that $F_n(x)$ exists, but the modified n-th derivative may exist for value of x for which there is no ordinary n-th derivative ($n > 1$).

THEOREM 3.1. *If $F_n(x)$ exists, it is of Baire's first class.*

For we have, by simple algebra,

$$F_n(x) = \lim_{h \to 0} \frac{F(x+nh) - {}_nC_1 F\{x+(n-1)h\} + \ldots + (-1)^n F(x)}{h^n},$$

the coefficients in the numerator being binomial coefficients.

To prove further theorems, we need a lemma.

Definition. We say that a sequence $\phi_n(x)$ tends to $\phi(x)$ upper-semi-uniformly if

(1) for each x, $\phi_n(x) \to \phi(x)$,

(2) given ϵ, there exists $n_0(\epsilon)$ independent of x such that $\phi_n(x) \geqslant \phi(x) - \epsilon$ for $n \geqslant n_0$ and all x.

LEMMA. *If, for each n, $\phi_n(x)$ is upper-semi-continuous, and $\phi_n(x) \to \phi(x)$ upper-semi-uniformly, then $\phi(x)$ is upper-semi-continuous.*

We have to prove that, for any x_0,

$$\phi(x_0) \geqslant \overline{\lim_{x \to x_0}}\, \phi(x).$$

If this is false, there is a positive number k and a sequence x_ν tending to x_0 for which

$$\phi(x_\nu) > \phi(x_0) + k,$$

Upper-semi-uniformity shows that there is a number n_0 such that

$$\phi_n(x_\nu) > \phi(x_0) + \tfrac{1}{2}k$$

for all x_ν and all $n \geqslant n_0$.

This gives, since $\phi_n(x)$ is upper-semi-continuous,

$$\phi_n(x_0) \geqslant \phi(x_0)+\tfrac{1}{2}k,$$

which contradicts the assumption that $\phi_n(x_0) \to \phi(x_0)$.

THEOREM 3.2. *If $F_{n+}(x) > K$ for all x in (a, b), then $F_{n-1}(x)$ is upper-semi-continuous.*

We have, from algebra similar to that of Theorem 3.1,

$$\frac{F\{x+(n-1)h\}-{}_{n-1}C_1 F\{x+(n-2)h\}+\ldots\pm F(x)}{h^{n-1}} = F_{n-1}(x)+\tfrac{1}{2}(n-1)hR(x, h),$$

where $R(x, h) > K$ if the positive number h is sufficiently small.

The left-hand side, which is a continuous function of x, thus tends to $F_{n-1}(x)$ upper-semi-uniformly and the result follows from the lemma.

Similarly we prove that, if $F_{n-}(x) > K$, then $F_{n-1}(x)$ is lower semi-continuous. Combining these two results, we have the corollary

If $F_{n*}(x) > K$, then $F_{n-1}(x)$ is continuous.

THEOREM 3.3. *If $F_n(x)$ is continuous, then d^nF/dx^n exists and is equal to $F_n(x)$.*

We require a lemma.

LEMMA 3.31. *If, for $1 \leqslant r \leqslant n-1$, $F_r(x) = d^rF/dx^r$, and, for $1 \leqslant r \leqslant n$, $F_r(x)$ is continuous, then d^nF/dx^n exists and is equal to $F_n(x)$.*

We have

$$F_n(x) = \lim \frac{F(x+h)-F(x)-h(dF/dx)-\ldots-\{h^{n-1}/(n-1)!\}d^{n-1}F/dx^{n-1}}{h^n/n!}$$

$$= \lim \frac{\phi(h)}{\psi(h)}, \quad \text{say.}$$

Now, for $h = 0$,

$$\phi = \frac{d\phi}{dh} = \ldots = \frac{d^{n-2}\phi}{dh^{n-2}} = 0,$$

$$\psi = \frac{d\psi}{dh} = \ldots = \frac{d^{n-2}\psi}{dh^{n-2}} = 0.$$

Also $$\frac{d^{n-1}\phi}{dh^{n-1}} = \left[\frac{d^{n-1}F}{dx^{n-1}}\right]_{x+h} - \left[\frac{d^{n-1}F}{dx^{n-1}}\right]_x$$

and $$\frac{d^{n-1}\psi}{dh^{n-1}} = h.$$

By the ordinary theory of undetermined forms, the limits of $\phi(h)/\psi(h)$ lie between those of

$$\frac{d^{n-1}\phi/dh^{n-1}}{d^{n-1}\psi/dh^{n-1}}.$$

Therefore $$D_+\left(\frac{d^{n-1}F}{dx^{n-1}}\right) \leqslant F_n(x) \leqslant D^+\left(\frac{d^{n-1}F}{dx^{n-1}}\right).$$

Writing $$G(x) = \int_a^x F_n(x)\,dx - \frac{d^{n-1}F}{dx^{n-1}},$$

we have $$D_+ G(x) \leqslant 0 \leqslant D^+ G(x).$$

Hence $G(x)$ is a constant, and the lemma follows.

We now prove the main theorem.

By Theorem 3.2, every $F_r(x)$ is continuous $(1 \leqslant r \leqslant n)$.

Now $F_1(x) = dF/dx$. Hence, taking $n = 2$ in Lemma 3.31, we see that d^2F/dx^2 exists and is equal to $F_2(x)$. Repeat this argument and the theorem follows.

COROLLARY. *If, throughout an interval, $F_n(x) = 0$, then $F(x)$ is a polynomial of degree at most $n-1$.*

THEOREM 3.4. *If, for $a \leqslant x \leqslant b$, $F_{n*}(x) \geqslant 0$, then $F_{n-1}(x)$ is an increasing function.*

By Theorem 3.2, $F_{n-1}(x)$ is continuous.

By Theorem 3.3, $d^{n-1}F/dx^{n-1}$ exists and is equal to $F_{n-1}(x)$.

By the theory of undetermined forms set out in Lemma 3.31,

$$D^+ F_{n-1}(x) \geqslant F_{n*}(x) \geqslant 0.$$

Therefore $F_{n-1}(x)$ is an increasing function.

4. *Definition and properties of the $C_r P$-integral.*

We say that $M(x)$ is a C_r-major-function of $f(x)$ in (a, b) if

(1) $M(x)$ is C_r-continuous,

(2) $M(a) = 0$,

(3) $C_r D_* M(x) > -\infty$,

(4) $C_r D_* M(x) \geqslant f(x)$.

From major and minor functions we define by Perron's method† the $C_r P$-integral

$$(C_r P) \int_a^b f(x)\, dx.$$

We note first the following theorem of consistency.

If $(C_{r-1} P) \int_a^b f(x)\, dx$ *exists, then* $(C_r P) \int_a^b f(x)\, dx$ *exists and has the same value.*

The proof of this is based on Theorem 2 . 1, which shows that a C_{r-1}-major function is also a C_r-major function.

The integral has the following properties, which have been given for the CP-integral in the paper cited‡. The reader will easily supply the proofs.

The indefinite $C_r P$-integral is C_r-continuous.

The $C_r P$-integral integrates any everywhere finite C_r-derivative.

The indefinite integral is C_r-differentiable p.p.

A positive function integrable $(C_r P)$ is integrable (Lebesgue) and the integrals agree.

If all the functions of a sequence lie between two functions integrable $(C_r P)$ the limit may be taken under the integral sign.

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† For details see CP.

‡ Theorems I–V, 319.

5. *Integration by parts.*

The theorem of integration by parts takes this form.

THEOREM 5. *If, for* $a \leqslant x \leqslant b$,

$$F(x) = (C_r P) \int_a^x f(x)\,dx,$$

$$G_r(x) = \int_a^x d\xi_1 \int_a^{\xi_1} d\xi_2 \ldots \int_a^{\xi_{r-1}} g(\xi_r)\,d\xi_r,$$

where $g(x)$ *has bounded variation, then* $f(x)\,G_r(x)$ *is* $C_r\,P$*-integrable in* (a, b) *and*

$$\int_a^b f G_r\,dx = \Big[F G_r\Big]_a^b - \int_a^b F G_{r-1}\,dx$$

(*where the integral on the right-hand side is a* $C_{r-1}\,P$*-integral*).

This is proved for $r = 1$ in the paper (CP).

We shall proceed by induction and assume the enunciation true with $r-1$ in place of r.

We require the following lemmas which give conditions for the product of two functions to be (1) C_r-continuous, (2) C_r-differentiable.

LEMMA 5.1. *If* $\Phi(x)$ *is* C_r*-continuous at* $x = x_0$ *and* $\Psi_r(x)$ *is the r-fold integral of a function* $\psi(x)$ *bounded near* $x = x_0$, *then* $\Phi(x)\,\Psi_r(x)$ *is* C_r*-continuous at* $x = x_0$.

LEMMA 5.2. *If* $C_r D\,\Phi(x_0) = \phi(x_0)$ *and* $\Psi_r(x)$ *is the r-fold integral of a function bounded near* $x = x_0$, *then*

$$C_r D(\Phi\Psi_r) = \Phi\Psi_{r-1} + \phi\Psi_r.$$

We prove Lemma 5.2. The proof of Lemma 5.1 is similar. For brevity we take $x_0 = 0$. If $\Psi_r(x)$ is a constant, the conclusion clearly holds. Also if $\Phi(x)$ is a constant, the conclusion holds. We may therefore suppose that Ψ_r, Φ have been adjusted by the addition of constants to make

$$\Phi(0) = \Psi_r(0) = 0.$$

Our hypothesis then is that

$$\frac{r(r+1)}{h^{r+1}} \int_0^h (h-\xi)^{r-1}\,\Phi(\xi)\,d\xi \to \phi(0),$$

and we wish to prove that

$$\frac{r(r+1)}{h^{r+1}}\int_0^h (h-\xi)^{r-1}\Phi(\xi)\,\Psi_r(\xi)\,d\xi \to 0,$$

the integrals being C_{r-1} P-integrals. We shall actually prove a slightly more general result, which we state as a separate lemma.

LEMMA 5.3. *If $\Phi(x)$ is C_{r-1} P-integrable, $\Psi_r(x)$ satisfies the conditions of Lemma 5.2, and*

$$(C_{r-1}P)\int_0^h (h-\xi)^{r-1}\Phi(\xi)\,d\xi = o(h^s) \quad (s>-1),$$

then

$$(C_{r-1}P)\int_0^h (h-\xi)^{r-1}\Phi(\xi)\,\Psi_r(\xi)\,d\xi = o(h^{s+1}).$$

We prove this first for $r=1$.

$$\begin{aligned}\int_0^h \Phi\Psi_1\,d\xi &= \Phi_1(h)\,\Psi_1(h) - \int_0^h \Phi_1\psi\,d\xi \\ &= o(h^s)\,O(h) - \int_0^h o(\xi^s)\,O(1)\,d\xi \\ &= o(h^{s+1}).\end{aligned}$$

In the general case, if we integrate the C_{r-1} P-integral by parts $r-1$ times, the hypothesis becomes that

$$\Phi_r(h) = o(h^s),$$

where the suffix r denotes r-fold integration from 0 to h, as in (2.3).

Similarly, $$\int_0^h (h-\xi)^{r-1}\Phi(\xi)\,\Psi_r(\xi)\,d\xi = \{\Phi\Psi_r\}_r(h).$$

When we integrate by parts, this is

$$\{\Phi_1\Psi_r\}_{r-1}(h) - \{\Phi_1\Psi_{r-1}\}_r(h).$$

Lemma 5.3, with $r-1$ in place of r and Φ_1 in place of Φ, shows that

$$\{\Phi_1\Psi_r\}_{r-1}(h) = o(h^{s+1}).$$

We thus need to prove that

$$\{\Phi_1 \Psi_{r-1}\}_r(h) = o(h^{s+1}).$$

$\Phi_1 \Psi_{r-1}$ is integrable $C_{r-2}\, P$ and so its $(r-1)$-fold integral is a continuous function. Hence it is enough to prove that

$$\{\Phi_1 \Psi_{r-1}\}_{r-1}(h) = o(h^s).$$

By integration by parts, the left-hand side becomes

$$\{\Phi_2 \Psi_{r-1}\}_{r-2}(h) - \{\Phi_2 \Psi_{r-2}\}_{r-1}(h).$$

Lemma 5.3, with $r-2$ in place of r, shows that the first term is $o(h^{s-1})$. The second term is the integral of a continuous function and so it is enough to prove that

$$\{\Phi_2 \Psi_{r-2}\}_{r-2}(h) = o(h^{s-1}).$$

Continuing this process of integrating by parts, dealing with the first term by this Lemma and reducing by one the number of integrations in the second term and the power of h under the o-sign, we have finally to prove that

$$\{\Phi_r \psi\}(h) = o(h^{s-r+1}),$$

which is true.

The following corollary to Lemma 5.2 is established by similar arguments.

COROLLARY. *If $C_r D_* \Phi(x_0) = \phi(x_0)$ and $\Psi_r(x)$ is the r-fold integral of a function bounded near $x = x_0$, then $\Phi\Psi_{r-1} + \phi\Psi_r = C_r D_*(\Phi\Psi_r)$ or $C_r D^*(\Phi\Psi_r)$ according as $\Psi_r(x_0)$ is positive or negative.*

We now give the proof of Theorem 5.

We may suppose, without loss of generality, that $G_{r-1}(x)$ and $G_r(x)$ are positive throughout (a, b). For $G_{r-1}(x)$ is the difference between two positive functions, $G^{(1)}_{r-1}$ and $G^{(2)}_{r-1}$ say.

Let $G^{(1)}_r$ and $G^{(2)}_r$ be the integrals of these functions. If the theorem is true for the pair $G^{(1)}_r$, $G^{(1)}_{r-1}$ and also for the pair $G^{(2)}_r$, $G^{(2)}_{r-1}$, it will be true for the pair G_r, G_{r-1}.

We now prove that $FG_{r-1}+fG_r$ is C_rP-integrable and that

$$\int_a^b (FG_{r-1}+fG_r)\,dx = \Big[FG_r\Big]_a^b. \tag{5.1}$$

Let $M(x)$ be a C_r-major function of $F(x)$. Then it is easy to see, from Lemma 5.1 and the corollary to Lemma 5.2, that MG_r is a major function of $FG_{r-1}+fG_r$. Similarly for minor functions $m(x)$, and (5.1) follows.

The theorem being supposed true for $r-1$, the function FG_{r-1} is $C_{r-1}P$-integrable and, *a fortiori*, C_rP-integrable.

Then fG_r, being the difference of two C_rP-integrable functions, is C_rP-integrable and the theorem follows.

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